"NECESSARY EXPERIMENT" IN PROTON-PROTON SCATTERING

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It is shown that the determination of n parameters of the proton-proton scattering matrix can be reduced with account of unitarity to n + 1 measurements in sufficiently accurate experiments on single, double, and triple scattering of protons.

THE difficulties in the experimental determination of the parameters of the proton-proton scattering matrix are due to the tremendous amount of labor required for the experiments on repeated scattering of particles, without which the scattering matrix cannot be determined. We therefore make an attempt in the present work to indicate a system of experiments, which would yield the scattering parameters for a fixed energy (below the pion production threshold) with a minimum amount of labor consumed in the experiments. We thus arrive at the "necessary experiment," the concept of which, as applied to other types of scattering, is formulated by the author in an earlier paper[1].

The scattering matrix for elastic proton-proton interaction, ignoring Coulomb parameters, has the form (see [2,3])

$$M = \tfrac{1}{2} a\,[1 + (\sigma_1 \mathbf{n})(\sigma_2 \mathbf{n})] + \tfrac{1}{2} b\,[1 - (\sigma_1 \mathbf{n})(\sigma_2 \mathbf{n})] + \tfrac{1}{2} c\,[(\sigma_1 \mathbf{m})(\sigma_2 \mathbf{m}) + (\sigma_1 \mathbf{l})(\sigma_2 \mathbf{l})] + \tfrac{1}{2} d\,[(\sigma_1 \mathbf{m})(\sigma_2 \mathbf{m}) - (\sigma_1 \mathbf{l})(\sigma_2 \mathbf{l})] + \tfrac{1}{2} e\,(\sigma_1 \mathbf{n} + \sigma_2 \mathbf{n}), \tag{1}$$

where [1)]

$$a = \frac{1}{k} \sum_{j=0,2,\ldots} \Big[\frac{2j-1}{j(j-1)}\, z\,(zP_j' - jP_j)\, E_{j-1}^{10} + \Big((2j+1)\, zP_j + \Big(2 - \frac{2j+1}{j} z^2\Big) P_j'\Big)(E_j^{1+}\cos^2\varepsilon_j + E_j^{1-}\sin^2\varepsilon_j) + \Big((2j+1)\, zP_j + \Big(-2 + \frac{2j+1}{j+1} z^2\Big) P_j'\Big)(E_j^{1+}\sin^2\varepsilon_j + E_j^{1-}\cos^2\varepsilon_j) + \frac{P_j'}{\sqrt{j(j+1)}}\sin 2\varepsilon_j\,(E_j^{1+} - E_j^{1-})\Big], \tag{2}$$

$$b = \frac{1}{k} \sum_{j=0,2,\ldots} \Big[(2j+1)\, P_j E_j^0 + (2j-1) \times \Big(\frac{j}{j-1} zP_j + \Big(\frac{1}{j} - \frac{z^2}{j-1}\Big) P_j'\Big) E_{j-1}^{10} + \frac{1}{j} P_j'(E_j^{1+}\cos^2\varepsilon_j + E_j^{1-}\sin^2\varepsilon_j) + \frac{1}{j+1} P_j'(E_j^{1+}\sin^2\varepsilon_j + E_j^{1-}\cos^2\varepsilon_j) - \frac{P_j'}{\sqrt{j(j+1)}}\sin 2\varepsilon_j\,(E_j^{1+} - E_j^{1-})\Big], \tag{3}$$

$$c = \frac{1}{k} \sum_{j=0,2,\ldots} \Big[-(2j+1)\, P_j E_j^0 + (2j-1)\Big(\frac{j}{j-1} zP_j + \Big(\frac{1}{j} - \frac{z^2}{j-1}\Big) P_j'\Big) E_{j-1}^{10} + \frac{1}{j} P_j'(E_j^{1+}\cos^2\varepsilon_j + E_j^{1-}\sin^2\varepsilon_j) + \frac{1}{j+1} P_j'(E_j^{1+}\sin^2\varepsilon_j + E_j^{1-}\cos^2\varepsilon_j) - \frac{P_j'}{\sqrt{j(j+1)}}\sin 2\varepsilon_j\,(E_j^{1+} - E_j^{1-})\Big], \tag{4}$$

