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TRANSVERSE DRIFT OSCILLATIONS IN AN INHOMOGENEOUS PLASMA

A. B. MIKHAĬLOVSKIĬ

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It is shown that an inhomogeneous plasma in a fixed magnetic field can support waves that propagate across the magnetic field with the same polarization as ordinary waves but at frequencies much lower than the electron plasma frequency. These waves are localized in regions in which the plasma pressure is nonuniform and their phase velocity is a sensitive function of the plasma electron Larmor drift velocity.

1. INTRODUCTION

IT is well known that a uniform plasma cannot support the propagation of the ordinary wave at frequencies below the electron plasma frequency $\omega < \omega_0$ when the wave vector is perpendicular to the external magnetic field $\mathbf{k} \perp \mathbf{H}_0$ (so long as cyclotron singularities are neglected[1]). We show in the present work, on the other hand, that such waves can propagate in an inhomogeneous plasma. These waves are intimately related with drift flows in an inhomogeneous plasma; the electric field of the wave is along the fixed magnetic field $\mathbf{E} \parallel \mathbf{H}_0$ and consequently is perpendicular to the wave vector, as for the usual ordinary wave characterized by $\omega > \omega_0$. Because of these two characteristics we call the waves investigated below "transverse drift waves" (or "transverse drift oscillations").

The analysis of the transverse drift oscillations is carried out using the dielectric tensor $\epsilon_{\alpha\beta}$ that has been derived earlier.[2] In Sec. 2 we derive the differential equation that describes the oscillations of interest.

The general properties of this equation are discussed in Sec. 3. We point out the analogy with the Schrödinger equation, discuss possible forms of the potentials, and show in general that all solutions of the differential equation correspond to perturbations that are localized in space; it is also shown that the wave frequencies form a discrete spectrum and that the phase velocity is always in the same direction as the electron Larmor drift velocity.

The short-wave approximation ($E'/kE \ll 1$) is treated in Sec. 4. In this approximation we investigate the relation between the drift oscillations and the ordinary wave that exists in a uniform plasma. Here we also analyze the excitation of oscillations near the ion-cyclotron harmonics. These oscillations can be excited if the plasma is sufficiently inhomogeneous and if the electron temperature is high enough.

In Sec. 5 the original equation is studied in the quasi-classical approximation,[3] which applies when the wavelength is small in the direction of the inhomogeneity. It is shown that the number of levels in the discrete frequency spectrum is finite. Making use of a quasi-classical quantization rule we show that the short-wave approximation corresponds to perturbations that are localized in the region of the potential minimum.

In Sec. 6 we consider the case in which the scale size of the inhomogeneity in the plasma is much smaller than the plasma skin depth and the wavelength in the direction in which the plasma is uniform. It is shown that under these conditions one of the solutions of the differential equation describes a transverse wave whose properties are independent of the detailed structure of the inhomogeneity.

In Sec. 7 we find an exact solution of the differential equation for a particular potential and analyze the transition from the exact expressions to the approximations discussed in the preceding sections.

In Sec. 8 we point out the relation between the oscillations investigated in the present work and other kinds of oscillations in an inhomogeneous plasma.

2. FORMULATION OF THE PROBLEM

It is well known from the theory of oscillations of a uniform plasma (for example, Ginzburg[3]) that the general dispersion equation separates into two equations when $\mathbf{k} \perp \mathbf{H}_0$ ($\mathbf{k}$ is the wave vector, $\mathbf{H}_0$ is the fixed magnetic field, $\mathbf{H}_0 \parallel z$); one of these equations

$$\varepsilon_{zz} - N^2 = 0$$

describes the ordinary wave while the other

$$\varepsilon_{xx}(N^2 - \varepsilon_{yy}) + \varepsilon_{xy} = 0$$

describes the extraordinary wave and the plasma wave. (Here, $\epsilon_{\alpha\beta} = \epsilon_{\alpha\beta}(\omega, k)$ is the dielectric tensor, $N^2 = c^2k^2/\omega^2$ is the square of the refractive index, ω is the oscillation frequency.)

