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ABSORPTION OF WAVES IN A PLASMA (QUASILINEAR APPROXIMATION)

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The absorption of electromagnetic waves propagating at an arbitrary angle with respect to a fixed magnetic field in a plasma is analyzed by means of the quasilinear theory. We take account of the interaction between particles and waves that leads to a redistribution of the particles in velocity, causing damping which differs from that appearing in the linear theory. We obtain general formulas that give the change in the distribution function and absorption of an arbitrary wave. The results are applied to a magnetic-sound wave propagating at an angle θ with respect to a fixed magnetic field, not close to $\pi/2$. We compute the electron Cerenkov absorption, the electron cyclotron absorption, and absorption at harmonics of the ion-cyclotron frequency.

It is well known that the absorption (or excitation) of electromagnetic waves in a plasma depends on the particle distribution function, and especially on its derivatives at resonance velocities. [1,2]

In many papers concerned with the absorption of waves in a plasma (for example, [1-5]) the particle velocity distribution is taken to be Maxwellian. However, the interaction between resonance particles and a wave modifies the distribution function and a small change in the function itself can result in a marked change in its derivatives; consequently the absorption in a plasma can change. If the wave amplitude is not too large (the form of the wave remains unchanged) and if the interaction of the wave with the particle appears only in a change in the distribution function (a redistribution of the particle velocities), the behavior of the plasma can be described by means of the quasilinear theory. [6,7] In this approach we separate the rapidly oscillating and slowly varying parts (in time) of the distribution function; the mean-square effect of the oscillations is then included in the equation for the distribution function.

Vedenov et al [6] have considered two specific problems in calculating wave absorption in a plasma by means of the quasilinear theory: the absorption of electron Langmuir oscillations and the absorption of transverse circularly polarized electromagnetic (extraordinary) waves propagating along a fixed magnetic field. In the present work the same method is used to obtain general formulas which give the absorption of electromagnetic waves propagating at an arbitrary angle with respect to a fixed magnetic field.

We assume that the criteria for the quasilinear approximation are satisfied: [6,7] the wave energy is much smaller than the total particle energy but, at the same time, greater than the level of the thermal noise; the width of the wave packet is large enough so that trapped particles can be neglected and the damping rate is much smaller than the oscillation frequency.

Assume that in a uniform plasma in a fixed magnetic field $\mathbf{H}_0$, parallel to the z axis, there propagates in the xz plane a one-dimensional wave packet, for which the wave vectors $\mathbf{k}$ form an angle θ with the direction of the field $\mathbf{H}_0$. Representing the electron distribution function f by a Fourier integral we have

$$f = f_0 + \int \frac{dk}{2\pi} f_k \exp\{ik(z\cos\theta + x\sin\theta) - i\omega_k t\}, \qquad (1)$$

where f_0 is a quantity that varies slowly in time. Inasmuch as trapped particles are neglected we can assume that the "background" f_0 is uniform in space.

The electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ are also written as Fourier integrals:

$$\mathbf{E} = \int \frac{dk}{2\pi} \widetilde{\mathbf{E}}_k \exp\{ik(z\cos\theta + x\sin\theta) - i\omega_k t\} \qquad (2)$$

(similarly for $\mathbf{H}$); in Eqs. (1) and (2)

$$f_{-k} = f_k^+, \qquad \widetilde{\mathbf{E}}_{-k} = \widetilde{\mathbf{E}}_k^+, \qquad \mathrm{Re}\,\omega_{-k} = -\mathrm{Re}\,\omega_k.$$

The linearized Boltzmann equation for the electrons is used to determine the f_k (assuming that they are small):

$$f_k = -\frac{ie}{m} e^{-i\mu\sin\varphi} \sum_{p=-\infty}^{\infty} \frac{e^{ip\varphi}}{p\omega_e + k_z v_\parallel - \omega} \hat{L}'_{kp} f_0, \qquad (3)$$

where we have for the linear operator

$$\hat{L}'_{kp} = \tilde{E}_{kz} J_p(\mu)\left[\left(1-\frac{p\omega_e}{\omega}\right)\frac{\partial}{\partial v_{\parallel}} + \frac{v_{\parallel}}{v_{\perp}}\frac{p\omega_e}{\omega}\frac{\partial}{\partial v_{\perp}}\right] + \left[\frac{p}{\mu}J_p(\mu)\tilde{E}_{kx} - iJ'_p(\mu)\tilde{E}_{ky}\right] \times \left[\left(1-\frac{k_z v_{\parallel}}{\omega}\right)\frac{\partial}{\partial v_{\perp}} + \frac{k_z v_{\perp}}{\omega}\frac{\partial}{\partial v_{\parallel}}\right]; \quad (4)$$