$$d = \frac{1}{k} \sum_{j=0,2,\ldots} \Big[-\frac{2j-1}{j(j-1)}(zP_j' - jP_j) E_{j-1}^{10} + \Big(\frac{z}{j} P_j' - P_j\Big)(E_j^{1+}\cos^2\varepsilon_j + E_j^{1-}\sin^2\varepsilon_j) + \Big(P_j + \frac{z}{j+1} P_j'\Big)(E_j^{1+}\sin^2\varepsilon_j + E_j^{1-}\cos^2\varepsilon_j) - \frac{zP_j' - 2j(j+1)P_j}{\sqrt{j(j+1)}}\sin 2\varepsilon_j\,(E_j^{1+} - E_j^{1-})\Big], \tag{5}$$

$$e = \frac{i}{k}\sqrt{1 - z^2} \sum_{j=0,2,\ldots} \Big[-\frac{2j-1}{j(j-1)}(zP_j' - jP_j) E_{j-1}^{10} + \frac{2j+1}{j}(zP_j' - jP_j)(E_j^{1+}\cos^2\varepsilon_j + E_j^{1-}\sin^2\varepsilon_j) - \frac{2j+1}{j+1}(zP_j' + (j+1) P_j)(E_j^{1+}\sin^2\varepsilon_j + E_j^{1-}\cos^2\varepsilon_j)\Big], \tag{6}$$

with $E = -\tfrac{1}{2} i\,(e^{2i\delta} - 1)$, $E_{-1}^{10} = E_0^{1+} = \epsilon_0 = P_0' \equiv 0$ and $z = \cos\theta_{\text{c.m.s.}}$ (for the remaining symbols see [3]).

[1)]We indicate here the connection between the coefficients of formula (1) and the coefficients in other methods of writing down the proton-proton scattering matrix: $a = \alpha + \beta = N$, $b = \alpha - \beta = (1/2)(B + G - N)$, $c = \delta + \varepsilon = (1/2)(G - B - N)$, $d = \delta - \varepsilon = H$, $e = 2\gamma = 2C$. Greek symbols were used, in particular, in [3], whereas capital Latin letters were used in [4].

The quantities observed in the "complete experiment" [3] are

$$\sigma(z) = \tfrac{1}{2}(|a|^2 + |b|^2 + |c|^2 + |d|^2 + |e|^2)$$

$$= \sum_{m=0,1,\dots} \sigma^{(2m)} z^{2m}, \quad (7)$$

$$\sigma(z) P_n(z) = \operatorname{Re} ae^* = z\sqrt{1-z^2} \sum_{m=0,1,\dots} (\sigma P_n)^{(2m)} z^{2m}, \quad (8)$$

$$\sigma(z) D_{nn}(z) = \tfrac{1}{2}(|a|^2 - |b|^2 + |c|^2 - |d|^2 + |e|^2)$$

$$= \sum_{m=0,1,\dots} (\sigma D_{nn})^{(m)} z^m, \quad (9)$$

$$\sigma(z) K_{nn}(z) = \tfrac{1}{2}(|a|^2 + |b|^2 - |c|^2 - |d|^2 + |e|^2)$$

$$= \sum_{m=0,1,\dots} (\sigma D_{nn})^{(m)} (-z)^m, \quad (10)$$

$$\sigma(z) C_{nn}(z) = \tfrac{1}{2}(|a|^2 - |b|^2 - |c|^2 + |d|^2 + |e|^2)$$

$$= \sum_{m=0,1,\dots} (\sigma C_{nn})^{(2m)} z^{2m}, \quad (11)$$

where $0 \le z \le 1$. It turns out, however, that when parametrization (2)—(6), which is based on the unitarity condition, is employed there is no need for using all five quantities (7)—(11) over the entire scattering-angle interval from 0 to $\pi/2$. Indeed, for a finite-value determination of n parameters it is sufficient to have n properly chosen experimental points. On the other hand, inasmuch as finite parametrization is unavoidable in the analysis of any finite experiment, it is clear that any parametrization of the quantities (7)—(11) not based on unitarity contains more parameters than a parametrization which is unitary; consequently, at a fixed volume of labor, these parameters will be determined with less accuracy than the phase shifts and mixing parameters.