A similar situation—separation of the Maxwell equations into two independent sets—occurs in an inhomogeneous plasma in a magnetic field in which the lines of force are straight. If a dielectric tensor is constructed for such a medium, as has been done earlier by the present author,[2] and if $k_z = 0$, the elements ϵ_{xz}, ϵ_{zx}, ϵ_{yz}, and ϵ_{zy} vanish and the Maxwell equations become

$$\frac{c^2}{\omega^2}\frac{d^2E_z}{dy^2} + (\varepsilon_{zz} - N^2)\,E_z = 0; \qquad \text{(I)}$$

$$\frac{c^2}{\omega^2}\frac{d^2E_x}{dy^2} - \frac{ic^2k}{\omega^2}\frac{dE_y}{dy} + \varepsilon_{xx}E_x + \varepsilon_{xy}E_y = 0;$$

$$-i\frac{c^2k}{\omega^2}\frac{dE_x}{dy} + \varepsilon_{yx}E_x + (\varepsilon_{yy} - N^2)\,E_y = 0. \qquad \text{(II)}$$

The wave vector k is along the x axis, the inhomogeneity direction is along the y axis and $E_\alpha = E_\alpha^0(y)\,e^{-i\omega t+ikx}$.

In the present work we shall be interested only in (I). Our analysis is limited to low frequencies ($\omega \ll \omega_c$, ω_c is the electron cyclotron frequency). The plasma pressure is assumed to be small compared with the magnetic pressure $\beta = 8\pi p/H_0^2 \ll 1$. We neglect the inhomogeneity in the magnetic field but take account of the inhomogeneity in density and temperature (the particle velocity distribution is assumed to be approximately Maxwellian). With these assumptions, the element of the dielectric tensor required below ϵ_{zz} assumes the form[2]

$$\varepsilon_{zz} = 1 - \frac{\omega_0^2}{\omega^2}\left(1 - \frac{kv_0}{\omega}\right) - \sum_{L=-\infty}^{+\infty}\left[1 + \frac{1}{k}\frac{\partial}{\partial y}\left(b\,\frac{\omega_{ci}}{\omega} - L\right)\frac{\omega_{0i}^2\,e^{-b}\,I_L(b)}{\omega - L\omega_{ci}}\right],$$

$$\omega_0^2 = \frac{4\pi e^2 n_0}{m}, \quad \omega_{0i}^2 = \frac{4\pi e_i^2 n_0}{m_i}, \quad \omega_{ci} = \frac{e_i H_0}{m_i c}, \quad b = \frac{k^2 T_i}{m_i\,\omega_{ci}^2}. \qquad (2.1)$$

Here, the operator $\partial/\partial y$ operates on the functions n_0 and T_i that appear to the right of it; I_L is the modified Bessel function; e, e_i, m, and m_i are the charges and masses of the electron and ion respectively, n_0 is the plasma density; T_i is the ion temperature and v_0 is the electron Larmor drift velocity:

$$v_0 = -\frac{1}{n_0 m\omega_c}\frac{d(nT)}{dy}, \qquad (2.2)$$

where T is the electron temperature and $\omega_c = eH_0/mc$.

The equations (I) and (2.1) are the starting equations for the analysis of the transverse drift oscillations given below.

We note that if the singularities close to the ion cyclotron harmonics are neglected the element ϵ_{zz} assumes the simple form

$$\varepsilon_{zz} = 1 - \frac{\omega_0^2}{\omega^2}\left(1 - \frac{kv_0}{\omega}\right) \qquad (2.1')$$

and can be obtained from the hydrodynamic analysis. Actually, the current induced by the wave field can be written $j_z = en_0v_z$ where v_z is determined from the equation of motion of the electrons in the wave field

$$-i\omega v_z = \frac{e}{m}E_z + \frac{ev_0}{mc}B_y \equiv \frac{e}{m}\left(1 - \frac{kv_0}{\omega}\right)E_z.$$

The quantity v_0, the equilibrium velocity of the electron component, appears everywhere and can be expressed in terms of the pressure gradient by means of the equilibrium equation [cf. (2.2)]

$$-dp/dy - eH_0c^{-1}n_0v_0 = 0.$$

The current j_z found in this way is

$$j_z = -\frac{ie^2n_0}{m\omega}\left(1 - \frac{kv_0}{\omega}\right)E_z,$$

and, when combined with the displacement current, gives the dielectric constant (2.1′).