Here, $\mu = k_x v_\perp/\omega_e$, $J_p(\mu)$ is the Bessel function, $J'_p(\mu) \equiv dJ_p(\mu)/d\mu$, e and m are respectively the charge and mass of the electron, $\omega_e = eH_0/mc$ is the electron cyclotron frequency, $k_x = k \sin\theta$, $k_z = k\cos\theta$, $v_\parallel$ is the velocity along the magnetic field $\mathbf{H}_0$, $\mathbf{v}_\perp$ is the velocity in the xy plane, φ is the angle in velocity space ($v_x = v_\perp \cos\varphi$, $v_y = v_\perp \sin\varphi$). In deriving Eq. (3) we have used the relation

$$e^{i\mu\sin\varphi} = \sum_{p=-\infty}^{\infty} e^{ip\varphi} J_p(\mu).$$

Equations (3) and (4) are obtained under the assumption that f_0 is independent of φ. If the waves propagate at an angle with respect to the magnetic field the distribution function may assume a dependence on φ in the quasilinear relaxation process; in this case f_0 is replaced by the sum

$$f_0 + \sum_q{}' f_{0q} e^{iq\varphi},$$

where $q = \pm 1, \pm 2, \ldots$ $\partial f_{0q}/\partial\varphi = 0$. If the wave amplitude is not too large and the quasilinear theory applies the distortion of f_0 is small and taking account of f_{0q} only yields small corrections to f_k as given by Eqs. (3) and (4).

Averaging the Boltzmann equation we obtain a diffusion equation in velocity space. Generally speaking, using this equation, which contains f_0 and all the f_{0q}, we can determine the time variation of the function f_{0q}. This approach, however, is not used in the present work. To compute the absorption of the energy of the wave packet within the framework of the quasilinear theory it is not necessary to consider the effects of small corrections f_{0q} on the form of the distribution function and on the damping factor; it is only necessary to take account of f_0.

Averaging the diffusion equation in velocity space over the angle φ, we obtain the following equation for f_0, which is independent of φ:

$$\frac{\partial f_0}{\partial t} + \frac{ie^2}{m^2}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\sum_{p=-\infty}^{\infty}\hat{L}''_{kp}\frac{1}{p\omega_e + k_z v_{\parallel} - \omega}\hat{L}_{kp} f_0 - S = 0 \quad (5)$$

where $S = \sum_\alpha \mathrm{St}(f_0, f_\alpha)$ is the collision integral, given by the linear operator

$$\hat{L}''_{kp} = \left[\left(\frac{\partial}{\partial v_\perp} + \frac{1}{v_\perp}\right)\frac{v_\parallel}{v_\perp}\frac{p\omega_e}{\omega} + \frac{\partial}{\partial v_\parallel}\left(1-\frac{p\omega_e}{\omega}\right)\right]E^+_{kz}J_p(\mu) + \left[\left(\frac{\partial}{\partial v_\perp} + \frac{1}{v_\perp}\right)\left(1-\frac{k_z v_\parallel}{\omega}\right) + \frac{\partial}{\partial v_\parallel}\frac{k_z v_\perp}{\omega}\right]\left[\frac{p}{\mu}E^+_{kx}J_p(\mu) + iE^+_{ky}J'_p(\mu)\right], \quad (6)$$

while the operator $\hat{L}_{kp}$ differs from $\hat{L}'_{kp}$ only in that $\tilde{\mathbf{E}}_k$ is replaced by $\mathbf{E}_k$ ($\tilde{\mathbf{E}}_k$ and $\mathbf{E}_k$ differ in their normalization):

$$\overline{\mathbf{E}^2} = \int_{-\infty}^{\infty}\frac{dk}{2\pi}|\mathbf{E}_k|^2 = \int_0^{\infty}\frac{dk}{\pi}|\mathbf{E}_k|^2.$$

If the damping rate γ is much smaller than the oscillation frequency ω, in carrying out the integration in Eq. (5) we can replace $(p\omega_e + k_z v_\parallel - \omega)^{-1}$ under the integral sign by the expression $i\pi\delta(p\omega_e + k_z v_\parallel - \omega)$. After integration Eq. (5) becomes

$$\frac{\partial f_0}{\partial t} - \frac{e^2}{m^2}\sum_{p=-\infty}^{\infty}\hat{L}''_{kp}\frac{1}{|\partial\omega/\partial k - v_\parallel\cos\theta|}\hat{L}_{kp}f_0\big|_{k=\alpha_p} - S = 0, \quad (7)$$

where $\alpha_p = (\omega - p\omega_e)/v_\parallel\cos\theta$ ($k > 0$, $\omega > 0$) in which the differentiation symbol $\partial/\partial v_\parallel$ in the operator $\hat{L}''_{kp}$ operates on both k and ω (by virtue of the condition $k = \alpha_p = (\omega - p\omega_e)/v_\parallel\cos\theta$).