Let us assume that only states with total momentum not exceeding $J = j_{max}$ participate in the scattering. We arrange the corresponding parameters in the sequence $\dots, \delta_{j-1}^{10}, \delta_j^{1+}, \epsilon_j, \delta_j^0, \delta_j^{1-}, \dots,$ where δ_j^0 is the phase of the singlet state, ϵ_j is the mixing parameter, and the triplet states are taken in sequence $l^0 = j^0 = j-1$, $j = l^+ + 1$, and $j = l^- - 1$. More accurately, we assume that the first n parameters in the series of groups of five with increasing j differ from zero (the last group of five may be incomplete). In other words, we assume that as the energy increases the scattering parameters are turned on in the indicated sequence. This ordering of the state parameters should be called natural, since it minimizes the number of terms in the power expansions of the quantities (7)—(11) (for fixed n). The remaining 23 bilinear combinations of the parameters (2)—(6), which enter into the determinations of all possible observed quantities (see [3]), also have for the same ordering of the parameters the smoothest variation in the number of terms. Consequently, the natural order of the parameters leads to the maximum concentration of information regarding the parameters in the coefficients for the expansions of the observed quantities.

Using the formulas $P_j(z)$ and $P_j'(z)$ and expressions (2)—(6) in decreasing powers of z, we readily note that the coefficients of $z^{J+1}E_J^{1+}$, $z^{J+1}E_J^{1-}$, and $z^{J-1}E_{J-1}^{10}$ in a differ only in sign from the corresponding coefficients of $z^J E_J^{1+}$, $z^J E_J^{1-}$, $z^{J-2}E_{J-1}^{10}$, and $z^{J-4}E_{J-1}^{10}$ in $e/i\sqrt{1-z^2}$. We notice also that when $\epsilon_J = 0$ the vanishing coefficients are those of $z^{J+1}E_J^{1+}$ in a, those of $z^J E_J^{1+}$ in $e/i\sqrt{1-z^2}$, and those of $z^J E_J^{1+}$ in d. The signs of the coefficients of the even powers in b and c are opposite, while those of the odd powers coincide.

By virtue of these relations, many terms in the expansions of (7)—(11) in powers of c vanish identically with respect to the values of δ and ϵ, and the number of equations obtained by comparing the expansions of (7)—(11) in linearly independent polynomials of z with the analogous experimentally-obtained expansions of these quantities turn out to be different for different n (and for identical J or l_{max}). This difference is not taken into account in the table given in [4]. The results of the analysis are indicated in Tables I and II of the present work.

Table II shows that to obtain an equation with n parameters neither the cross sections σ, nor combinations of the cross sections σ and the polarization P_n, nor likewise combinations of the cross section σ and the correlation C_{nn} (for $n \ge 7$) are sufficient, all the more since the equations obtained from σ and σC_{nn} are in part linearly dependent. In addition, inasmuch as $K_{nn}(z) = D_{nn}(-z)$, all the equations obtained from K_{nn} coincide with the corresponding equations obtained from D_{nn}. A sufficient number of equations can be obtained by combining the cross sections σ for $0 \le z \le 1$ and the depolarization D_{nn} for $-1 \le z \le 1$ (three curves of five quantities of the "complete experiment"). To be sure, such a combination gives more relations than parameters. This means only that there is no need for measuring the entire D_{nn} curve. It is sufficient to measure it respectively in $\tfrac{3}{2}J - 2$, $\tfrac{3}{2}J - 1$, $\tfrac{3}{2}J - 1$, $\tfrac{3}{2}J$, and $\tfrac{3}{2}J + 1$ points and reduce the data simultaneously with J, J, J+1, J+1, and J+1 points

