MULTIPLE DISPERSION RELATIONS

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We consider a form of multiple dispersion relations that replace the Mandelstam representation for diagrams with anomalous mass relations and are appropriate for application to many-point functions. The integral Bergmann-Weil representation is employed for this purpose. The structure of the representation is explained on the particular example of the triangle diagram of the vertex part. A new singularity of this diagram is found, not obtainable by the Landau method.

1. Multiple dispersion relations have been first introduced by Mandelstam,[1] who postulated double dispersion relations for the scattering amplitudes. On the basis of these dispersion relations, referred to as the Mandelstam representation, and the unitarity property of the amplitudes a new general approach was developed, allowing one to write a system of simultaneous equations for the scattering amplitudes (four-point functions). Production amplitudes (five-point functions), six-point functions, and all remaining many-particle amplitudes should also be included in this system of equations.

It became clear, however, that for the production amplitudes and other many-point functions a representation of the Mandelstam type cannot be written. Moreover, even for the scattering amplitude the Mandelstam representation is not always valid.[2] The representation is also false for the vertex part if written in terms of the squares of the external momenta (external masses).

It is therefore of great interest whether a representation can be found, which would be a generalization of the Mandelstam representation and which would be applicable also to amplitudes with complicated analytic properties. In this work we clarify the question of what form of multiple dispersion relations for the amplitudes in perturbation theory would be applicable to the general case of many-point functions.

To this end we make use of the Bergmann-Weil representation, whose general features are given in Sec. 2 and in Appendix 1. As a first example we study further the simplest diagram for the vertex part—the triangle (Fig. 1). The interest in this diagram has to do with the fact that singularities of the triangle type must necessarily appear among the singularities of the production amplitude (five-point function).

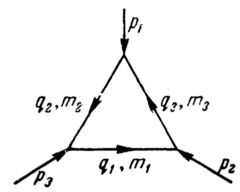

FIG. 1

In Sec. 3 we discuss the singularities of the amplitude corresponding to the triangle diagram in all three variables—squares of the external momenta. It is shown that for certain values of the variables a new singularity arises, which cannot be obtained by the Landau method.[3,4] By a special choice of the cut this singularity may be removed from the physical sheet. In Sec. 4 we write the Bergmann-Weil representation for this amplitude in two variables. It is shown that in the Bergmann-Weil representation in three variables the integration is only over a region where the variables are real.

2. Consider the amplitude for some process as a function of the invariants constructed out of the external momenta. For simplicity we choose two variables and keep the remaining ones fixed. Then the physical amplitude is the boundary value of an analytic function of two complex variables, $f(z_1, z_2)$, with $z_1 = x_1 - i\epsilon$ and $z_2 = x_2 - i\epsilon$. All the following results are easily generalized to the case of a larger number of variables. The function f is defined in the four-dimensional space of two complex variables and is analytic in some domain D of this space. The singularities of the function f—poles and branch points—represent two-dimensional surfaces. The branch point surfaces should be connected by cuts which form three-dimensional hypersurfaces. The boundary of the domain of analyticity D consists of both sides of these three-dimensional cuts.

Let the equations for these cuts be written in the form $Z_i(z_1, z_2) = r$, where r is a real parameter. There are two possibilities. The first possibility is that the Z_i depend on one variable (some on z_1, others on z_2). Then the domain D is the direct product of the domains D_1 in the variable z_1, and D_2 in the variable z_2. In that case Cauchy's theorem may be applied independently in each variable with the result that we obtain the Mandelstam representation for the function $f(z_1, z_2)$. The other possibility is that at least one of the Z_i depends on both variables, repeated application of Cauchy's theorem is not possible, and the Mandelstam representation is violated. To illustrate the above we consider the simplest diagram for the scattering amplitude—the "box" diagram. If the relations among the masses are not anomalous the singularities of this amplitude consist of the two-particle thresholds $s = (m_1 + m_3)^2$, $t = (m_2 + m_4)^2$, and of the Landau curve, which is singular only for real s and t.[2] The boundary of the domain of analyticity consists of the cuts $s = (m_1 + m_3)^2 + r$, $t = (m_2 + m_4)^2 + r'$, with the Landau curve (and the corresponding cut) lying inside the boundary. In this case, as is well known, the Mandelstam representation is valid.[5]

For an anomalous relation between the masses in the box, when the anomalous branch of the Landau curve becomes singular, to the boundary of the domain of analyticity is added a new cut, whose equation depends on both the variables s and t. The Mandelstam representation is violated in this case and in its place one may use the Weil representation given below.