The change in the energy density of the electromagnetic field ϵ (energy exchange between the wave and particle) is determined by the current density j and the electric field $\mathbf{E}$. After averaging we obtain the expression

$$\dot{\varepsilon} \equiv \frac{\partial\varepsilon}{\partial t} = -2\gamma\varepsilon = \int_0^{\infty}\frac{dk}{\pi}\dot{\varepsilon}_k,$$

$$\dot{\varepsilon}_k = \frac{\pi^2 e^2}{k_z m}\sum_{p=-\infty}^{\infty}\int_0^{\infty} v_\perp^2\,dv_\perp\left\{\frac{v_\parallel}{v_\perp}E^+_{kz}J_p(\mu) + \frac{p}{\mu}E^+_{kx}J_p(\mu) + iE^+_{ky}J'_p(\mu)\right\}\hat{L}_{kp}f_0\big|_{v_\parallel=\beta_p}, \quad (8)$$

where $\beta_p = (\omega - p\omega_e)/k\cos\theta$, $\epsilon = (\overline{E^2} + \overline{H^2})/8\pi$.

Equations (7) and (8) are of general nature and apply for an electromagnetic wave of any kind propagating at an arbitrary angle with respect to a fixed magnetic field. In any given case one must use the appropriate dispersion equation and the relation between the electric field components in Eqs. (7) and (8) in order to obtain the required results (provided, as indicated above, the width of the packet is large enough, the wave amplitude is not too great, and the damping is weak). Obviously these expressions can only be used when the wave can actually propagate in a plasma.

Equation (7) gives the change in the distribution function that results from the particle-wave interaction. It is clear from Eqs. (7) and (8) that in the absence of collisions a steady state would be established in which all wave absorption would vanish. Collisions mean that the stationary state is changed.

As in the work of Vedenov [6] the collision integral is taken in the Landau form [8] and linearized (this procedure is possible when the resonance velocities are much greater than the mean thermal velocities of the particles). The effect of collisions of particles of type a with particles of type b on the distribution function $f_a(v)$ is expressed in the linear approximation (if $v^2 \gg 2T_b/m_b$ and the velocity distribution of the b particles is Maxwellian):

$$\mathrm{St}\,(f_a(\mathbf{v}),\, f_b) = \frac{2\pi\lambda e_a^2 e_b^2 n_b}{m_a}\frac{\partial}{\partial v_\beta}\left\{\frac{2v_\beta}{m_b} f_a(\mathbf{v}) + \frac{1}{m_a}\left[v^2\delta_{\beta\gamma} - v_\beta v_\gamma + \frac{T_b(3v_\beta v_\gamma - v^2\delta_{\beta\gamma})}{m_b v^2}\right]\frac{\partial f_a(\mathbf{v})}{\partial v_\gamma}\right\} \quad (9)$$

(e_a, n_a, m_a, and T_a are respectively the charge, density, mass, and temperature of particles of type a; λ is the Coulomb logarithm). The wave packet interacts with particles whose longitudinal velocities satisfy the condition $k_z v_\parallel = \omega - p\omega_e$ (the width of the region of resonance velocities is related to the width of the wave packet by this expression).

The effect of collisions outside the resonance regions is to establish a Maxwellian distribution function in velocity;[1] however, f_0 is continuous at the boundaries of the resonance regions. It may thus be assumed that the distribution in $v_\perp$ remains Maxwellian while the particle-wave interaction only changes the dependence of f_0 on $v_\parallel$, i.e.,

$$f_0 = n\,(m/2\pi T_e)^{3/2}\exp\,(-mv_\perp^2/2T_e)\,\psi\,(v_\parallel). \quad (10)$$