Table I. Degrees present in the expansions of the "complete experiment" curves

	$^5/_2\ J-2$	$^5/_2\ J-1$	$^5/_2\ J$	$^5/_2\ J+1$	$^5/_2\ J+2$
σ	$0.2, \ldots, 2J-4, 2J-2$	$0.2, \ldots, 2J-4, 2J-2$	$0.2, \ldots, 2J-2, 2J$	$0.2, \ldots, 2J-2, 2J$	$0.2, \ldots, 2J-2, 2J$
$\sigma P_n/z\sqrt{1-z^2}$	$0,2, \ldots, 2J-8, 2J-6$	$0.2, \ldots, 2J-6, 2J-4$	$0.2, \ldots, 2J-6, 2J-4$	$0.2, \ldots, 2J-6, 2J-4$	$0.2, \ldots, 2J-4, 2J-2$
σD_{nn}, σK_{nn}	$0.1, \ldots, 2J-4, 2J-3$	$0.1, \ldots, 2J-3, 2J-2$	$0.1, \ldots, 2J-2, 2J$	$0.1, \ldots, 2J-1, 2J$	$0.1, \ldots, 2J-1, 2J$
σC_{nn}	$0.2, \ldots, 2J-4, 2J-2$	$0.2, \ldots, 2J-4, 2J-2$	$0.2, \ldots, 2J-4, 2J-2$	$0.2, \ldots, 2J-2, 2J$	$0.2, \ldots, 2J-2, 2J$

Table II. Number of equations given by the experimental values of the "complete experiment"

	$^5/_2\ J-2$	$^5/_2\ J-1$	$^5/_2\ J$	$^5/_2\ J+1$	$^5/_2\ J+2$
$\sigma(z)$	J	J	$J+1$	$J+1$	$J+1$
$\sigma(z)P_n(z)/z\sqrt{1-z^2}$	$J-2$	$J-1$	$J-1$	$J-1$	J
$\sigma(z)D_{nn}(z)$, $\sigma(z)D_{nn}(-z)$	$2J-2$	$2J-2$	$2J$	$2J+1$	$2J+1$
$\sigma(z)C_{nn}(z)$	J	J	$J+1$	$J+1$	$J+1$
Total number of equations	$5J-4$	$5J-2$	$5J+1$	$5J+2$	$5J+3$
Constraints	$\sigma^{(2J-2)}=-(\sigma C_{nn})^{(2J-2)}$	$\sigma^{(2J-2)}+(\sigma C_{nn})^{(2J-2)}=2(\sigma D_{nn})^{(2J-2)}$	$\sigma^{(2J)}=(\sigma C_{nn})^{(2J)}$	—*	—
Linearly independent equations	$5J-5$	$5J-3$	$5J$	$5J+2$	$5J+3$

*When J = 0 we have $\sigma^{(0)} = -(\sigma C_{nn})^{(0)}$.

for σ. As already noted in [1], it is possible to use in this case the methods described in [5] and find such an ordering of the measurement points and such a distribution of the observation times as to maximize the information on the measured quantities, which is proportional to the logarithm of the determinant of their error matrix. We recall that for a specified volume of work measurements at n points give more information on n parameters than measurements at a larger number of points.

Of course, the determination of n parameters from n measurements is not unique. The multiplicity of this solution can be determined by using the result of an investigation of systems of similar equations, given in the appendix of [1], and recognizing that all scattering phases and mixing parameters tend to zero as $k \to 0$. Almost all the equations encountered in the analysis of proton-proton scattering have the same form as previously considered by us in [1], and a system of n such equations has consequently 2^n solutions (including complex ones).