We note that the term in ϵ_{zz} containing the quantity kv_0/ω is due to the interaction of electrons with the magnetic field of the wave and is a specific property of the inhomogeneous plasma.

3. GENERAL PROPERTIES OF TRANSVERSE DRIFT OSCILLATIONS

For the time being, we shall not be interested in waves with very high velocity ($\omega/k \sim c$) and assume $\omega/k \ll c$. In addition we assume $\omega \ll \omega_{ci}$. (In Sec. 4 we remove this restriction). Equation (I) then assumes the form

$$\psi'' - \left(k^2 + \frac{\omega_0^2}{c^2} - \frac{v_0}{V}\frac{\omega_0^2}{c^2}\right)\psi = 0, \qquad (3.1)$$

where $V \equiv \omega/k$ is the phase velocity of the wave. We introduce the notation $E_z \equiv \psi$, having in mind the analogy between our equation and the quantum-mechanical Schrödinger equation. In the present problem the role of the quantum mechanical potential U is played by a quantity related to the plasma density, the Larmor drift velocity, and the phase velocity of the wave:

$$U \sim \frac{\omega_0^2}{c^2}\left(1 - \frac{v_0}{V}\right), \qquad (3.2)$$

while the role of the energy E is played by a quantity which is the negative of the square of the wave number:

$$E \sim -k^2. \qquad (3.3)$$

We assume reasonably that when $y \to \pm\infty$ the quantity $\nabla(n_0 T) \to 0$. It then follows that the "potential" U approaches the finite positive values U_1 and U_2 as $y \to \pm\infty$, where

$$U_1 \sim c^{-2}(\omega_0^2)_{y=-\infty}, \qquad U_2 \sim c^{-2}(\omega_0^2)_{y=+\infty}. \qquad (3.4)$$

According to Eq. (3.3) the "particle energy" must be negative so that all solutions of Eq. (3.1) correspond to bound states (cf. [4], Sec. 19). Thus, all the solutions of Eq. (3.1) (if they exist) are local: the field ψ decays exponentially at infinity.

A necessary and sufficient condition for the existence of a solution is that the plasma pressure be inhomogeneous at some point, $\nabla p \neq 0$, i.e., $v_0 \neq 0$. If the pressure variation is monotonic, in which case v_0 has the same sign everywhere, all the solutions of Eq. (3.1) correspond to waves with a phase velocity $V \equiv \omega/k$ in the same direction as v_0 with magnitude smaller than the maximum drift velocity $v_{0\,max}$ so that

$$\frac{\omega}{k v_{0max}} < \frac{1}{1 + c^2k^2/\omega_0^{*2}}, \qquad (3.5)$$

where ω_0^* corresponds to some effective density at the point where $v_0 \neq 0$.

If the pressure variation is not monotonic the different regions (characterized by different directions of the drift velocity) support waves characterized by different directions of the phase velocity (the potential well for $V > 0$ appears as a hill for $V < 0$ and vice versa).

4. SHORT-WAVE APPROXIMATION ($\partial E/\partial y \ll kE$)

A. In certain cases an approximate solution can be obtained for Eq. (3.1) as well as for the more general equation (I). At the present time, investigations of oscillations of an inhomogeneous plasma make wide use of the short-wave approximation (in the x direction) in which case $\partial E/\partial y \ll kE$. [5-7] In this section we investigate Eq. (I) in this approximation. It follows from the structure of Eq. (I) that the contribution of the second derivative E_z'' is small in this case and can be neglected if we are not concerned with the spatial dependence of the field in the direction of the plasma inhomogeneity y. The differential equation then becomes algebraic. We obtain the dispersion equation by setting the coefficient of E_z equal to zero. When $\omega \ll \omega_{ci}$ the dispersion equation becomes

$$1 - \frac{c^2k^2 + \omega_0^2}{\omega^2} + \frac{\omega_0^2}{\omega^2}\frac{kv_0}{\omega} = 0. \qquad (4.1)$$

