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MULTIPOLE EXPANSIONS IN DECAY PROCESSES

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A covariant form of expansion of an arbitrary electromagnetic vertex in terms of relativistic multipoles is presented. Use is made of the irreducible tensors of the little Lorentz group L_u (u is the four-velocity of the center of inertia), which are the analogs of the multipole moments of the classical theory. These tensors are constructed from the wave amplitudes of the particles participating in the decay process with the help of a transformation which is connected with the parallel transfer of vectors and spinors in Lobachevskiĭ space. [5] The same technique is employed to investigate the structure of the β decay Hamiltonian in the universal weak interaction theory. The number of different types of form factors for β decay of particles with arbitrary spin is computed and a relativistically invariant classification of β transitions in terms of degrees of forbiddenness is given.

SMORODINSKIĬ and the author [1] have considered the multipole radiation of a classical system of charges in a relativistically covariant manner. Use was made of the irreducible tensors of the little Lorentz group L_u, where u is the four-velocity of the center of inertia. It is of interest to apply this method to the investigation of the phenomenological structure of electromagnetic vertices in relativistic quantum field theory. In expanding the electromagnetic vertex in terms of irreducible representations of the group L_u it is important that the velocities of the particles before and after the decay are different. The method applied here to the study of electromagnetic radiation and β decay of particles with arbitrary spin can also be used for other processes without essential modifications.

1. IRREDUCIBLE TENSORS OF THE LITTLE GROUP L_u

In the majority of papers [2-4] the S matrix for various processes has been studied in the center of mass system (c.m.s.). On the other hand, the spinor functions of the particles and their polarization parameters are always given in the rest system of each of the particles.

In order to add the spins in the c.m.s. it is natural to transform the basis functions of the little groups $L_{u_{(i)}}$, where the $u_{(i)}$ are the four-velocities of the particles, into the equivalent basis functions of the little group L_u (u is the four-velocity of the center of inertia). Let us write this transformation in the form

$$\varphi'' = T(u_{(i)} u)\varphi. \tag{1}$$

As shown by Smorodinskiĭ, [5] the transformation (1) corresponds to the parallel transfer of a tensor (or spinor) along the geodetic line of the hyperboloid $u_0^2 - u_x^2 - u_y^2 - u_z^2 = 1$ connecting the points $u_{(i)}$ and u. If the spatial axes of the reference systems $\mathbf{u}_{(i)} = 0$ and $\mathbf{u} = 0$ are parallel then obviously

$$\varphi''_{\mathbf{u}=0} = \varphi_{\mathbf{u}_{(i)}=0}. \tag{2}$$

The equality (2) can be used for a relativistically covariant formulation of the transformation (1).

For a particle with integer spin the quantity φ is a symmetric irreducible tensor of rank s which is orthogonal to $u_{(i)}$. The transformation from φ to φ'' is given by the following simple formula (everywhere $AB = \mathbf{AB} + A_4B_4$, $A_4 = iA_0$):

$$\varphi''_{\alpha\beta\gamma\ldots\delta} = T_{\alpha k}T_{\beta l}T_{\gamma m}\ldots T_{\delta n}\varphi_{klm\ldots n}, \tag{3}$$

where

$$T_{\alpha k} = \delta_{\alpha k} + (u_{(i)} + u)_\alpha (u_{(i)} + u)_k/(1 + \gamma_{(i)}) \tag{4}$$

[in our notation $\gamma_{(i)} = -u_{(i)}u$].

We note that formula (4) can be obtained immediately by analogy with the parallel transfer of three-vectors referred to the surface of the unit sphere.

It is easily verified that

$$\varphi''_\alpha\varphi''_\alpha = \varphi_\alpha\varphi_\alpha, \quad \varphi''_\alpha u_\alpha = 0. \tag{5}$$

A particle with half-integer spin is, in the Rarita-Schwinger formalism, [6] represented by a bispinor all components of which are irreducible tensors of the group $L_{u_{(i)}}$ of the rank s − 1/2. For a Dirac bispinor

$$T(u_{(i)}u) = \sqrt{\frac{1+\gamma_{(i)}}{2}}\left(1 - \frac{\sigma_{\alpha\beta}u_{(i)\alpha}u_\beta}{1+\gamma_{(i)}}\right), \tag{6}$$

where $\sigma_{\alpha\beta} = (\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha)/2i$. In the c.m.s. the two small components of φ'' are zero in accordance with (2).

Formulas (3), (4), and (6) allow us to write down the transformation (1) for particles with arbitrary spin. The parallel transfer of vectors was discussed in detail by Smorodinskiĭ,[5] cf. also[7].

Using the spinor amplitudes φ''_{S_2} and φ''_{S_1} connected with φ_{S_1} and φ_{S_2} by the parallel transfer transformation (1), we can easily construct irreducible tensors and pseudotensors which are orthogonal to the four-velocity of the center of inertia u. As we shall see below, these are in a certain sense equivalent to the multipole moments of the classical theory.[1]

In the following we shall denote the irreducible tensors of the little group L_u by $\langle\varphi''^*_{S_2}\varphi_{S_1}\rangle^{(L)}$ and the pseudotensors by $\langle\langle\varphi''^*_{S_2}\varphi_{S_1}\rangle\rangle^{(L)}$ (L indicates the rank of the tensor). The relation between the relativistic tensors and the wave functions in the space of the spinor indices[8,9] is given by the standard Clebsch-Gordan formula:

$$\langle\varphi''^*_{s_2}\varphi_{s_1}\rangle^{(L)}_m = \sum_{m_1m_2} C^{s_1m_1}_{Lms_2m_2}\varphi''^*_{s_2m_2}\varphi_{s_1m_1}, \qquad (7)$$

where, according to the covariant Michel theory of polarization,[9,10] m_1 and m_2 are the projections of the four-vectors $S''_1 = TS_1$ and $S''_2 = TS_2$ on the unit four-vector along the axis of quantization in the c.m.s.