The value of the collision integral averaged over $v_\perp$ is given by the following expression (when $v_\parallel^2 \gg T_a/m_a$):

$$\overline{\mathrm{St}\,(f_a,\, f_b)} = \frac{4\pi\lambda e_a^2 e_b^2 n_a n_b}{m_a^2}\left(\frac{m_a}{2\pi T_a}\right)^{1/2} \times\left\{\left(1 + \frac{T_b m_a}{T_a m_b}\right)\frac{\partial}{\partial v_\parallel}\frac{1}{|v_\parallel|^3}\left(v_\parallel\psi + \frac{T_a}{m_a}\frac{d\psi}{dv_\parallel}\right) + \frac{m_a}{m_b}\left(1 - \frac{T_b}{T_a}\right)\frac{\partial}{\partial v_\parallel}\frac{v_\parallel\psi}{|v_\parallel|^3}\right\}. \quad (11)$$

If

$$k_x v_{Te}/\omega_e \ll 1 \quad (v_{Te}^2 = 2T_e/m)$$

using Eqs. (7), (10), and (11) we obtain an equation for the distribution function $\psi(v_\parallel)$ in the steady state that takes account of collisions of resonance electrons with electrons and ions:

$$\frac{1}{\pi}\sum_{p=-\infty}^{\infty}\left\{\frac{D_p\,(d\psi/dv_\parallel - \psi mp\omega_e/T_e k_z)}{|\partial\omega/\partial k - v_\parallel\cos\theta|}\right\}\Bigg|_{k=\alpha_p} + \frac{4\lambda ne^2(2+Z)}{|v_\parallel|^3}\left(v_\parallel\psi + \frac{T_e}{m}\frac{d\psi}{dv_\parallel}\right) = 0, \quad (12)$$

$$D_p = \frac{k_x^{2|p|}(T_e/m)^{|p|}(1-p\omega_e/\omega)^2\,|E_{kz} + (pE_{kx} - i\,|p|\,E_{ky})\,\omega_e/k_x v_\parallel|^2}{2^{|p|}\omega_e^{2|p|}\,|p|!}. \quad (12a)$$

It is assumed that $n = n_e = Zn_i$ and $ZT_i/M \ll T_e/m$ (Ze is the ion charge).

The following expression is obtained for the energy absorption:

$$\dot{\varepsilon}_k = \frac{\sqrt{\pi}ne^2\omega}{mk_z^2 v_{Te}^3}\sum_{p=-\infty}^{\infty} D_p\left(\frac{T_e}{m}\frac{d\psi}{dv_\parallel} - \frac{p\omega_e}{k_z}\psi\right)\Bigg|_{v_\parallel=\beta_p}. \quad (13)$$

If $|\omega - p\omega_e| < |\omega - (p \pm 1)\omega_e|$ the absorption is determined by the p-th term of the summation in Eq. (13). It is evident that in this case ψ is determined by the p-th term in the sum and the collision integral in Eq. (12). Thus, we have from Eqs. (12) and (13)

$$\dot{\varepsilon}_k = -\frac{\sqrt{\pi}ne^2\omega^2 D_p\psi(v_\parallel)}{mk_z^3 v_{Te}^3} \times\left(1 + \frac{|v_\parallel|^3 D_p m}{4\pi\lambda ne^2(2+Z)\,T_e\left|\dfrac{\partial\omega}{\partial k} - v_\parallel\cos\theta\right|}\right)^{-1}\Bigg|_{v_\parallel=\beta_p}. \quad (14)$$

Damping occurs by virtue of the interaction of the waves with the particles in the resonance velocity ranges. If the amplitude is not too large in this range ($E^2 \ll 8\pi nT$) we find that the distribution function $\psi(v_\parallel)$ is not changed greatly; however $d\psi/dv_\parallel$ is changed. Using Eqs. (12) and (13) we obtain Eq. (14) in which ψ appears but $d\psi/dv_\parallel$ does not. Since ψ does not change greatly, in Eq. (14) we can write $\psi = \exp(-mv_\parallel^2/2T_e) = \exp(-m(\omega - p\omega_e)^2/2k_z^2 T_e)$ as in the linear case. The reduction in wave damping caused by the change in ψ and $d\psi/dv_\parallel$ is taken into account by the denominator in Eq. (14) and there is no necessity for considering the change in ψ more precisely (in this expression).