B. If we assume in addition that $\omega^2 \ll \omega_0^2$, the dispersion equation describes oscillations at a frequency

$$\omega = \frac{kv_0}{1 + c^2k^2/\omega_0^2}, \qquad (4.2)$$

which is a sensitive function of the electron Larmor drift velocity. For this reason we call these drift waves.

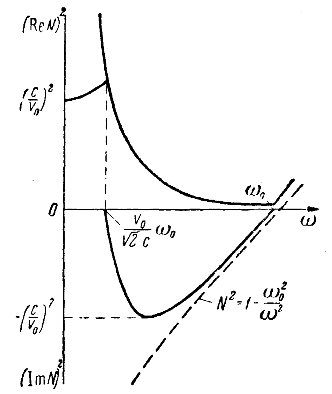
Equation (4.2) is analogous to Eq. (3) of [8], which also describes drift waves. However, the two wave modes are very much different from each other: the waves described by (4.2) are transverse $\mathbf{E} \perp \mathbf{k}$ whereas the waves in [8] are longitudinal $\mathbf{E} \parallel \mathbf{k}$ (plasma drift oscillations).

It is evident that the other two solutions of Eq. (4.1), $\omega = \pm\sqrt{\omega_0^2 + c^2k^2}$, correspond to the ordinary wave that exists in a uniform plasma. The phase velocity of this wave exceeds the velocity of light whereas the phase velocity of the drift wave is appreciably smaller than the thermal velocity of the electrons. Hence, the two wave modes are very different from each other (in frequency) so that all three roots of Eq. (4.1) are always real —the oscillations are neither damped nor amplified.

C. It is interesting to note that the inhomogeneous plasma appears as a medium with a complex refractive index for the waves described by Eq. (4.1). Assuming that ω is given, for $N = ck/\omega$ we have

$$N^2 - \frac{\omega_0^2}{\omega^2}\frac{v_0}{c}N + \frac{\omega_0^2}{\omega^2} - 1 = 0. \qquad (4.3)$$

A sketch of $N = N(\omega)$ is shown in the figure. For



The refractive index as a function of frequency for the ordinary wave in an inhomogeneous plasma. The square of the real part of the refractive index is plotted in the direction of positive ordinates and the square of the imaginary part is plotted in the direction of negative ordinates. The dashed curve shows N^2 for a uniform plasma.

purposes of comparison we also show the curve $N^2 = N^2(\omega)$ for a uniform plasma. In the frequency range

$$\tfrac{1}{4}\, v_0^2/c^2 < \omega^2/\omega_0^2 < 1 \tag{4.4}$$

N has a real part and an imaginary part whereas the quantity N is always imaginary in a uniform plasma for $\omega < \omega_0$.

D. As noted above, the drift oscillations (4.2) are neither amplified nor damped ($\mathrm{Im}\ \omega = 0$). In certain cases, however, the oscillation frequency can be of the order of the ion cyclotron frequency, in which case it is no longer valid to neglect ion effects. When $|\omega - L\omega_{ci}| \ll \omega_{ci}$, where L is some fixed number, the dispersion equation analogous to Eq. (4.2) is

$$\frac{1}{\omega} - \zeta \frac{kv_0}{\omega^2} + \zeta \frac{m}{m_i} \frac{I_L e^{-b}}{\omega - L\omega_{ci}} \left(1 - \frac{kv_{0i}}{\omega}\right) = 0,$$

$$\zeta^{-1} = 1 + \frac{c^2k^2}{\omega_0^2}, \qquad v_{0i} = -\frac{T_i}{m_i\, \omega_{ci}} \frac{\partial}{\partial y} \ln\left[n_0 be^{-b} I_L(b)\right]. \tag{4.5}$$