It is interesting to note in this connection that the states of polarization of particles A and B with identical spin are to be regarded as equivalent if their wave functions and polarization parameters are related by the transformation (1):

$$\varphi''_A = T(u_1u_2)\varphi_A = \varphi_B, \quad S''_\alpha = T_{\alpha k}(u_1u_2)S_{kA} = S_{\alpha B}. \quad (1a)$$

2. THE DECAY A = B + γ

Let us consider an electromagnetic vertex with three external lines. In the most general case the transition of particle A to particle B with emission of a photon is described by the S matrix element

$$\langle p_1s_1|S|p_2s_2ke\rangle$$
$$= i\sqrt{4\pi}(2\pi)^4(\Gamma_\alpha e^*_\alpha)\,[-2(ku)]^{-1/2}\delta^4(p_1 - p_2 - k). \quad (8)$$

Here e is the polarization four-vector of the photon, and the wave functions of the particles A and B are normalized to unit volume in the c.m.s., which in this case coincides with the rest system of the particle A: $u = p_1/m_1$.

In the rest system of particle A the emission rate per unit time is

$$dW = \frac{1}{2\pi}\frac{m_2}{E_2}\overline{(\Gamma_\alpha e^*_\alpha)^2}\,\omega\, d\Omega_{\mathbf{k}}.$$

It follows from energy-momentum conservation that the masses of the particles are different for $k^2 = 0$. The frequency of the emitted radiation is

$$\omega = -(ku) = m_1 + (p_2u).$$

From this we obtain easily

$$\omega = \frac{m_1^2 - m_2^2}{2m_1} = \left(1 - \frac{\Delta m}{2m_1}\right)\Delta m, \qquad (9)$$

where $\Delta m = m_1 - m_2$.

The frequency ω is independent of the direction of the momentum $\mathbf{n} = \mathbf{k}/k_0$. The recoil velocity of particle B is uniquely related to the mass difference:

$$\mathbf{v} = -\mathbf{n}\left(1 - \frac{\Delta m}{2m_1}\right)\Delta m \Big/ \left(m_1 - \Delta m + \frac{(\Delta m)^2}{2m_1}\right). \qquad (10)$$

Let us now return to formula (8). In virtue of gauge invariance Γ satisfies the continuity equation

$$\Gamma_\alpha k_\alpha = 0. \qquad (11)$$

We can therefore write in the most general case

$$\Gamma_\alpha = M_{\alpha\beta}k_\beta, \qquad (12)$$

where $M_{\alpha\beta}$ is an antisymmetric four-tensor which depends only on the spins of the particles A and B and the four-momentum of the photon, k. Let us write $M_{\alpha\beta}$ as a sum of two tensors, one of which is orthogonal to the four-velocity of the center of inertia:

$$M_{\alpha\beta} = i\{u_\alpha N_\beta - u_\beta N_\alpha + \varepsilon_{\alpha\beta\gamma\delta}M_\gamma u_\delta\}. \qquad (13)$$

Here $\epsilon_{\alpha\beta\gamma\delta}$ is the completely antisymmetric unit tensor ($\epsilon_{1234} = i\epsilon_{1230} = i$), N is a polar four-vector, and M is an axial four-vector, where

$$Nu = Mu = 0. \qquad (14)$$

It is easily seen that (13) corresponds to separating the transition current into an electric and a magnetic part.

In the system where * $\mathbf{u} = 0$, $N = \{\mathbf{N}, 0\}$, $M = \{\mathbf{M}, 0\}$,

$$\langle|S|\rangle = \sqrt{4\pi}(2\pi)^4\{\omega(\mathbf{N}\mathbf{e}^*)$$
$$+ \omega([\mathbf{M}\mathbf{n}]\mathbf{e}^*)\}\,\delta^3(\mathbf{p}_2 + \mathbf{k})\,\delta(m_1 - E_2 - \omega).$$

Each of the vectors N and M depends in general on the direction $\mathbf{n} = \mathbf{k}/k_0$ and on the spin states of the particles A and B. In order to express this dependence in covariant form, we introduce as before[1] the irreducible tensors of the group L_u, $T^{(L)}$, which

*$[\mathbf{Mn}] = \mathbf{M}\times\mathbf{n}$.

are constructed with the help of the space-like unit vector

$$n=\frac{(k+u\,(ku))}{(-ku)}=\frac{k+u\,(ku)}{\sqrt{k^2+(ku)^2}}\bigg|_{k^2=0}\qquad (nu=0). \tag{15}$$

In the c.m.s. the usual relation between the components of the tensor $T^{(L)}$ and the spherical functions Y_{LM} obtains (see, e.g., [8,1]).

In the most general case we can exhibit the dependence of N and M on the direction n in the following way:

$$\begin{aligned} N_\alpha &= \sum_{L\geqslant 1} (-ku)^{L-1} C\,(L-1)\; N^{(L)}_{\alpha\beta\gamma\ldots\delta}\, T^{(L-1)}_{\beta\gamma\ldots\delta} + a n_\alpha, \\ M_\alpha &= \sum_{L'\geqslant 1} (-ku)^{L-1} C\,(L-1)\; M^{(L)}_{\alpha\beta\gamma\ldots\delta}\, T^{(L-1)}_{\beta\gamma\ldots\delta} + b n_\alpha. \end{aligned} \tag{16}$$

Here $T^{(L)}_{\beta\gamma\ldots\delta}\, u_\beta = 0$,

$$C\,(L)=\sqrt{\frac{4\pi}{2L+1}}\, i^L \frac{\sqrt{(2L)!}}{(2L-1)!!}, \tag{17}$$

and the tensors $N^{(L)}$ and pseudotensors $M^{(L)}$ are proportional to the irreducible tensors of the little Lorentz group L_u constructed from the spinor amplitudes of the particles A and B. Thus

$$\begin{aligned} N^{(L)}_{\alpha\beta\gamma\ldots\delta} &= G^{(L)}\,(m_1^2 m_2^2 0)\, \langle \varphi''^{*}_{s_2}\, \varphi_{s_1}\rangle^{(L)}_{\alpha\beta\gamma\ldots\delta}, \\ M_{\alpha\beta\gamma\ldots\delta} &= D^{(L')}\,(m_1^2 m_2^2 0)\, \langle\langle \varphi''^{*}_{s_2}\, \varphi_{s_1}\rangle\rangle^{(L')}_{\alpha\beta\gamma\ldots\delta}, \end{aligned} \tag{18}$$

where $G^{(L)}$ and $D^{(L)}$ are invariant functions of the variables m_1^2, m_2^2, and k^2 taken at a physical point with given m_1, m_2, and $k^2 = 0$.

