STATIONARY PLASMA FLOW IN A MAGNETIC FIELD

A. P. KAZANTSEV

Institute for Radiophysics and Electronics, Siberian Division, Academy of Sciences, U.S.S.R.

Submitted to JETP editor October 20, 1962

J. Exptl. Theoret. Phys. (U.S.S.R.) **44**, 1283-1288 (April, 1963)

Isolated waves of finite amplitude propagating in a cold plasma at an arbitrary angle with respect to the magnetic field are analyzed. It is shown that three waves of this kind are possible: large scale compressional shocks (Alfvén waves), rarefaction shocks (magnetic sound waves), and small-scale compressional shocks corresponding to high-frequency magnetic-sound waves. The relation between the wave amplitude and the Mach number is established. The critical magnetic field at which the solution for the compressional wave becomes unstable is also determined.

1. INTRODUCTION

MEDIA in which the dispersion is nonlinear for small oscillations can support the propagation of stationary isolated waves: if the phase velocity of the waves diminishes (increases) with increasing wave number, the stationary isolated wave is a density compression (rarefaction).[1]

In the present work we consider an isolated wave propagating in a cold low-density plasma (so that dissipation processes can be neglected) at an arbitrary angle with respect to the unperturbed magnetic field $\mathbf{H}_0$. The case being treated is of interest as it exhibits several oscillation branches. In the low frequency region ($\omega \lesssim \Omega_i = eH_0/m_i c$, Ω_i is the ion Larmor frequency) these are accelerated waves (''magnetic'' sound) with a minimum phase velocity $v_+ = H_0/\sqrt{4\pi n_0 m_i}$, and retarded Alfvén waves with a maximum phase velocity $v_- = v_+ \cos\theta$ (here θ is the angle between the direction of propagation and the magnetic field).

The qualitative behavior of ω/k as a function of frequency is shown in Fig. 1 ($\cos\theta \gg \sqrt{m_e/m_i}$, cf. [2]). The existence of two branches means that the isolated wave can be a compression or rarefaction shock.

In the high-frequency region ($\Omega_i \ll \omega \ll \Omega_e$) there is one branch with a high phase velocity $\omega/k = v_+\sqrt{\omega/\Omega_i}$. The characteristic frequency here is $\sqrt{\Omega_e\Omega_i}$ while the effective value of the magnetic field in the wave is of order $H_0\sqrt{m_i/m_e}$ so that $\omega/k \sim v_+\sqrt{m_i/m_e}$. Higher propagation velocities mean that the change in the various physical quantities in the wave can be anomalously large. The theorem given in [1] does not hold here and if the fine structure of the wave at its base is neglected the wave is always a compression shock.

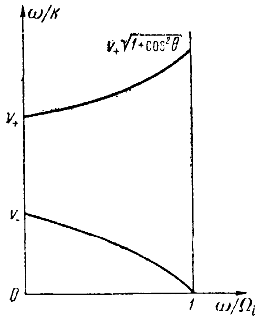

FIG. 1

The limiting cases indicated above differ appreciably in their propagation scales. The flow of plasma must be symmetric with respect to the peak of the wave in the coordinate system moving with the wave since there is no dissipation. It is clear that in the case of Alfvén and magnetic-sound shocks (large-scale waves) the ions cannot execute very many rotations in the magnetic field of the shock without disturbing the correlation between the trajectories of the heavy particles at the ''input'' and ''output'' of the wave. Taking the ion Larmor radius as a characteristic dimension of the wave we find

$$v_+/\Omega_i = \sqrt{m_i c^2/4\pi e^2 n_0}.$$

The wavelength of the high-frequency shock (small-scale wave) is determined by the electron Larmor radius and is of order $\sqrt{m_e c^2/4\pi e^2 n_0}$.

In this work we consider the structure of the isolated wave indicated above and establish the relation between wave amplitude and Mach number. We also find the limiting values for various physical quantities and the limitations on the Mach number. We note that the problem of the isolated wave has been considered earlier for the case $\theta = \pi/2$

in [1,3,4] and for the case $\theta = 0$ in [5-7]. Finally, a low-amplitude rarefaction shock has been investigated by Galeev and Karpman.[8]