We find therefore that when $\zeta kv_0 \approx L\omega_{ci}$ the oscillation frequency is complex with

$$\mathrm{Re}\,\omega \approx L\omega_{ci},$$

$$\mathrm{Im}\,\omega \equiv \gamma \approx \pm |L\omega_{ci}| \left(\frac{m}{m_i}\zeta I_L e^{-b}\right)^{1/2} \left(1 + \frac{1}{\zeta}\left|\frac{v_{0i}}{v_0}\right|\right)^{1/2}. \tag{4.6}$$

Thus, oscillations can be excited (the plus sign in front of the right side) close to the ion cyclotron harmonics if the plasma inhomogeneity is large enough, i.e., if

$$\rho/a \gtrsim m/m_i \sqrt{\beta_e}, \qquad \beta_e = 8\pi n_0 T/H_0^2, \tag{4.7}$$

where $\rho = \sqrt{T/m\omega_c^2}$ is the electron Larmor radius while a is the characteristic scale size of the inhomogeneity. The maximum growth rate is of order

$$\gamma_{max} \approx (m/m_i)^{1/2}\, \omega_{ci}, \tag{4.8}$$

while the characteristic wave numbers $k \approx k^*$ are of order

$$k^* \sim \omega_0/c. \tag{4.9}$$

These waves can be excited in highly inhomogeneous transition layers containing hot electrons.

5. QUASI-CLASSICAL APPROXIMATION

If the wave has a sufficiently small scale size in the direction of the inhomogeneity so that

$$E_z^{-1}\partial E_z/\partial y \gg n_0^{-1}\partial n_0/\partial y, \quad T^{-1}\partial T/\partial y,$$

the quasi-classical approximation can be used to solve Eq. (3.1).[3,4] In this case the role of the dispersion equation is played by the Bohr quantization rule

$$\int_{y_1}^{y_2} \left(\frac{v_0}{V}\frac{\omega_0^2}{c^2} - \frac{\omega_0^2}{c^2} - k^2\right)^{1/2} dy = \pi\left(n + \frac{1}{2}\right), \tag{5.1}$$

where n is the number of the level $n \gg 1$. The field distribution in space is given by the following expressions: when $y_1 < y < y_2$

$$\psi_n \sim \left(\frac{v_0}{V}\frac{\omega_0^2}{c^2} - \frac{\omega_0^2}{c^2} - k^2\right)^{-1/4}$$

$$\times \cos\left[\int_{y_1}^{y_2} \left(\frac{v_0}{V}\frac{\omega_0^2}{c^2} - \frac{\omega_0^2}{c^2} - k^2\right)^{1/2} dy + \frac{\pi}{4}\right], \tag{5.2}$$

and when $y > y_2$

$$\psi_n \sim \frac{1}{2}\left(k^2 + \frac{\omega_0^2}{c^2} - \frac{v_0}{V}\frac{\omega_0^2}{c^2}\right)^{-1/4}$$

$$\times \exp\left[-\int_{y_2}^{y} \left(k^2 + \frac{\omega_0^2}{c^2} - \frac{v_0}{V}\frac{\omega_0^2}{c^2}\right)^{1/2} dy\right], \tag{5.2'}$$

and similarly for $y < y_1$. The quantities y_1 and y_2 are the turning points and are functions of the quantum number n, that is to say, they depend on the height of the level; for example, for y_1 we have

$$V(n) = v_0(y_1)/[1 + c^2k^2/\omega_0^2(y_1)], \tag{5.3}$$

and similarly for y_2.

Using Eq. (5.1) we can obtain the number of levels in the oscillation spectrum and the dependence of phase velocity $V \equiv \omega/k$ on level number n. It is evident that Eq. (5.1) can be satisfied for arbitrarily large n since the integrand on the left contains the parameter 1/V, which can increase without limit. For sufficiently high n, in which case $\overline{v}_0/V \gg 1 + c^2k^2/\overline{\omega_0^2}$, this equation can be written

$$|V| = \frac{1}{\pi^2 n^2 c^2}\left[\int_{-\infty}^{+\infty} \omega_0 |v_0|^{1/2} dy\right]^2. \tag{5.4}$$

(It is assumed here that ∇p falls off sufficiently fast at infinity.) It follows that when $n \to \infty$ the quantity $|V|$ approaches zero, diminishing as the square of the level number.