We note that the four-vectors an_α and bn_α (a and b are relativistic scalars) do not contribute to the radiation, since the three-dimensional transversality condition [1] (ek = 0, eu = 0) leads to the relation

$$en=0. \tag{19}$$

We shall therefore leave them out in the following.

It is easily seen that the quantities L and L′ in (16) are restricted by the inequalities

$$|s_1-s_2|\leqslant L;\;\; L'\leqslant |s_1+s_2|;\;\; L, L'\geqslant 1. \tag{20}$$

If the relative intrinsic parity of particles A and B is positive ($\eta = 1$), L takes only even and L′ only odd values within the limits (20). If the relative intrinsic parity of A and B is negative ($\eta = -1$), then L takes only odd and L′ only even values. In both cases the number of form factors $G^{(L)}$ and $D^{(L)}$ is equal to $2s_1 + 1$ if $s_1 < s_2$ or to $2s_2 + 1$ if $s_2 < s_1$. If $s_1 = s_2 = s$, the number of form factors is equal to 2s.

The unitarity condition for the S matrix leads in first approximation in $e^2/\hbar c$ to real values for all form factors $G^{(L)}$ and $D^{(L)}$.

Substituting (16) in (12) and (13), we can write with the help of the relativistic electric and magnetic multipoles introduced in [1]:

$$\begin{aligned} \Gamma_\alpha e^*_\alpha &= \sum_{L\geqslant 1} K\,(L)\,(-ku)^L\, N^{(L)}_{\beta\gamma\ldots\delta}\,(E^{(L)}_{\alpha\beta\gamma\ldots\delta}\, e^*_\alpha) \\ &+ \sum_{L'\geqslant 1} K\,(L')\,(-ku)^{L'}\, M^{(L')}_{\beta\gamma\ldots\delta}\,(\mathfrak{M}^{(L)}_{\alpha\beta\gamma\ldots\delta} e^*_\alpha), \\ K\,(L) &= \sqrt{(L+1)/L}\; C(L). \end{aligned} \tag{21}$$

The relativistic multipoles are given by the formulas

$$E^{(L)}_{\alpha\beta\gamma\ldots\delta}=(-ku)\frac{\partial}{\partial k_\alpha} T^{(L)}_{\beta\gamma\ldots\delta}\,\Big|_{k^2=0},$$

$$\mathfrak{M}^{(L)}_{\alpha\beta\gamma\ldots\delta}=\varepsilon_{\alpha lmn}\, k_l \left(\frac{\partial}{\partial k_m} T^{(L)}_{\beta\gamma\ldots\delta}\right)\bigg|_{k^2=0} u_n$$

and satisfy the relations

$$E^{(L)}_{\alpha\beta\gamma\ldots\delta}\, k_\alpha = E^{(L)}_{\alpha\beta\gamma\ldots\delta}\, u_\alpha = 0;\qquad \mathfrak{M}^{(L)}_{\alpha\beta\gamma\ldots\delta}\, k_\alpha = \mathfrak{M}^{(L)}_{\alpha\beta\gamma\ldots\delta}\, u_\alpha = 0.$$

It is immediately seen from (21) that the tensors $N^{(L)}$ describe electric 2^L pole radiation and the tensors $M^{(L)}$ magnetic 2^L pole radiation. The terms proportional to n in Γ do not contribute to the radiation by virtue of gauge invariance [cf. (19)].

We note that the form factors $G^{(L)}$ and $D^{(L)}$ have the dimensions of eR^L, where e is the charge and R a characteristic invariant length parameter. The wavelength approximation in the system where u = 0 corresponds to

$$\omega=(-ku)\ll 1/R\ll 1/m. \tag{22}$$

In this case

$$G^{(L)}\approx g_0^{(L)} R^L,\quad D^{(L)}\approx d_0^{(L)} R^L$$

and only the first nonvanishing terms with the lowest L are retained in the sums (21).

If the inequality (22) is satisfied the radiation of the photon can be described by a semiclassical theory. In this case there is a unique connection between $N^{(L)}$ and $M^{(L)}$ and the matrix elements of the tensors of the electric and magnetic 2^L moments. [1]

3. ELECTROMAGNETIC TRANSITIONS OF ELEMENTARY PARTICLES

The decay probability and the polarization parameters for the photon and the recoil particle can be expressed in terms of the set of form factors $G^{(L)}$ and $D^{(L)}$. In squaring the expansion (21) it is natural to use formula (7). In the final expressions we obtain products of the type

$$\langle \varphi''^{*}_{s_2}\, \varphi_{s_1}\rangle^{(L)}_m \langle \varphi''^{*}_{s_2}\, \varphi_{s_1}\rangle^{*(L')}_{m'} = \sum_{m_1 m_1' m_2 m_2'} C^{s_1 m_1}_{L m s_2 m_2'}\, C^{s_1 m_1'}_{L m s_2 m_2}\, \rho^{(s_1)}_{m_1 m_1'}\, \rho^{(s_2)}_{m_2' m_2}, \tag{23}$$

where $\rho^{(S_1)}_{m_1 m_1'}$ and $\rho^{(S_2)}_{m_2 m_2'}$ are respectively elements of the covariant density matrices of particles A and B, and m_1 and m_2 are the projections of the polarization four-vectors S_1 and S_2 on the four-dimensional vectors n_1 and $n''_{1\alpha} = T_{\alpha\beta} n_1$ (cf. Sec. 1).[9]