2. EQUATIONS OF MOTION AND CONSERVATION LAWS

In the present work we limit ourselves to the nonrelativistic case: $v_0/c \ll 1$, where v_0 is the propagation velocity of the wave. Then, the initial system of equations of motion for the electrons and ions and Maxwell's equations are

$$\frac{\partial \mathbf{v}_{e,i}}{\partial t} + (\mathbf{v}_{e,i}\nabla)\mathbf{v}_{e,i} = \mp \frac{e}{m_{e,i}}\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}_{e,i}\mathbf{H}]\right), \quad (1)^*$$

$$\partial n_{e,i}/\partial t + \operatorname{div} n_{e,i}\mathbf{v}_{e,i} = 0, \quad (2)$$

$$\operatorname{rot}\mathbf{H} = \frac{4\pi e}{c}(n_i\mathbf{v}_i - n_e\mathbf{v}_e), \qquad \operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \quad (3)^\dagger$$

$$\operatorname{div}\mathbf{E} = 4\pi e(n_i - n_e), \quad \operatorname{div}\mathbf{H} = 0. \quad (4)$$

We assume further that all quantities depend only on $\xi = z + v_0 t$, with $H \to H_0$, $v_{e,i} \to 0$, and $n_{e,i} \to n_0$ as $\xi \to \pm\infty$. The system in (1)–(4) can now be written conveniently in complex form:

$$\rho_{e,i}\frac{dv_{e,i}}{d\xi} = \mp\frac{ie}{m_{e,i}c}(H\rho_{e,i} - H_\perp^0 - H_\parallel^0 v_{e,i}/v_0), \quad (5)$$

$$\rho_{e,i}\frac{dv_\parallel^{e,i}}{d\xi} = \mp\frac{e}{m_{e,i}v_0}\left[E_\parallel + \frac{1}{c}\operatorname{Im}(v_{e,i}^* H)\right], \quad (6)$$

$$i dH/d\xi = 4\pi e n_0 c^{-1}(v_i/\rho_i - v_e/\rho_e), \quad (7)$$

$$dE_\parallel/d\xi = 4\pi e n_0(1/\rho_e - 1/\rho_i),$$

$$\rho_{e,i} \equiv n_0/n_{e,i} = 1 + v_\parallel^{e,i}/v_0; \quad \mathbf{v}_{e,i} \equiv (\operatorname{Re} v_{e,i}, \operatorname{Im} v_{e,i}, v_\parallel^{e,i}),$$

$$\mathbf{H} \equiv (\operatorname{Re} H, \operatorname{Im} H, H_\parallel^0). \quad (8)$$

The projections of various vectors on the z axis are denoted by $\parallel$ while the projections on the x axis are denoted by $\perp$.

The equations in (5)–(8) have the following first integrals (conservation of momentum):

$$m_e v_e + m_i v_i = H_\parallel^0 (H - H_\perp^0)/4\pi n_0 v_0, \quad (9)$$

$$m_e v_e + m_i v_i = \{E_\parallel^2 + |H_\perp^0|^2 - |H|^2\}/8\pi n_0 v_0. \quad (10)$$

We shall treat in detail the quasi-neutral case:

$$\rho_e \approx \rho_i = \rho, \quad v_\parallel^e \approx v_\parallel^i \approx v_\parallel, \quad E_\parallel \approx 0.$$

The quasi-neutrality of a plasma is guaranteed over a wide range of angles θ by the fact that the problem is nonrelativistic. It follows from (10) that in this case

$$\rho = 1 + (|H_\perp^0|^2 - |H|^2)/8\pi n_0 m_i v_0^2. \quad (11)$$

*$[\mathbf{vH}] = \mathbf{v}\times\mathbf{H}$.

†rot = curl.

3. LARGE-SCALE WAVES

The drift approximation can be used for the description of electron motion in large-scale waves. The only limitation is on the range of variation of θ: the condition $\cos\theta \gg \sqrt{m_e/m_i}$ must hold.

From (5) and (10) we find that the electron and ion velocities and the magnetic fields are related by

$$v_e = v_0(H\rho - H_\perp^0)/H_\parallel^0, \quad (12)$$

$$v_i = \mu_\parallel^2 v_0 (H - H_\perp^0)/H_\parallel^0, \quad (13)$$

where $\mu = v_+/v_0$ is the reciprocal Mach number $(\mu_\parallel, \mu_\perp \equiv \mu\cos\theta, \mu\sin\theta$.