Using Eq. (5.1) we can derive the physical picture corresponding to the short-wave approximation discussed in the preceding section. We must take account of the fact that the number n remains fixed while k can be made arbitrarily large. In this case Eq. (5.1) will be satisfied only when the integrand remains finite everywhere in the interval from y_1 to y_2 so that

$$\left(\frac{v_0}{V}\frac{\omega_0^2}{c^2} - \frac{\omega_0^2}{c^2} - k^2\right) \ll k^2. \tag{5.5}$$

This requirement can be satisfied if we consider levels located at the very bottom of the potential well, for which $y_2 - y_1 \ll a$. In this case, the integrand can be expanded in terms of a small parameter such as $(y_2 - y_1)/a$ and the integral in Eq. (5.1) can be computed easily. As a result Eq. (5.1) assumes the form

$$\frac{v_0^*}{V}\frac{\omega_0^{*2}}{c^2} - \frac{\omega_0^{*2}}{c^2} - k^2 = \left(n + \frac{1}{2}\right)\frac{\omega_0^*}{c}\frac{1}{\tilde{a}}\left(\frac{\tilde{v}_0}{V} - 1\right)^{1/2}, \quad (5.6)$$

where v_0^* and ω_0^* are the drift velocity and plasma frequency at the potential minimum, $y = y^*$, while the quantities $\tilde{a}$ and $\tilde{v}_0$, which are of the same order as a and v_0, are of the form

$$(\tilde{a})^{-1} = \left[\frac{1}{2n_0(y^*)}\left(\frac{\partial^2 n_0}{\partial y^2}\right)_{y=y^*}\right]^{1/2}, \quad \tilde{v}_0 = \frac{\tilde{a}^2}{2n_0(y^*)}\left(\frac{\partial^2 (n_0 v_0)}{\partial y^2}\right)_{y=y^*}.$$

Neglecting terms containing n we have

$$V_0 = v_0^*(1 + c^2k^2/\omega_0^{*2})^{-1}, \quad (5.7)$$

which coincides with Eq. (4.2) if v_0 and ω_0 in Eq. (4.2) are regarded as functions of y^* rather than the arbitrary point y. Thus, the oscillation frequency given by Eq. (4.2) is not itself a function of coordinates.

Introducting terms of order n in Eq. (5.6) we obtain the following correction to the phase velocity:

$$\delta V = -V_0 \frac{c(n + 1/2)}{\omega_0^* \tilde{a}(1 + c^2k^2/\omega_0^{*2})}\left[\left(1 + \frac{c^2k^2}{\omega_0^{*2}}\right)\frac{\tilde{v}_0}{v_0^*} - 1\right]^{1/2}. \quad (5.8)$$

It is evident that for fixed k the short-wave approximation is valid up to values of n such that

$$n + 1/2 \lesssim ak. \quad (5.9)$$

We note that in the present problem another large parameter, in addition to ka, is the quantity $\omega_0 a/c$. If the level number is not too large, so that $n + 1/2 \lesssim \omega_0 a/c$, we can use the local solution of Eq. (4.2) regardless of the order of ka.

6. NARROW TRANSITION-LAYER APPROXIMATION

Assume that the scale size of the inhomogeneity a is small compared with the wave length in the x direction, $ka \ll 1$, and with the thickness of the plasma skin depth, $a\omega_0/c \ll 1$. Then, inside the well $\psi'' \gg (k^2 + \omega_0^2/c^2)\psi$ and consequently [cf. (3.1)]

$$\psi'' + \frac{\omega_0^2}{c^2}\frac{v_0}{V}\psi = 0; \quad (6.1)$$

outside the well, where v_0 is small compared with the phase velocity corresponding to some level $v_0/V \ll 1$, we have