The total emission probability per unit time does not contain terms corresponding to the interference of different multipoles:

$$P = \frac{m_2}{(-p_2 u)}(1-v^2)^{1/2}\Big\{\sum_L (-ku)^{2L+1}\frac{(L+1)(L-1)!}{(2L+1)(2L-1)!!}\frac{1}{2^{L-1}} \times (G^{(L)^2})^2 + \sum_{L'}(-ku)^{2L'+1}\frac{(L'+1)(L'-1)!}{(2L'+1)(2L'-1)!!}\frac{1}{2^{L-1}}(D^{(L)})\Big\}. \quad (24)$$

Here v is the velocity of particle A.

Expressions for the angular distribution and the polarizations of the radiation and of the recoil particle for a pure multipole (wave length approximation) are given in the book of Akhiezer and Berestetskiĭ.[8] If all form factors are taken into account interference terms appear, which we shall not write down in view of their complexity. We indicate, however, the basic characteristics of the angular distribution and the polarization of the photon.

a) The angular distribution has the form $dW = Wk_0^2(ku)^{-2}d\Omega$, where

$$W = 1 + \sum_l \alpha_{2l} P^{(2l)}_{\alpha\beta\gamma\ldots\delta} n_\alpha n_\beta n_\gamma \ldots n_\delta, \quad (25)$$

and n is given by (15), $P^{(2l)}$ are the four-dimensional polarization moments of the particle A [$P^{(2l)}u = 0$], and the coefficients α_{2l} are expressed in terms of products of the form factors $G^{(L')}$ and $D^{(L)}$ and Racah coefficients. We see that the angular distribution of the radiation depends linearly on the even polarization moments alone, where $0 < l < s_1$.

b) The linear polarization of the photon also depends only on the even polarization moments, but in contrast to (25) $1 < l < s_1$ since it is always zero for the decay of a particle with spin $s_1 = 1/2$.

c) The circular polarization, on the other hand, depends only on the odd polarization moments and vanishes if the particle is unpolarized:

$$\varepsilon_2 = \sum_l \gamma^{(2l+1)} P^{(2l+1)}_{\alpha\beta\ldots\gamma} n_\alpha n_\beta \ldots n_\gamma \quad (26)$$

We give a summary of the polarization parameters characterizing the decay $\Sigma^0 \to \Lambda^0 + \gamma$, written in four-dimensional form. If $\eta = 1$, we have magnetic dipole radiation, and if $\eta = -1$, electric dipole radiation.

After a simple calculation we find ($\eta = \pm 1$):

$$S''_{\alpha(\Lambda)} = T_{\alpha\beta} S_{\beta(\Lambda)} = -n_\alpha (S_{(\Sigma)} n), \quad (27)$$

$$\varepsilon_1 = \varepsilon_3 = 0, \quad \varepsilon_2 = (S_{(\Sigma)} n), \quad (28)$$

where $S_{(\Sigma)}$ is the polarization four-vector of the Σ^0 particle, $S_{(\Lambda)}$ is the polarization four-vector of the Λ^0 particle, ϵ_1, ϵ_3 are the parameters of the linear, and ϵ_2 of the circular polarization of the photon.

The correlation between the circular polarization of the photon and the polarization of the Λ^0 particle is given by

$$K_\alpha = \langle | \varepsilon_2 S''_{\alpha(\Lambda)} | \rangle = n_\alpha. \quad (29)$$

The correlation between $S_{(\Lambda)}$ and the linear polarization vector of the photon is given by[10]

$$S''_{(\Lambda)} = -\{S_{(\Sigma)} - 2e'(e'S_{(\Sigma)})\}, \quad \eta = 1,$$
$$S''_{(\Lambda)} = -\{S_{(\Sigma)} - 2e(eS_{(\Sigma)})\}, \quad \eta = -1 \quad (30)$$

($e'_\alpha = \epsilon_{\alpha\beta\gamma\delta} k_\beta e_\gamma u_\delta$). The radiation in the transition from Σ^0 to Λ^0 is independent of the polarization of the Σ^0 particle and isotropic in the rest system of the Σ^0 particle.

4. ELECTROMAGNETIC FORM FACTORS FOR $k^2 \neq 0$

For $k^2 \neq 0$, the structure of the electromagnetic vertex can be found on the basis of considerations completely analogous to the preceding ones (Sec. 2, cf. also[1)]). As before, we shall start from the gauge invariance of the theory and from the representation (12). However, for $k^2 \neq 0$ we must, in contrast to (21), take into account the contribution of the longitudinal scalar term an_α to [see (16)], where ($n = (k + u(ku))/\sqrt{k^2 + (ku)^2}$). The final expression for Γ can be written in the following form:*

$$\Gamma_\alpha = \Gamma^{(e)}_\alpha + \Gamma^{(m)}_\alpha + \left(u - k\frac{ku}{k^2}\right)_\alpha H,$$

$$\Gamma^{(e)}_\alpha = \sum_{L\geqslant 1}\sqrt{\frac{L+1}{L}}\,C(L)(-ku)\left(\sqrt{(k^2+(ku)^2}\right)^{L-1} G^{(L)}(k^2) \times \{\langle \varphi''^*_{s_2}\varphi''_{s_1}\rangle^{(L)}_{\beta\gamma\ldots\delta} E^{(L)}_{\alpha\beta\gamma\ldots\delta}\},$$