If the wave amplitude is so large that the effect of collisions on the distribution function in the resonance region is negligible compared with the in-

[1]In the present work we treat absorption of waves in the steady state (in which case $df_0/dt = 0$). It should be noted, however, that this state is actually quasistationary because slow damping of the oscillations also causes the function to change.

teraction between the wave and the electrons Eq. (14) becomes

$$\dot{\varepsilon}_k = -\frac{(2\pi^3)^{1/2}\lambda n^2 e^4 \omega^2 (2+Z)\,|\,\partial\omega/\partial k - (\omega - p\omega_e)/k\,|}{|\,\omega - p\omega_e\,|^3 (mT_e)^{1/2}}\,\psi\left(\frac{\omega - p\omega_e}{k_z}\right) \tag{15}$$

or

$$\dot{\varepsilon}_k = -\frac{\pi(2+Z)\,\omega^2\,|\,\partial\omega/\partial k - (\omega - p\omega_e)/k\,|\,\nu n T_e}{2\,|\,\omega - p\omega_e\,|^3}\,\psi\left(\frac{\omega - p\omega_e}{k_z}\right), \tag{16}$$

where $\nu = (8\pi)^{1/2}\lambda n e^4/(mT_e^3)^{1/2}$ is the electron-ion collision frequency.

Similar calculations can be carried out for the ion distribution function. The procedure is exactly the same as for the electrons. In computing the ion collision integral, however, we cannot neglect certain terms in Eq. (11) that were neglected in obtaining Eq. (12).

Carrying out calculations similar to those given above we determine the wave absorption due to the ion interaction:

$$\dot{\varepsilon}_k = -\frac{\sqrt{\pi}\,Ze^2\omega^2 D_{pi}}{Mk_z^3 v_{Ti}^3}\left[\frac{T_i}{T_e} + \frac{p\omega_i}{\omega}\left(1 - \frac{T_i}{T_e}\right)\right]\psi(v_\parallel) \times \left\{1 + \frac{|\,v_\parallel\,|^3 D_{pi}\, m}{4\pi\lambda n e^2 T_e\,|\,\partial\omega/\partial k - v_\parallel \cos\theta\,|}\right\}^{-1}\Bigg|_{v_\parallel = \beta_{pi}}; \tag{17}$$

$$D_{pi} = \frac{k_x^{2|p|}(T_i/M)^{|p|}(1 - p\omega_i/\omega)^2}{2^{|p|}\,\omega_i^{2|p|}\,|\,p\,|!}\,\Big|\,E_{kz} + \frac{\omega_i}{k_x v_\parallel}(pE_{kx} + i\,|\,p\,|\,E_{ky})\Big|^2. \tag{17a}$$

Here, $\beta_{pi} = (\omega - p\omega_i)/k\cos\theta$, $\omega_i = ZeH_0/Mc$ is the ion cyclotron frequency (as before we assume that $ZT_i/M \ll T_e/m$).

At large wave amplitudes Eq. (17) becomes

$$\dot{\varepsilon}_k = -\frac{(2\pi^3)^{1/2}\lambda Z n^2 e^4 M^{1/2}\omega^2}{mT_i^{3/2}\,|\,\omega - p\omega_i\,|^3}\left|\frac{\partial\omega}{\partial k} - \frac{\omega - p\omega_i}{k}\right| \times \left[\frac{T_i}{T_e} + \frac{p\omega_i}{\omega}\left(1 - \frac{T_i}{T_e}\right)\right] T_e\,\psi\left(\frac{\omega - p\omega_i}{k_z}\right). \tag{18}$$

Thus, at large wave amplitudes the absorption of energy in the wave is proportional to the collision frequency.

After integration over the wave packet Eqs. (14) and (17) can be written

$$\gamma = \gamma_0/(1 + \eta\varepsilon/nT_e),$$

where γ_0 is the damping rate given by the linear theory while the coefficient η depends on the fixed magnetic field, the temperature, the plasma density, and the frequencies of the waves forming the wave packet. If $\eta\epsilon/nT_e \ll 1$, $\gamma \cong \gamma_0$; however, γ differs considerably from γ_0 if $\eta\epsilon/nT_e \gtrsim 1$. If $\eta\epsilon/nT_e \gg 1$ Eqs. (14) and (17) become Eqs. (15) and (18) respectively.

Thus, if $\eta \gg 1$ the damping rate γ can be very different from γ_0 for oscillation amplitudes that satisfy the condition

$$\varepsilon/nT_e \ll 1 \ll \eta\varepsilon/nT_e.$$

If η is not too large the damping changes appreciably compared with γ_0 only for large wave amplitudes ($\epsilon \gtrsim nT_e$) in which case the quasilinear theory no longer applies. Hence, Eqs. (15) and (18), obtained by means of the quasilinear theory, apply only when $\eta \gg 1$.