$$\psi'' - (k^2 + \omega_0^2/c^2)\psi = 0. \quad (6.2)$$

We consider that level for which the function ψ does not have a node (ground state). It is to be expected, as follows qualitatively from Sec. 5, that this level corresponds to the maximum phase velocity. The dispersion equation for this case can be found by the same method used to find the energy levels of a particle in a narrow well ([4], problem 1 of Sec. 45). We find

$$V = \frac{\omega_{01}^2}{c^2}\frac{T_1}{m\omega_c}\frac{1 - p_2/p_1}{(k^2 + \omega_{01}^2/c^2)^{1/2} + (k^2 + \omega_{02}^2/c^2)^{1/2}}, \quad (6.3)$$

where the subscript 1 corresponds to quantities to the left of the well while the subscript 2 corresponds to quantities to the right.

It is noteworthy that V depends only on the magnitude of the jump in density and temperature but not on the structure of the well. Thus, if the dimensions of the inhomogeneous layer are smaller than the other characteristic dimensions of the problem it is possible to propagate a wave with a finite phase velocity that is independent of the width and shape of the layer (surface wave).[1)]

We note that all the remaining solutions of Eqs. (6.1) and (6.2), which correspond to wave functions with nodes, yield phase velocities such that $V \to 0$ when $a \to 0$. Under these conditions $\psi'' \gtrsim \psi/a^2$ and it then follows from Eq. (6.1) that

$$(V)_{a\to 0} \sim \omega_0^2 c^{-2}(v_0 a^2)_{a\to 0} \sim a\Delta p \to 0. \quad (6.4)$$

7. EXAMPLE OF AN EXACT SOLUTION FOR A SPATIAL PROBLEM (CASE OF FIXED DENSITY AND TEMPERATURE VARYING LINEARLY IN SOME PORTION)

We now consider a concrete potential, using an example which is a good illustration of the general properties of the drift oscillations noted above. Assume that $T = T_1$ for $y < 0$ and $T = T_2$ for $y > a$ where T_1 and T_2 are constants and that in the range $0 < y < a$ the temperature varies in linear fashion from T_1 to T_2. Then

$$v_0 = \begin{cases} -(T_2 - T_1)/ma\omega_c, & 0 < y < a \\ 0, & y < 0,\ y > a \end{cases}. \quad (7.1)$$

In this case Eq. (7.1) corresponds to the quantum mechanical problem of a particle in a rectangular potential well, the solution of which is well known (cf. [4], problem 2 of Sec. 20). Outside the well the field ψ falls off exponentially in a distance of order $[k^2 + \omega_0^2/c^2]^{-1/2}$:

[1)]The possible existence of a surface wave has been pointed out by A. A. Vedenov.

$$\psi(y>a)\sim\exp\{-y\sqrt{k^2+\omega_0^2/c^2}\},$$

$$\psi(y<0)\sim\exp\{y\sqrt{k^2+\omega_0^2/c^2}\}, \qquad (7.2)$$

while there is a standing wave inside the well:

$$\psi(0<y<a)\sim\sin(k_0y+\delta). \qquad (7.3)$$

Here, $\delta = \arctan[k_0^2/(k^2+\omega_0^2/c^2)^{1/2}]$ while the quantity k_0 is by definition

$$k_0=\left[\frac{\omega_0^2}{c^2}\frac{v_0}{V}-\left(k^2+\frac{\omega_0^2}{c^2}\right)\right]^{1/2}, \qquad (7.4)$$

and is found from the equation for the characteristic values, which is obtained by matching the solutions in different regions of y; this equation plays the role of the dispersion equation:

$$k_0a\equiv\frac{a\omega_0}{c}\left(\frac{v_0}{V}-1-\frac{c^2k^2}{\omega_0^2}\right)^{1/2}$$
$$=\pi n-2\arccos\left[\frac{V}{v_0}\left(1+\frac{c^2k^2}{\omega_0^2}\right)\right]^{1/2},$$
$$n=1,2,3,\ldots \qquad (7.5)$$

Here, the short-wave approximation corresponds to the case $k_0^2 \ll k^2$ although, as we have noted above in Sec. 5, the dispersion equation corresponding to this case (4.2) remains valid even if $k_0^2 \gtrsim k^2$ but $k_0^2 \ll \omega_0^2/c^2$. Equation (7.5) illustrates this feature.