1)The electromagnetic vertex explicitly enters, e.g., in the matrix element for the scattering of electrons on nuclei (with $m_1 = m_2$, $s_1 = s_2$) or for the excitation of nuclei by electrons ($m_1 \neq m_2$). In the one-photon approximation

$$T_{12\to 1'2'} = (2\pi)^4 \frac{\Gamma_i \bar\psi_2 \gamma_i \psi_1}{k^2}\delta^4(p_1 + k_1 - p_2 - k_2),$$

where p_1 and p_2 are the four-momenta of the particles A and B, k_1 and k_2 are the four-momenta of the electrons, and $\bar\psi_2\gamma_i\psi_1$ is the current density of the electron.

$$\Gamma_\alpha^{(m)} = \sum_{L'\geqslant 1} \sqrt{\frac{L'+1}{L'}} C(L') \left(\sqrt{k^2+(ku)^2}\right)^{L'} D^{(L')}(k^2)$$
$$\times \{\langle\langle \varphi''^*_{s_2} \varphi''_{s_1} \rangle\rangle^{(L')}_{\beta\gamma\ldots\delta} \mathfrak{M}^{(L')}_{\alpha\beta\gamma\ldots\delta}\},$$
$$H = \sum_{L''\geqslant 0} C(L'') \left(\sqrt{k^2+(ku)^2}\right)^{L''} F^{(L'')}(k^2) \{\langle \varphi''^*_{s_2} \varphi''_{s_1} \rangle^{(L)}_{\beta\gamma\ldots\delta} T^{(L)}_{\beta\gamma\ldots\delta}\}. \quad (31)$$

Here C(L) is given by (17), and $E^{(L)}$ are the electric and $\mathfrak{M}^{(L)}$ the magnetic multipoles related to the irreducible tensors $T^{(L)}$ by

$$E^{(L)}_{\alpha\beta\gamma\ldots\delta} = \frac{1}{\sqrt{L(L+1)}} \sqrt{k^2+(ku)^2} \frac{\partial}{\partial k_\alpha} T^{(L)}_{\beta\gamma\ldots\delta},$$
$$\mathfrak{M}^{(L)}_{\alpha\beta\gamma\ldots\delta} = \frac{1}{\sqrt{L(L+1)}} \varepsilon_{\alpha lmn} \left(\frac{\partial}{\partial k_l} T^{(L)}_{\beta\gamma\ldots\delta}\right) u_m k_n. \quad (32)$$

It is convenient to set u equal to p_1/m_1.

We note that in the classical limit with $q = \sqrt{k^2 + (ku)^2} \ll 1/R$ the longitudinal scalar form factors coincide with the electric ones: $F^{(L)} \approx G^{(L)}$. L, L′, and L″ satisfy (20) in all sums.

If A and B are the same particle, the term with $L'' = 0$ in (31) is nonvanishing and $k^2 > 0$. Here L and L″ take even and L′ odd values, $u - k(ku)/k^2 = (p_1 + p_2)/2m$, and the total number of form factors in (31) is equal to 3s + 1 for integer spin and to 3s + 1/2 for half-integer spin (see the table below). In the static limit $(ku)^2/k^2 = k^2/2m^2 \ll 1$ and the terms $\Gamma^{(e)}$ in (31) can be neglected. The number of form factors is then equal to 2s+1.

For a scalar particle ($s_1 = s_2 = 0$)

$$\Gamma = F^{(0)}(k^2) \frac{p_1+p_2}{2m}. \quad (33)$$

If $s_1 = s_2 = 1/2$, both the magnetic dipole and monopole terms are present in the sums (31).

In our formalism [$\psi'' = T\psi$, see (6)] we have for the proton

$$\Gamma_\alpha = \left(\frac{p_1+p_2}{2m}\right)_\alpha F^{(0)}(k^2) (\bar{\psi}''_2 \psi_1)$$
$$+ iD^{(1)}(k^2) \varepsilon_{\alpha\beta\gamma\delta} (\bar{\psi}''_2 \gamma_5 \gamma_\beta \psi_1) k_\gamma u_\delta. \quad (34)$$

Usually Γ for nucleons is written in terms of Dirac spinors without use of the transformation (6): [11]

$$\Gamma_\alpha = iF'(k^2)(\bar{\psi}_2 \gamma_\alpha \psi_1) + iD'(k^2)(\bar{\psi}_2 \sigma_{\alpha k} k_k \psi_1).$$

Here

$$F^{(0)}(k^2) = F'(k^2)\sqrt{\frac{1+\gamma}{2\gamma}} + 2mD'(k^2)\sqrt{\frac{\gamma^2-1}{2\gamma^3}},$$
$$D^{(1)}(k^2) = \left(D' + \frac{F'}{2m}\right)\sqrt{\frac{1+\gamma}{2\gamma}}\left(1 - \frac{\sqrt{\gamma-1}}{\gamma}\right) \quad (\gamma = -(u_1 u_2)).$$

Evidently $F^{(0)}(k^2) \to e$, $D^{(1)}(k^2) \to eg/2m$ if $k^2/m^2 \to 0$, where e is the proton charge and g the gyromagnetic ratio including the normal and anomalous magnetic moments.

5. STRUCTURE OF THE HAMILTONIAN FOR β DECAY

As is well known, the weak interaction Hamiltonian is written as a product of two currents, the nuclear and the leptonic: [12]

$$H = Gj^{(n)} j^{(l)}/\sqrt{2}, \quad (35)$$

where the matrix element for the β transition is [13]

$$S = (2\pi)^4 iH\delta^4(p_1 - p_2 - k_1 - k_2). \quad (36)$$

Here p_1, p_2, k_1, and k_2 are respectively the four-momenta of the decaying particle, the recoiling particle, the electron, and the antineutrino.