In the general case when $Z_i(z_1, z_2)$ depends on two variables the following formula of Weil[6] is valid:

$$(2\pi i)^2 f(z_1, z_2) = \sum_{i<j} \iint \frac{dz_1' dz_2' (P_i Q_j - Q_i P_j) f(z_1', z_2')}{[Z_i(z_1', z_2') - Z_i(z_1, z_2)][Z_j(z_1', z_2') - Z_j(z_1, z_2)]} \qquad (1)$$

Here P_i and Q_i are analytic in D functions of z_1, z_2 and z_1', z_2' defined by the equation

$$Z_i(z_1', z_2') - Z_i(z_1, z_2) = (z_1' - z_1) P_i + (z_2' - z_2) Q_i.$$

The integration in Eq. (1) is over those values of z_1 and z_2 that satisfy the set of equations

$$Z_i(z_1, z_2) = r, \qquad Z_j(z_1, z_2) = r.$$

For each cut the integration is carried out over both sides in the positive direction. Therefore the integrand in Eq. (1) contains, generally speaking, the jump in the function f across the cuts, provided only that the functions Z_i themselves are not singular on the corresponding cuts.

Let us see now how we might find the functions entering the integrand of Eq. (1). In perturbation theory, when f represents an amplitude corresponding to some diagram, the functions Z_i may sometimes be found if the equation for the singularities of the function f is known. Let the equation for the singularities, which may be obtained for example by the Landau method,[4] be of the form $F(z_1, z_2) = 0$. Then the equation for the cut is obtained by setting $F(z_1, z_2) = r$, $0 \le r < \infty$.

Another method consists of variation of the internal masses which appear as parameters in the equation $F(z_1, z_2) = 0$. One may, for example, set $m_i = m_i^0 + f_i(r)$, $0 \le r < \infty$, with f_i a monotonically increasing function. Both methods for determining the cut are discussed in Secs. 3 and 4 on the example of the triangle diagram.

We remark that the Landau equation for the singularities of the diagram has several branches and on some of these branches the amplitude $f(z_1, z_2)$ is actually not singular (the corresponding Feynman parameters α_i are negative). In that case the equation for the cut obtained by one of the above indicated methods also has a superfluous branch, that does not correspond to a true singularity. This gives rise to a fictitious difficulty. In the usual method for the determining of the singularities and jumps of the integral of Eq. (1) we find that the jumps arise everywhere when $Z_i(z_1, z_2) = r$ and $Z_j(z_1, z_2) = r'$, regardless of which branch of the equation for the cut is used for the integration in Eq. (1). In fact, when the structure of the functions P_i and Q_i is taken into account it is seen that the integral (1) taken over the true branches of the cuts Z_i has the required analytic properties. This we show in Appendix 1.

The functions Z_i may themselves have branch points, but in that case no difficulties arise with the application of the Weil formula. For simplicity we restrict ourselves to regular functions Z_i, as is usually the case if the Z_i are obtained from perturbation theory. Then into the integrand in Eq. (1) enter the jumps of the function $f(z_1, z_2)$ across the cuts; these may be determined with the help of the unitarity condition or its analytic continuation.

In a number of cases instead of the Weil representation a multiple Cauchy integral may be written, however only over the boundary of that part of the space which contains no anomalous cuts [i.e., cuts $Z_i(z_1, z_2)$ that depend on both variables]. For example, for the triangle amplitude (Fig. 1) for real p_3^2 all cuts lie either on the real axes in the plane of p_1^2 and p_2^2, or in the region where $\mathrm{Im}\, p_1^2\, \mathrm{Im}\, p_2^2 < 0$.

Therefore we may write using the Cauchy theorem

$$F(p_1^2, p_2^2, p_3^2) = \frac{1}{(2\pi i)^2}\int_{-\infty}^{\infty}\frac{da}{a-p_2^2-i\varepsilon}\int_{-\infty}^{\infty}\frac{db}{b-p_1^2-i\varepsilon}F(b, a, p^2{}_3).$$

This integral determines the function F for p_1^2 and p_2^2 lying in the upper half planes. The drawback of this representation has to do with the fact that the integrand can no longer be determined with the help of the unitarity condition.

3. Let us consider the simplest diagram for the vertex function—the "triangle" (Fig. 1). Its contribution is given by the expression

$$F(p_1^2, p_2^2, p_3^2) = \frac{1}{2\pi}\int\frac{d\alpha_1 d\alpha_2 d\alpha_3 \delta(1-\alpha_1-\alpha_2-\alpha_3)}{\alpha_1\alpha_2 p_3^2+\alpha_2\alpha_3 p_1^2+\alpha_3\alpha_1 p_2^2-\sigma}, \quad (2)$$

where

$$\sigma = \sum_{i=1}^{3}\alpha_i m_i^2,$$

and F is related to the vertex function Γ by

$$\Gamma(p_1^2, p_2^2, p_3^2) = g_1 g_2 g_3 F(p_1^2, p_2^2, p_3^2)/8\pi.$$

It is convenient to replace the p_i^2 by the "cosines" y_{ik} and to go over to the parameters (see [3])