For example, for electron Langmuir oscillations the condition $\eta \gg 1$ means $2\pi m\omega^3/3k^4\lambda e^2\Delta(\omega/k) \gg 1$ where $\Delta(\omega/k)$ is the spread in phase velocity in the wave packet.

The results that have been obtained can be used to analyze specific problems. We first treat a magnetic-sound wave (for simplicity we consider the frequency region $\omega_i/\cos\theta \ll \omega \ll \omega_e\cos\theta$; $\cos\theta \gg \sqrt{m/M}$. In this case the dispersion equation is

$$N^2 = \frac{k^2c^2}{\omega^2} = \frac{\omega_0^2}{\omega(\omega_e\cos\theta - \omega)}, \quad \text{if} \quad \omega_0^2 \gg \omega_e^2 \quad \left(\omega_0^2 = \frac{4\pi n e^2}{m}\right),$$

$$E_z = -\frac{i\omega\sin\theta}{\omega_e\cos\theta - \omega}E_y, \qquad E_x = -\frac{i}{\cos\theta}E_y.$$

If $\omega < \omega_e/2$ we need consider only the electron Cerenkov absorption caused by the interaction of the wave with particles having longitudinal velocities $v_\parallel = \omega/k_z$; particles in resonance with the cyclotron harmonics have higher velocities and we can neglect the effect of the tail of the Maxwell distribution. Hence, the change in ψ and the absorption are given by Eqs. (12), (13), and (14) with $p = 0$:

$$\frac{d\psi}{dv_\parallel} = -\frac{mT_e^{-1}v_\parallel\,\psi(v_\parallel)}{1 + v_\parallel^3\,\xi\varepsilon_k}\Bigg|_{k=\varkappa_0} \tag{19}$$

(the distribution is Maxwellian for small oscillation amplitude)

$$\dot{\varepsilon}_k = -\frac{\sqrt{\pi}\,\omega^5\sin^2\theta\,\psi(\omega/k_z)\,\varepsilon_k}{k_z^3 v_{Te}^3(\omega_e\cos\theta - \omega)(1 + \omega^3\xi\varepsilon_k/k_z^3)} = -2\gamma_k\varepsilon_k, \tag{20}$$

where

$$\xi = \frac{4\pi\omega^3\sin^2\theta}{\omega_0^4\lambda(2+Z)T_e(\omega_e\cos\theta - \omega)\,|\,\partial\omega/\partial k - \omega/k\,|},$$

$\epsilon_k = (|\,E_k\,|^2 + |\,H_k\,|^2)/8\pi$. In the frequency region being considered $|\,E\,|^2 + |\,H\,|^2 \cong 2N^2|\,E_y\,|^2$ for the magnetic-sound branch. It is easily shown that at small wave amplitudes Eq. (20) gives the familiar result of the linear theory for the Cerenkov damping.[4]

As the wave amplitude increases the velocity distribution function is leveled in the resonance region. At the boundaries of the resonance band ($v_{\|} = v_1$) $\psi = \exp(-mv_1^2/2T_e)$ and as the velocity increases ψ diminishes more slowly than a Maxwellian function (remember that we are treating the case in which the resonance velocities are much greater than the mean thermal velocity; it is only in this case that the expressions we have obtained for the collision integral apply). At high oscillation amplitudes the energy absorption is given by

$$\dot{\varepsilon}_k = -\frac{(2\pi^3)^{1/2}\lambda n^2 e^4 (2+Z)\,|\partial\omega/\partial k - \omega/k|\,\psi(\omega/k_z)}{m^{1/2} T_e^{1/2}\omega}$$
$$= -\frac{(2\pi^3)^{1/2}\lambda n^2 e^4 k_z c^2 (2+Z)\,\omega_e\,|\omega_0^2 - k^2c^2|\,\psi(\omega/k_z)}{(mT_e)^{1/2}\omega\,(k^2c^2+\omega_0^2)^2}. \quad (21)$$

The dispersion equation written above does not apply near $\omega = \omega_e \cos\theta$ since the refractive index increases sharply at this frequency and the thermal corrections to the real and imaginary parts of N^2 become important. If ω is not close to $\omega_e\cos\theta$ ($\omega \ll \omega_e\cos\theta - \omega$) then $\omega_0^2 \gg k^2c^2$ and

$$\dot{\varepsilon}_k = -(\pi m/8T_e)^{1/2}\lambda n e^2 (2+Z)\,\omega_e\,\omega^{-1} k_z c^2 \psi(\omega/k_z). \quad (22)$$