In the rest system of the decaying particle the decay rate is equal to

$$dw = \frac{1}{(2\pi)^5}|H|^2 \frac{m_2}{E_2}(m_1 - E_2 - k_{10})^2 |\mathbf{k}_1|^2 k_{10}\, d\Omega_{\mathbf{k}}\, d\Omega_{(\mathbf{k}_1\mathbf{k}_2)}. \quad (36a)$$

In (36a), $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$ is the total momentum of the electron and the antineutrino. If we are interested in the electron-neutrino correlation we must integrate (36a) over $d\Omega_{\mathbf{k}} = 2\pi \sin\theta' d\theta'$, where θ′ is the angle between **k** and the polarization vector of particle A.

According to the universal weak interaction theory [12]

$$j^{(l)}_\alpha = \bar{\psi}_1 \gamma_\alpha (1 + \gamma_5) \psi_2.$$

Summing $|H|^2$ over the polarizations of the electron and the antineutrino, we obtain

$$|H|^2 = \frac{G^2}{(k_1 u)(k_2 u)} \{ j^{(n)}_\beta j^{(n)*}_\beta (-k_1 k_2) + 2\mathrm{Re}(j^{(n)}_\beta k_{1\beta})(j^{(n)}_\alpha k_{2\alpha})^* \}. \quad (37)$$

In the following we shall only be interested in the nuclear current. Using the method of the parallel transfer of the tensors of the little group (Sec. 1 of the present paper and also [5]), we attempt to expand the nuclear current in terms of irreducible tensors of the group L_u ($u = p_1/m_1$), thus effecting a relativistic classification of β decays by degrees of forbiddenness.

The program of describing the β decay of an arbitrary system with the help of form factors has been proposed by Smorodinskiĭ at the Conference on Nuclear Spectroscopy at Kharkov. [14]

Let us write the current $j^{(n)}$ as the sum of two currents: the vector current V and the axial vector current A. The total momentum of the electron and the antineutrino plays in this case the role of

k^2 (Sec. 4). For β decay k is a time-like vector and the physical region of k^2 is given by the inequality

$$-m_{(l)}^2 \leqslant k^2 \leqslant -\Lambda^2, \qquad \Lambda = m_1 - m_2. \tag{38}$$

Let us separate out from V and A the transverse components $V^{(0)}$ and $A^{(0)}$ which satisfy the equalities $V^{(0)}k = A^{(0)}k = V^{(0)}u = A^{(0)}u = 0$ and write V and A in a form convenient for comparison with the electromagnetic vertex for $p_1^2 = p_2^2$ (see below):

$$V = V^{(0)} + \frac{p_1 + p_2}{2m_1} S + \frac{k}{\sqrt{k^2 + (ku)^2}} s,$$

$$A = A^{(0)} + \frac{p_1 + p_2}{2m_1} P + \frac{k}{\sqrt{k^2 + (ku)^2}} p. \tag{39}$$

The terms proportional to $(p_1 + p_2)$ are orthogonal to $k = p_1 - p_2$ in the unphysical region $p_1^2 = p_2^2$, S and s are scalars and P and p are pseudoscalars.

The expansion of the four-vector $V^{(0)}$ is completely analogous to the expansion of the transverse part of the electromagnetic vertex in terms of electric and magnetic multipoles. The scalars S and s are expanded in the same way as H in (31).

Using the irreducible tensors $T^{(L)}$ of the little group and the relativistic multipoles (31), we can write the covariant expansion of V in the form

$$V_\alpha = \sum_{L\geqslant 1} \sqrt{\frac{L+1}{L}} C(L)(-ku) \times (\sqrt{k^2 + (ku)^2})^{L-1} g_V^{(L)}(k^2) \{\langle \varphi_{s_2}^{''*} \varphi_{s_1} \rangle_{\beta\gamma\ldots\delta}^{(L)} E_{\alpha\beta\gamma\ldots\delta}^{(L)}\}$$
$$+ \sum_{L'\geqslant 1} \sqrt{\frac{L'+1}{L'}} C(L') (\sqrt{k^2 + (ku)^2})^{L'} d_V^{(L')}(k^2) \times \{\langle\langle \varphi_{s_2}^{''*} \varphi_{s_1} \rangle\rangle_{\beta\gamma\ldots\delta}^{(L')} \mathfrak{M}_{\alpha\beta\gamma\ldots\delta}^{(L)}\}, \tag{40}$$

$$S = [f_V^{(0)}(k^2) \langle \varphi_{s_2}^{''*} \varphi_{s_1} \rangle^{(0)} + \sum_{L\geqslant 1} C(L) (\sqrt{k^2 + (ku)^2})^L f_V^{(L)}(k^2) \{\langle \varphi_{s_2}^{''*} \varphi_{s_1} \rangle_{\beta\gamma\ldots\delta}^{(L)} T_{\beta\gamma\ldots\delta}^{(L)}\}], \tag{41}$$

$$s = [n_V^{(0)}(k^2) \langle \varphi_{s_2}^{''*} \varphi_{s_1} \rangle^{(0)} + \sum_{L\geqslant 1} C(L) (\sqrt{k^2 + (ku)^2})^L n_V^{(L)}(k^2) \{\langle \varphi_{s_2}^{''*} \varphi_{s_1} \rangle_{\beta\gamma\ldots\delta}^{(L)} T_{\beta\gamma\ldots\delta}^{(L)}\}]. \tag{42}$$