However, in the third and fourth cases out of the five considered in the tables, equations are encountered with left halves containing a factor $|E_J^{1+}|\sin^2\epsilon_J$, with a coefficient that does not vanish when $k \to 0$. Such equations give a factor 4 in place of 2 in the multiplicity of the solution of the system (in the third case one such equation is contained in σ and σC_{nn} and another in D_{nn} and K_{nn}, while in the fourth case it is contained only in σD_{nn} and σK_{nn}). When each such equation replaces the remaining ones in the system, the multiplicity of the solution doubles. This is apparently why in the third and fourth case the total number of linearly independent equations, resulting from the curves of the "complete experiment," exceeds the number $2n-1$ by unity, whereas in the remaining cases these numbers coincide. Introduction of any number of known scattering parameters, which have the property that they vanish as $k \to 0$, does not change the multiplicity of the solution. The depth of the minima for different real solutions is the same, and for real combinations of the parameters close to the complex solution it is less.

At sufficiently high measurement accuracy the ambiguity of the analysis can be eliminated without a detailed measurement of the remaining P_n and C_{nn} curves. It is sufficient only to calculate these curves for all variants of the analysis and to find measurement places either on P_n or on C_{nn}, or on both curves (not more than one point on each) such as lift the ambiguity. We can use also the D_{nn} curve which is different for different solutions, and passes through the measured points. Such a method of eliminating the ambiguity of the analysis necessitates that the error corridors of the compared quantities not overlap at the points of additional measurements. If this condition cannot be satisfied, it is necessary to make more measurements on the curves P_n, C_{nn}, and D_{nn}.

The results of the work by Smorodinskiĭ and the author [6] show that in this way it is impossible to discard the two solutions which are obtained from each other by the helicity-inversion transformation. To discriminate between these two solutions experiments using an electric or magnetic field between the scatterers are necessary. If the accuracy is sufficient, the same result is obtained by taking into account the Coulomb interaction in (2)—(6), for this disturbs the equivalence of the real solutions from among all the 2^n or 2^{n+1} solutions of the system of equations; this is connected with the fact that when $k \to 0$ the Coulomb phases tend not to zero but to infinity.

The proposed measurement sequence for the determination of the scattering matrix is particularly effective in experiments aimed at determining the matrix elements with high accuracy and calling for a large amount of work. At the early preliminary stages of the research, trial measurements of all the observable quantities in a large number of rough points may be more effective.

By way of comparison let us point out that in scattering of particles with spin 0 and $\frac{1}{2}$ the natural ordering of the parameters has the form $\ldots, \delta^+, \delta^- \ldots$ ($l + \frac{1}{2}$ and $j = l - \frac{1}{2}$ respectively). The coefficients of $z^{J+1}\,{}^2E_J^-$ in a and in $b/i\sqrt{1-z^2}$ ($M = a + b\sigma\cdot\mathbf{n}$) differ in sign. Therefore, as was noted earlier [7], when $n = 2J + 1$ the quantities σ and $\sigma P/\sqrt{1-z^2}$ contain the highest powers z^{2J} and z^{2J-1} respectively, and the number of terms is $2J+1$ and $2J$; when $n = 2J$ these quantities include the higher powers z^{2J-1} and z^{2J-2} and lead to $2J$ and $2J-1$ equations. In both cases σ yields n equations and P yields $n-1$ equations, making a total of $2n-1$. In the scattering of spinless particles the scattering cross section yields $2n-1$ equations for n phases.

Comparison of the number of equations given by the "complete experiment" for particles with spin 0, 0 and 0, $\frac{1}{2}$ and identical particles with spin $\frac{1}{2}$, $\frac{1}{2}$, with the number of measurements necessary to determine the scattering matrix with account of its unitarity, shows that unitarity not only reduces by one-half (as was noted in [3]) the number of functions to be measured, but also imposes on the measured quantities certain compatibility conditions, which will be considered in another article.

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