If $\omega > \omega_e/2$ the cyclotron absorption becomes more important than the Cerenkov absorption (there are more particles whose velocities are in resonance with the electron cyclotron frequency ω_e than with velocities that satisfy the Cerenkov absorption condition $v_{\|} = \omega/k_z$). We consider electron cyclotron absorption in the magnetic-sound branch. In this case, p = 1 in Eqs. (12)—(14) and the absorption is given by

$$\dot{\varepsilon}_k = -\frac{\sqrt{\pi}\,G\varepsilon_k}{v_{Te}(1+F\varepsilon_k)}\,\psi\left(\frac{\omega-\omega_e}{k_z}\right), \quad (23)$$

where

$$G = \frac{|\omega^2\sin^2\theta + \omega_e^2\cos\theta(1+\cos\theta) - \omega\omega_e(1+\sin^2\theta+\cos\theta)|^2\omega}{4k\cos^3\theta\,\omega_e^2(\omega_e\cos\theta-\omega)},$$

$$F = \frac{8\pi\,|\omega-\omega_e|^3 G}{m\lambda\omega_0^4\omega^2(2+Z)\,|\partial\omega/\partial k - (\omega-\omega_e)/k|}.$$

At large wave amplitudes ψ in the resonance region is given by

$$[(\omega-\omega_e)\,d\psi/\partial v_{\|} - mv_{\|}T_e^{-1}\omega_e\psi]\,|_{k=\alpha_1} = 0. \quad (24)$$

From the equation $k = \alpha_1 = (\omega - \omega_e)/v_{\|}\cos\theta$ and the dispersion equation we have

$$\omega - \omega_e = -\omega_e\,\frac{k^2c^2(1-\cos\theta)+\omega_0^2}{k^2c^2+\omega_0^2}.$$

The cyclotron resonance does not distort the Maxwellian distribution at frequencies $\omega \ll \omega_e$. However, if

$$\omega_e\cos\theta > \omega_e \gg \omega_e\cos\theta - \omega, \text{then } \omega_0^2 \ll k^2c^2.$$

If $(1-\cos\theta) \gg \omega_0^2/k^2c^2$ in this case then

$$\psi = \exp\left[-\frac{mv_1^2}{2T_e} + \frac{m(v_1^2 - v_{\|}^2)}{2T_e(1-\cos\theta)}\right] \quad (25)$$

[v_1 is the boundary of the resonance region ($0 > v_1 \geq v_{\|}$); when $v_{\|} = v_1$ the function ψ goes over continuously into the Maxwell function].

However, if the angle θ is not close to 0, when $\omega_e > \omega \gg \omega_e - \omega$

$$\psi = \exp\left\{-\frac{mv_1^2}{2T_e} + \frac{3}{4}\frac{m}{T_e}\left(\frac{\omega_e c}{\omega_0}\right)^{2/3}(v_1^{4/3} - v_{\|}^{4/3})\right\}$$
$$= \exp\left\{-\frac{mv_1^2}{2T_e} - \frac{3}{4}\frac{m\omega_e}{T_e}\left(\frac{v_1}{k_1} - \frac{v_{\|}}{k}\right)\right\}. \quad (26)$$

Equation (26) coincides with Eq. (63) of Vedenov et al[6] (in the last formula the factor 3/4 is missing in the exponential). It should be noted once again that the expressions for the collision integral have been obtained for values $|v_{\|}| \gg (T/m)^{1/2}$ and if ω is not close to ω_e a more accurate expression is required for the collisions. In the case of oblique propagation the thermal corrections can have a marked effect on the refractive index if ω is not close to $\omega_e\cos\theta$.

The following relation is satisfied in the frequency range $\omega_i/\cos\theta \ll \omega \ll \omega_e\cos\theta$:

$$\frac{\partial\omega}{\partial k} - \frac{\omega-\omega_e}{k}$$
$$= \frac{\omega_e[k^4c^4(1-\cos\theta) + \omega_0^2 k^2c^2(2+\cos\theta) + \omega_0^4]}{k(k^2c^2+\omega_0^2)^2}. \quad (27)$$

The cyclotron absorption is determined by Eq. (23) with Eq. (27) taken into account. At high wave amplitudes the energy absorption is given by Eq. (15).