It is clear that an expansion for A is obtained by interchanging the tensors and pseudotensors in (40) to (42). Taking into account that usually $|A_4^2| \ll \mathbf{A}^2$, we relabel the form factors (40) to (42) somewhat in order to make them more readily comparable with the β decay matrix elements of the usual theory.[15] Thus

$$A_\alpha^{(0)} = \sum_{L'\geqslant 1} (-i) \sqrt{\frac{L'+1}{L'}} C(L') d_A^{(L'-1)}(k^2) (\sqrt{k^2 + (ku)^2})^{L'-1} \times \{\langle\langle \varphi_{s_2}^{''*} \varphi_{s_1} \rangle\rangle_{\beta\gamma\ldots\delta}^{(L')} E_{\alpha\beta\gamma\ldots\delta}^{(L')}\}$$
$$+ \sum_{L\geqslant 1} (-i) \sqrt{\frac{L+1}{L}} C(L) g_A^{(L)}(k^2) (\sqrt{k^2 + (ku)^2})^L \times \{\langle \varphi_{s_2}^{''*} \varphi_{s_1} \rangle_{\beta\gamma\ldots\delta}^{(L)} \mathfrak{M}_{\alpha\beta\gamma\ldots\delta}^{(L)}\}, \tag{43}$$

$$P = -i \Big[f_A^{'(0)}(k^2) \langle\langle \varphi_{s_2}^{''*} \varphi_{s_1} \rangle\rangle^{(0)} + \sum_{L'\geqslant 1} C(L') f_A^{(L'-1)}(k^2) \times (\sqrt{k^2 + (ku)^2})^{L'-1} \{\langle\langle \varphi_{s_2}^{''*} \varphi_{s_1} \rangle\rangle_{\beta\gamma\ldots\delta}^{(L')} T_{\beta\gamma\ldots\delta}^{(L')}\} \Big], \tag{44}$$

$$p = -i \Big[n_A^{'(0)}(k^2) \langle\langle \varphi_{s_2}^{''*} \varphi_{s_1} \rangle\rangle^{(0)} + \sum_{L'\geqslant 1} C(L') n_A^{(L'-1)}(k^2) (\sqrt{k^2 + (ku)^2})^{L'-1} \times \{\langle\langle \varphi_{s_2}^{''*} \varphi_{s_1} \rangle\rangle_{\beta\gamma\ldots\delta}^{(L')} T_{\beta\gamma\ldots\delta}^{(L')}\} \Big] \tag{45}$$

[the dimensions of the quantities $d^{(L)}$, $f^{(L)}$, $g^{(L)}$, and $n^{(L)}$ are those of $R^{(L)}$].

As in Sec. 2, L and L′ take only even or odd values depending on the relative parity of the particles A and B. If no definite value can be assigned to the relative parity there is not any longer any sense in writing V and A in terms of the tensors $\langle\ \rangle^{(L)}$ and pseudotensors $\langle\langle\ \rangle\rangle^{(L)}$. The expansion of the total nuclear current V + A is then given by formulas (40) to (42), where L and L′ take all possible values within the limits (20). The total number of form factors is equal to $8s_2 + 4$ if $s_2 < s_1$ and to $8s + 1$ if $s_1 = s_2$. In particular, the decay $\Lambda^0 \rightarrow p + e^- + \bar{\nu}$ is described by six form factors in complete accordance with [14,16], and the decay $\pi^0 \rightarrow \pi^+ + e^- + \bar{\nu}$ by two form factors.

The dependence of the number of form factors of different type on the spin and the relative parity of the particles A and B is conveniently shown in the form of a table.

We note that in first approximation in the coupling constant all form factors (40) to (45) are real (invariance under time reversal, see [13]). In the wavelength approximation $q = \sqrt{k^2 + (ku)^2} \ll 1/R$ the representation of the β decay current in the form of the sums (40) to (45) is nothing but the expansion of the β decay current in terms of degrees of forbiddenness ($f_V^{(0)} \sim 1$, $f_V^{(L)} \sim R^L$, $d_A^{(0)} \sim 1$, $d_A^{(L)} \sim R^L$ etc.).

We also give the formula for the electron-antineutrino correlation in the decay of a particle with arbitrary spin. Substituting (40) to (45) in (36a) and (37), taking account of (24), we obtain readily after summing over the polarizations of the recoiling particle and integrating over $d\Omega_{\mathbf{k}}$ ($\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$)

$$dW = \frac{1}{(2\pi)^5} \frac{m_2}{E_2} (m_1 - E_2 - k_{10})^2 |\mathbf{k}_1|^2 k_{10} dk_{10} d\Omega_{(\mathbf{k}_1, \mathbf{k}_2)} \times \Big\{ (1 + v\cos\theta) \Big[4\pi \Big| \Big(1 + \frac{E_2}{m_2}\Big) f^{(0)} + \frac{k_0}{|\mathbf{k}|} n^{(0)} \Big|^2$$

s	j	Number of form factors								Total number of form factors	
		electric type		magnetic type		longitudinal-scalar		not current conserving			
		V	A	V	A	V	A	V	A	β-decay	γ-radiation
Integer number	1	s + 1 (s)	s	s	s + 1 (s)	s + 1	s	s + 1	s	8s + 4 (8s + 2)	2s + 1 (2s)
	−1	s	s + 1 (s)	s + 1 (s)	s	s	s + 1	s	s + 1	8s + 4 (8s + 2)	2s + 1 (2s]
Half-integer number	1	s + 1/2 (s − 1/2)	s + 1/2	s + 1/2	s + 1/2 (s − 1/2)	s + 1/2	s + 1/2	s + 1/2	s + 1/2	8s + 4 (8s + 2)	2s + 1 (2s)
	−1	s + 1/2	s + 1/2 (s − 1/2)	s + 1/2 (s − 1/2)	s + 1/2	s + 1/2	s + 1/2	s + 1/2	s + 1/2	8s + 4 (8s + 2)	2s + 1 (2s)

Note. The figure in parentheses is the number of form factors for $\Delta s = 0$ ($\Delta s = |s_1 - s_2|$), if it is different from the number of form factors in the case $\Delta s \neq 0$ ($s = s_2$, if $s_2 < s$; $s = s_1$, if $s_1 < s_2$); $j = (-1)^{\Delta s}\eta$, where η is the relative parity.