Solutions corresponding to perturbations localized near the minimum point of U are not present here because of the potential shape that has been assumed—the derivative of the potential becomes infinite at the turning point and is zero at other points. For this reason the localization region corresponds to the width of the well a if $a^2 \times (k^2+\omega_0^2/c^2) \gtrsim 1$ and to the damping length $(k^2+\omega_0^2/c^2)^{-1/2}$ if $a^2(k^2+\omega_0^2/c^2)\lesssim 1$.

The quasi-classical dispersion equation ($n \gg 1$) analogous to Eq. (5.1) is of the form

$$\frac{\omega}{k}=v_0\Bigg/\left[1+\frac{c^2}{\omega_0^2}\left(k^2+\frac{\pi^2n^2}{a^2}\right)\right]. \qquad (7.6)$$

The quantity $\pi n/a$ has the meaning of a wave number in the y direction, $\pi n/a = k_y$, so that the denominator contains the square of the total wave number $k^2 = k_x^2 + k_y^2$. At very high values of n we obtain from Eq. (7.6) the particular case described by Eq. (5.5):

$$V=\frac{1}{\pi^2n^2}\frac{a^2\omega_0^2}{c^2}v_0. \qquad (7.7)$$

In the narrow-well approximation, in which case $a^2(k^2+\omega_0^2/c^2)\ll 1$, we obtain the following relations from Eq. (7.5):

$$V=\frac{v_0}{2}\frac{a\omega_0}{c}\left(1+\frac{c^2k^2}{\omega_0^2}\right)^{-1/2}, \quad n=1, \qquad (7.8)$$

$$V=\frac{v_0}{\pi^2(n-1)^2}\left(\frac{a\omega_0}{c}\right)^2, \quad n=2,3,\ldots, \qquad (7.9)$$

which are analogous to the general expressions (6.3) and (6.4).

8. RELATION BETWEEN TRANSVERSE WAVES AND OTHER KINDS OF INHOMOGENEOUS PLASMA WAVES

The case $\mathbf{k} \perp \mathbf{H}_0$, for which the system of Maxwell's equations separates into two subsystems (I) and (II), is singular (degenerate). When $k_z \neq 0$ waves described by (I) become mixed with the waves of (II). The relation between the different waves when $k_z \neq 0$ can be seen easily when the short-wave approximation of Sec. 4 applies. In this case the general equation (dispersion equation, cf. Sec. 4) becomes the following equation if the additional assumptions are made $\omega \ll \omega_{ci}$, $k_z \ll k_x$, $\omega/k \ll v_A$, where v_A is the Alfvén velocity:

$$(\varepsilon_{zz}-N^2)\left(\varepsilon_{xx}-\frac{i}{k}\frac{\partial\varepsilon_{yx}}{\partial y}\right)-N^2\cos^2\theta\,\varepsilon_{zz}=0,$$
$$k\approx k_x, \qquad \cos\theta=k_z/k.$$

As has been pointed out earlier,[2] this equation describes Alfvén waves and ion acoustic waves; in [6] we have given dispersion curves $\omega = \omega(k_z)$ corresponding to these wave modes. If one takes $k_z \to 0$ at appropriate places in [2] and [6] it is found that our transverse drift branch goes over to one of the ion acoustic waves. In this case the longitudinal field component E_x diminishes as $\cos\theta$ and $E_x = 0$ in the limit $\cos\theta = 0$. The range of $\cos\theta$ in which our analysis applies is of order v_0/v_T (v_T is the electron thermal velocity) for a dense plasma $\beta \gtrsim m/m_i$ while $\cos\theta \sim v_0/v_A$ for $\beta \lesssim m/m_i$.

Thus, the transverse drift oscillations considered in this work represent the limiting case of ion acoustic oscillations of an inhomogeneous plasma.

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