It will now be convenient to write all quantities in dimensionless form. We introduce the notation:

$$H = H_\perp^0\sqrt{\lambda}\,e^{i\varphi}, \quad |H/H_\perp^0| = \sqrt{\lambda}, \quad \xi = \xi_i\tau,$$

$$\xi_i = \mu_\parallel\sqrt{m_i c^2/4\pi e^2 n_0}.$$

Equation (7) now becomes

$$i\rho(\lambda)\frac{d}{d\tau}(\sqrt{\lambda}\,e^{i\varphi}) + \sqrt{\lambda}\,e^{i\varphi}(\rho(\lambda) - \mu_\parallel^2) + \mu_\parallel^2 - 1 = 0. \quad (14)$$

Here $\rho(\lambda) = 1 + \frac{1}{2}\mu_\perp^2(1-\lambda)$. Multiplying Eq. (14) by $d(\sqrt{\lambda}\,e^{-i\varphi})$ and taking the real part, we have after integrating using the boundary conditions $\lambda(\pm\infty) = 1$ and $\varphi(\pm\infty) = 0$

$$(1-\mu_\parallel^2)(\lambda - 2\sqrt{\lambda}\cos\varphi + 1) = \mu_\perp^2(\lambda-1)^2/4. \quad (15)$$

Using the last expression we can eliminate the function of φ in (14). Thus we obtain the following equation for λ:[1)]

$$\rho(\lambda)\,d\lambda/d\tau = \pm\tfrac{1}{4}\mu_\perp^2|\lambda - 1|\sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)},$$

$$\lambda_\pm = 1 + 4\mu_\perp^{-2}(1 - \mu_\parallel^2 \pm \mu_\perp\sqrt{1-\mu_\parallel^2}). \quad (16)$$

Here, $\sqrt{\lambda_+}$ is the maximum value of the magnetic field in the compressional wave, $\sqrt{\lambda_-}$ is the minimum magnetic field in the rarefaction wave. It is evident from Eq. (16) that the isolated wave can be either a compression shock or a rarefaction shock. Since λ is real and $\lambda \to 1$ when $\tau \to \pm\infty$, we must satisfy the condition $\operatorname{Im}\lambda_\pm = 0$ and $\lambda_\pm \gtrless 1$, which can be rewritten:

$$v_- < v_0 < v_+. \quad (17)$$

One of the conditions in (17) is obvious and the other means that the velocity of the compression (rarefaction) wave must be smaller (larger) than the phase velocity of the accelerated (retarded) waves.

The qualitative behavior of $\sqrt{\lambda_\pm(\mu)}$ is shown

1)In Eq. (16) we must use both signs since we require a symmetric solution and $d\lambda/d\tau$ has different signs with respect to the center of symmetry $\tau = 0$.

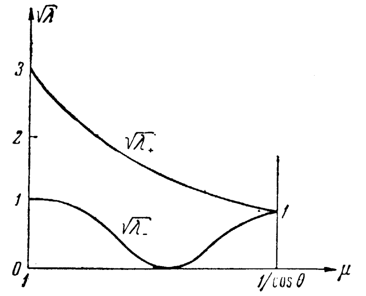

FIG. 2

in Fig. 2. It is evident that the relation between amplitude and Mach number for the $(\rho > 1)$ is not unique.

In the case of the rarefaction waves (diamagnetic wave or $\lambda < 1$) with $\mu = 2/\sqrt{1+3\cos^2\theta}$ the amplitude of the transverse magnetic field vanishes at the extremum while the density is given by the expression

$$(n/n_0)_{min} = (1 + 3\cos^2\theta)/(3 + \cos^2\theta). \qquad (18)$$

In the paramagnetic wave $(\lambda > 1)$ the amplification of the transverse magnetic field can reach a maximum of a factor of three. In this case $\mu = 1$ while the density is given by

$$(n/n_0)_{max} = (1 - 4\sin^2\theta)^{-1}. \qquad (19)$$

The density vanishes at $\theta = 30°$. At this point the solution no longer holds. This situation arises because we have neglected the thermal spread in the velocities of the plasma particles. Actually, the plasma flux at the compression point is retarded and transfers its energy to the magnetic field. If the retardation is strong (the plasma essentially comes to rest $v_\parallel + v_0 \approx 0$), because of the thermal spread there is flux of reflected particles and this is obviously not taken into account by the equations of motion (1). The criterion for the single-velocity approximation can be written as follows:

$$H_0^2 \gg 8\pi n_0 T \rho^{-\gamma}. \qquad (20)$$

Here T and γ are the temperature and adiabatic index of the plasma.