$$+ \sum_{L=|s_1-s_2|+\lambda}^{|s_1+s_2|} |C(L)|^2 |\mathbf{k}|^{2L} \left| \left(1 + \frac{E_2}{m_2}\right) f^{(L)} + \frac{k_0}{|\mathbf{k}|} n^{(L)} \right.$$

$$+ \left(1 - \frac{v\cos\theta}{3}\right) \left[4\pi \left| n^{(0)} + \frac{|k|}{2m_1} f^{(0)} \right|^2\right.$$

$$+ \sum_{L=|s_1-s_2|+\lambda}^{|s_1+s_2|} |C(L)|^2 |\mathbf{k}|^{2L} \left| n^{(L)} + \frac{|\mathbf{k}|}{2m_1} f^{(L)} \right|^2$$

$$+ \sum_{L=|s_1-s_2|+\lambda}^{|s_1+s_2|} |C(L)|^2 |g^{(L)}|^2 |\mathbf{k}|^{2L-2} (-ku)^2 \frac{L+1}{L}$$

$$+ \sum_{L=|s_1-s_2|+\lambda}^{|s_1+s_2|} |C(L)|^2 |d^{(L)}|^2 |\mathbf{k}|^{2L} \frac{L+1}{L} \Big]\Big\}. \qquad (46)$$

Here $v = |\mathbf{k}_1|/k_{10}$ is the velocity of the electron and θ is the angle between $\mathbf{k}_1$ and $\mathbf{k}_2$. The summation in (46) goes over all values of L within the indicated limits, with $\lambda = 0$ if $s_1 \neq s_2$ and $\lambda = 1$ if $s_1 = s_2$. The form factors entering in (46) refer either to the vector or to the axial vector part of the nuclear current depending on L and the relative intrinsic parity of particles A and B

$$(f^{(0)} \to f_V^{(0)},\ f_A^{\prime(0)};\ n^{(0)} \to n_V^{(0)},\ n_A^{\prime(0)};\ f^{(L)} \to f_V^{(L)},\ f_A^{(L-1)};$$
$$n^{(L)} \to n_V^{(L)},\ n_A^{(L-1)};\ g^{(L)} \to g_V^{(L)},\ d^{(L-1)}/(-ku)).$$

As should be expected, the general formula for the electron-antineutrino correlation does not contain terms corresponding to the interference of V and A.

The interference between V and A (nonconservation of parity) gives rise to a linear dependence of the angular distribution of the leptons on the even as well as on the odd polarization moments of the decaying particle A, in contrast to the angular distribution of the photons.

If isotopic invariance holds the vector current is conserved when $m_1 = m_2$.[14,12] It is seen from (39) that for $p_1^2 = p_2^2$ the nonconserved current corresponds to the term proportional to k. It follows from this that the form factors $n_V^{(L)}$ are of the order of ξR^L in the region $|k^2|/m^2 \ll 1$, where ξ is the degree of accuracy to which the isotopic spin is conserved[17] (for light nuclei $\xi \sim 0.01$, for nucleons $\xi \sim 10^{-4}$ [13]).

It should be noted that the longitudinal scalar term in (39) is conserved in the nonphysical region $p_1^2 = p_2^2$. For time-like k (physical region) it is plainly not conserved, since $p_1 + p_2$ is also a time-like vector.

In formula (39) we have replaced the four-vector $u - k\,(ku)/k^2$ [see (31)] which is always orthogonal to k, by $p_1 + p_2$, since the latter four-vector does not change, in contrast to the first, as we go from $k^2 > 0$ to $k^2 < 0$ in the region $|k^2/m^2| \ll 1$ (there is no singularity on the light cone).

This allows us to estimate the form factors of the weak vector current for small time-like k in analogy with the isotopic vector part of the electromagnetic form factors for small space-like k.[12] If the particles A and B belong to the same isotopic multiplet we can easily obtain a simple relation between the form factors $g_V^{(L)}$, $d_V^{(L)}$, and $f_V^{(L)}$ and the corresponding electromagnetic form factors of the 2T + 1 particles belonging to the multiplet:

$$d_V^{(L)}(0) = \frac{1}{e} d_1^{(L)}(0), \qquad f_V^{(L)} = \frac{f_1^{(L)}(0)}{e} \qquad \text{etc.},$$

where $d_1^{(L)}$ [$f_1^{(L)}$] are determined by the system of 2T + 1 equations

$$\square_{(i)}^{(L)} = \sum_{n=0}^{2T} d_n T_{3(i)}^n \qquad (i = 1, 2, \ldots, 2T+1)$$

(e is the proton charge). In particular, for the decay $\pi^0 \to \pi^+ + e^- + \bar{\nu}$ (T = 0) we have

$$f_V^{(0)} \approx 1,\ n_V^{(0)} \sim \xi,\ V \approx (p_1 + p_2)/2m \ [12,16].$$

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Note added in proof (December 3, 1962). After this paper had been submitted to press, the authors became acquainted with a paper of Durand et al[18] in which the kinematic structure of an arbitrary electromagnetic vertex for $k^2 \neq 0$ is studied by a different method. The results concerning the number of independent vertex functions are in complete agreement with Sec. 4 of our paper.

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ERRATA

Article by V. L. Lyuboshitz Multiple Expansions in Decay Processes [Soviet Phys. JETP 17, 382 (1963)]

Page	col.	line	reads	should read
382		Date of submittal to editor	December 3, 1962	July 16; resubmitted September 7, 1962
385	1.	Eq. (24), last factor	$(D^{(L)})$	$(D^{(L)})^2$
386	1.	3rd from bottom	$\sqrt{\frac{\gamma^2-1}{2\gamma^3}}$	$\sqrt{(\gamma-1)\frac{\gamma^2-1}{2\gamma^3}}$
387	r.	23rd from top	$\pi^0 \to \pi^+ + e^- + \bar{\nu}$	$\pi^- \to \pi^0 + e^- + \bar{\nu}$
388	r.	5th from bottom	$\pi^0 \to \pi^+ + e^- + \bar{\nu}$	$\pi^- \to \pi^0 + e^- + \bar{\nu}$
388	r.	7th from bottom	$\square_{(l)}^{(L)}$	$D_{(l)}^{(L)}$