Absorption at harmonics of the ion cyclotron frequency is given by Eqs. (17) and (18). At low wave amplitudes Eq. (17) yields the results of the linear theory.[9,10] The results obtained for wave damping by means of the quasilinear theory can be used to estimate the absorption of monochromatic waves. If the wave packet is sufficiently narrow so that

$$\varepsilon = \int_0^\infty \frac{dk}{\pi}\varepsilon_k = \frac{\Delta k}{\pi}\varepsilon_k, \qquad \varepsilon_k \sim \frac{\pi\varepsilon}{|\partial k/\partial v_{\|}|\,\Delta v_{\|}},$$

where Δk is the width of the wave packet (which we assume to be small), $\Delta v_{\|}$ is the velocity range that is in resonance with the wave.

If the width of the wave packet Δk is reduced so that the spread in phase velocities $\Delta(\omega/k)$ becomes small the resonance particles are trapped. Strictly speaking, the quasilinear theory does not apply in this case (in order for this theory to apply we require $\Delta(\omega/k) \gg \Delta v_{\text{trap}}$ where Δv_{trap} is the velocity range of the particles trapped by the monochromatic wave). However, the absorption of a monochromatic wave can be estimated roughly by the quasilinear theory by taking $\Delta v_{||} = \Delta v_{\text{trap}}$. It is probable that the "diffusion coefficient" obtained by the quasilinear theory is not too different from the true value.

If $\Delta v_{||} = \Delta v_{\text{trap}}$ and we use the fact that

$$(\partial\omega/\partial k - v_{\parallel} \cos\theta)\, \partial k/\partial v_{\parallel} = k_z,$$

then the following expression is obtained for the Cerenkov damping [assuming that $\Delta v_{||} \sim 2(2eE_z/mk_z)^{1/2}$] on the magnetic-sound branch

$$\dot{\varepsilon} \sim -\frac{\sqrt{\pi}\,\omega^5 \sin^2\theta\,\psi(\omega/k_z)\,\varepsilon}{k_z^3 v_{Te}^3 (\omega_e \cos\theta - \omega)} \times \left\{1 + \frac{2^{1/2}\pi^2 (mk_z)^{1/2} n\omega^6 \sin^2\theta\varepsilon}{k_z^4 \omega_0^4 \lambda (2+Z)(\omega_e\cos\theta - \omega)(eE_z)^{1/2} nT_e}\right\}^{-1}. \quad (28)$$

If the wave amplitude is large

$$\dot{\varepsilon} = -2\gamma\varepsilon \sim -\frac{\omega_0^4 \lambda (2+Z)(ek_z E_z)^{1/2} \psi(\omega/k_z)}{(2\pi^3)^{1/2} n v_{Te}^3 \omega m^{1/2}} nT_e. \quad (29)$$

The absorption of the electron Langmuir wave is also given by Eq. (29) ($E_z = E$, $\theta = 0$, $\epsilon = E^2/8\pi$).

For cyclotron absorption of the extraordinary wave ($p = 1$, $\theta = 0$)

$$\Delta v_{\parallel} \sim T_e^{1/4} m^{-3/4} (eE/\omega)^{1/2},$$

$$\dot{\varepsilon} \sim -\frac{\pi^{1/2}\omega(\omega_e - \omega)}{kv_{T_e}} \psi\left(\frac{\omega - \omega_e}{k}\right) \times \varepsilon \left\{1 + \frac{m^{7/4}(\omega_e - \omega)^4 \varepsilon}{2k^2\omega^{1/2}\lambda n^2 e^4 (2+Z) T_e^{1/4} (eE)^{1/2}}\right\}^{-1}, \quad (30)$$

and for large wave amplitudes it follows from Eq. (30) that

$$\dot{\varepsilon} \sim -\frac{(2\pi E)^{1/2} k\omega^{3/2} \lambda n^2 e^{9/2} (2+Z)}{T_e^{1/4} m^{5/4} (\omega_e - \omega)^3} \psi\left(\frac{\omega - \omega_e}{k}\right). \quad (31)$$

The absorption at harmonics of ω_e and ω_i for $\theta \neq 0$ can be estimated in the same way (having determined $\Delta v_{||}$ in suitable fashion).

In order to make an exact analysis of the damping of the monochromatic waves we must take account of trapped particles and the variation of f_0 in space. The damping rate for a monochromatic wave computed in this way[11] (to within a factor $(7\pi + 6)/8\sqrt{2}$) coincides with the damping rate γ given by Eq. (29).

Thus, at high wave amplitudes the particles are redistributed in velocity and the absorption is determined by the frequency of collisions that disturb the quasistationary state. If the wave amplitude is of the order of the thermal noise the absorption is given by the formulas of the linear theory.

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