Thus, when $\theta \geq 30°$ the maximum magnetic field in the paramagnetic wave (and this also applies to the wave velocity v_0) must be rather small. From the condition $\rho > 0$ we find that inequality (17) and $1 < \lambda_+ < 9$ for the compression wave must be replaced by

$$v_- < v_0 < v_+\sqrt{\frac{2\cos^2\theta}{1+\sin\theta}}, \qquad 1 < \lambda_+ < 1 + \frac{4\,\mathrm{ctg}^2\,\theta}{1+\sin\theta}, \qquad \theta \geq 30°. \qquad (21)^*$$

*ctg = cot.

The upper limits on v_0 and λ_+ in the last inequalities are determined taking account of (20).

Equation (16) can be integrated in terms of elementary functions. Since $(\lambda > 1)$ we have for the compression wave

$$\frac{\mathrm{arc\,ch}\,[1 + 2(1-\lambda_-)(\lambda_+-\lambda)/(\lambda-1)(\lambda_+-\lambda_-)]}{2\sqrt{(\mu^2-1)(1-\mu_\parallel^2)}} - \arccos\left(1 + 2\frac{\lambda-\lambda_+}{\lambda_+-\lambda_-}\right) = \tau. \qquad (22)^*$$

We now write expressions for the physical quantities in dimensional form. The following formulas hold for a low amplitude compression wave $(1-\mu_\parallel^2 = \frac{1}{4}\delta^2\tan^2\theta)$:

$$n = n_0\left(1 + \frac{\delta\,\mathrm{tg}^2\,\theta}{\mathrm{ch}\,\alpha}\right), \qquad v_\parallel = -v_0\frac{\delta\,\mathrm{tg}^2\,\theta}{\mathrm{ch}\,\alpha}, \qquad v_0 = v_-, \qquad (23)^\dagger$$

$$v_e \approx v_i = \sin\theta\, v_+ F(\xi), \qquad (24)$$

$$H = H_\perp^0(1 + \delta/\mathrm{ch}\,\alpha)(1 + F(\xi)),$$

$$F(\xi) = \frac{1}{\mathrm{ch}^2\,\alpha}(|\mathrm{sh}^2\,\alpha - 1| + 2i\,\mathrm{sh}\,\alpha) - 1, \quad \alpha = \delta\,\mathrm{tg}^2\theta\cdot\xi/\xi_i. \qquad (25)^\ddagger$$

The solution for the rarefaction wave is obtained from (22) by replacing $\lambda_\pm$ by $\lambda_\mp$ and $\lambda - 1$ by $1 - \lambda$.

We now find the quasi-neutrality condition for the plasma. It follows from Eq. (6) that

$$E_\parallel \sim v_0 H_0/c\cos\theta. \qquad (26)$$

Substituting $E_\parallel$ in Eq. (8) we find that the quasi-neutrality condition for the plasma is

$$|\rho_e - \rho_i| \sim (v_0/c\cos\theta)^2 \ll 1. \qquad (27)$$

4. SMALL-SCALE WAVES

We now consider small-scale waves. We assume at the outset that the parameter $\epsilon = \sqrt{m_e/m_i\cos^2\theta}$ is small. In this case the drift approximation cannot be used to describe the electron motion since the wave length is of the order of the electron Larmor radius. However, the calculations can be simplified considerably if we limit ourselves to the analysis of waves whose amplitude is not too small.

Since the electrons execute about one orbit in the magnetic field of the wave for the conditions that hold here, the electron angular momentum cannot be smaller than the ion angular momentum: $m_e v_e \sim m_i v_i$.

For this reason we can neglect the ion current in Maxwell's equations. The characteristic value of the induced magnetic field in the wave is H

*ch = cosh.
†tg = tan.
‡sh = sinh.

$\sim H_0/\epsilon$. Hence, to accuracy of order ϵ we can neglect $H_\perp^0$ everywhere compared with H and take the condition $H(\pm\infty) = 0$ as the boundary condition on H. Thus, H satisfies the following equation approximately

$$\rho\frac{d}{d\xi}\left(\rho\frac{dH}{d\xi}\right)+\frac{ieH_\parallel^0}{m_e c v_0}\rho\frac{dH}{d\xi}-\frac{4\pi e^2 n_0}{m_e c^2}\rho H=0, \quad (28)$$

which is equivalent to the equation of motion of a nonlinear oscillator in a magnetic field.

From momentum and energy conservation we have

$$\left|\rho\frac{dH}{d\xi}\right|^2=\frac{4\pi e^2 n_0}{m_e c^2}|H|^2\left(1-\frac{|H|^2}{16\pi n_0 m_i v_0^2}\right), \quad \rho\frac{d\arg H}{d\xi}=\frac{eH_\parallel^0}{2m_e c v_0}. \quad (29)$$

Turning to dimensionless variables we write

$$|H|=H_0\nu/\varepsilon M; \quad M=H_\parallel^0/\sqrt{16\pi n_0 m_e v_0^2},$$
$$\xi=\xi_e\eta, \quad \xi_e=\sqrt{m_e c^2/4\pi e^2 n_0}. \quad (30)$$

From Eq. (29) we obtain the following equation for ν

$$\rho(\nu)\,d\nu/d\eta=\pm\nu\sqrt{\nu_+^2-\nu^2}, \quad (31)$$

where $\nu_+ = \sqrt{1-M^2}$ is the maximum value of the magnetic field in the wave.

In the approximation used here we have $\rho(\nu) = 1-2\nu^2$. From the conditions $\rho > 0$ and $\mathrm{Im}\,\nu_+ = 0$ we obtain the following limitations on the reciprocal Mach number:

$$1/\sqrt{2}<M<1. \quad (32)$$

When ν_+ is approximately $1/\sqrt{2}$ the particle density at the maximum again becomes large. The condition that must be satisfied if one is to neglect the thermal spread of the velocities in this case is less stringent than in Eq. (20):

$$H_0^2\gg 8\pi n_0\varepsilon^2 T\rho^{-\gamma}. \quad (33)$$

Integrating Eq. (31) we have

$$\operatorname{arcch}\nu_+/\nu-2\nu_+\sqrt{\nu_+^2-\nu^2}=\nu_+|\eta|. \quad (34)$$

For amplitudes that satisfy the condition $\epsilon \ll \nu_+ \ll 1$ the second term in the left side of Eq. (34) can be neglected. Returning to the physical variables, we obtain the following expressions:

$$v_\parallel=-2v_0|f|^2, \quad n=n_0(1+2|f|^2), \quad v_0=H_\parallel^0/\sqrt{16\pi n_0 m_e},$$
$$H=\sqrt{m_i/m_e}\,H_\parallel^0 f(\xi), \quad v_e=2\sqrt{m_i/m_e}\,v_0 f(\xi),$$
$$v_i\sim\sqrt{m_e/m_i}\,v_0 f(\xi), \quad (35)$$

where $f(\xi) = \nu_+ e^{i\xi/\xi_e}/\coth(\nu_+\xi/\xi_e)$. Here, we have neglected the difference in the velocity of rotation of the phase of the magnetic field and the electrons, a procedure that is valid only for low-amplitude waves. It is evident from Eq. (35) that the energy in the wave is distributed equally between the magnetic field and the translational and rotational motion of the plasma. Since the electron velocity is assumed to be nonrelativistic, the following condition must be satisfied:

$$H_0\ll\sqrt{4\pi n_0 m_e^2 c^2/m_i}. \quad (36)$$

Estimates analogous to those in Eqs. (26) and (27) show that the quasi-neutrality of the plasma is also provided by the nonrelativistic condition for the problem (36).

The author is indebted to V. L. Pokrovskiĭ for a number of valuable remarks.

[1] Vedenov, Velikhov, and Sagdeev, Nuclear Fusion **1**, 82 (1961).

[2] V. D. Shafranov, Fizika plazmy i problema upravlyaemykh termoyadernykh reaktsiĭ (Plasma Physics and the Problem of a Controlled Thermonuclear Reaction) AN SSSR 1958, Vol. 4, p. 424.

[3] R. Z. Sagdeev, ibid, p. 384.

[4] V. N. Tsytovich, ZhTF **32**, 1042 (1962), Soviet Phys. Tech. Phys. **7**, 760 (1963).

[5] B. A. Tverskoĭ, JETP **42**, 833 (1962), Soviet Phys. JETP **15**, 581 (1962).

[6] P. Saffman, J. Fluid Mech. **11**, 16 (1961).

[7] A. D. Pataraya, ZhTF **32**, 139 (1962), Soviet Phys. Tech. Phys. **7**, 97 (1962).

[8] A. Galeev and B. I. Karpman, Preprint, Siberia Div. Acad. Sci., U.S.S.R.

Translated by H. Lashinsky
206