$$\beta_i = \alpha_i m_i \Big/ \sum_k \alpha_k m_k;$$

$$y_{12} = (m_1^2+m_2^2-p_3^2)/2m_1m_2 \equiv z,$$

$$y_{13} = (m_1^2+m_3^2-p_2^2)/2m_1m_3 \equiv y,$$

$$y_{23} = (m_2^2+m_3^2-p_1^2)/2m_2m_3 \equiv x, \quad y_{ii} = 1.$$

Then

$$F(x, y, z) = \frac{1}{2\pi}\int\frac{\Pi_i d\beta_i \delta(1-\beta_1-\beta_2-\beta_3)}{[m_1m_2\beta_3+m_1m_3\beta_2+m_2m_3\beta_1][-\beta_i\beta_k y_{ik}]}. \quad (3)$$

After evaluation the integral (3) becomes a set of Spence functions. The derivative of the integral (2) with respect to the masses $\Sigma_i \partial F/\partial m_i^2$ has been evaluated in terms of elementary functions by Kallen and Wightman; [7] the simple expression for this function in terms of the variables x, y, and z is given in Appendix 2.

In the case $y + z > 0$ [3] the function F satisfies a simple dispersion relation in x:

$$F(x, y, z) = \frac{1}{\pi}\int_{-\infty}^{-1}\frac{A(x', y, z)\,dx'}{x'-x}. \quad (4)$$

A(x, y, z) may be obtained either from the unitarity condition for the diagram of Fig. 1, or directly as the imaginary part of the integral (3).

The expression for A(x, y, z) in the case $m_1 = m_2 = m_3$ has the form

$$A(x, y, z) = \frac{1}{2\sqrt{\Lambda}}\ln\frac{(1-x)(y+z-x-1)+\sqrt{(x^2-1)\Lambda}}{(1-x)(y+z-x-1)-\sqrt{(x^2-1)\Lambda}},$$

$$\Lambda = (1-x)^2+(1-y)^2+(1-z)^2-2(1-x)(1-y) - 2(1-x)(1-z)-2(1-y)(1-z). \quad (5)$$

The expression for A in the case of unequal masses is given in Appendix 3.

Let us consider the function F(x, y, z) as a function of the three complex variables x, y, z in their six-dimensional space. We see from Eqs. (2) and (3) that there exists a region of sufficiently large positive values of x, y, z, where F has no singularities. For all remaining x, y, z the function F is obtained by means of analytic continuation. The physical values of F correspond, as can be seen from Eq. (2), to the limit $F(\bar{x}-i\epsilon, \bar{y}-i\epsilon, \bar{z}-i\epsilon)$.

The nearest singularities of F(x, y, z) were obtained in [3]. They represent four four-dimensional surfaces:

$$x = -1, \quad y = -1, \quad z = -1;$$
$$x^2+y^2+z^2-2xyz-1 = 0.$$

The structure of the last surface has been studied in detail by Bonnevay et al. [8] On the physical sheet, defined above, only part of this surface is singular. The singular points of the hypersurface

$$k(x, y, z) \equiv x^2+y^2+z^2-2xyz-1 = 0 \quad (6)$$

are conveniently characterized as follows.

We introduce the variables $x = \cos\theta_x$, $y = \cos\theta_y$, $z = \cos\theta_z$. The case of real θ_i was considered in [3] and it was shown that the singularities of F are determined by the equation $\theta_x + \theta_y + \theta_z = 2\pi$. We define the branch of the inverse cosine in such a way that values of θ_x, θ_y, θ_z lie in the complex strips shown in Fig. 2: $\theta_i = \bar{\theta}_i + i\vartheta_i$; $0 \le \bar{\theta}_i \le \pi$; $-\infty < \vartheta_i < \infty$. At that x, y, z are defined in the planes with the usual cuts. It is easy to verify, by making use of the analysis given in [8], that for arbitrary complex x, y, z the equation determining the singularities is given as before by:[1)]

$$\theta_x + \theta_y + \theta_z = 2\pi. \quad (7)$$

Let us choose some z: $-1 < z < 1$, $0 \le \theta_z \le \pi$. Since Re θ_x, Re $\theta_y \le \pi$, the singularities may occur only if Re $\theta_x \ge \pi - \theta_z$ and simultaneously Re $\theta_y \ge \pi - \theta_z$. The dashed lines in Fig. 2 correspond to Re $\theta_x = \pi - \theta_z$ and Re $\theta_y = \pi - \theta_z$. In the complex planes of x and y the dashed lines of Fig. 2 cor-

[1)]This is also easily seen from the explicit form of the mass derivative of F given in Appendix 2.

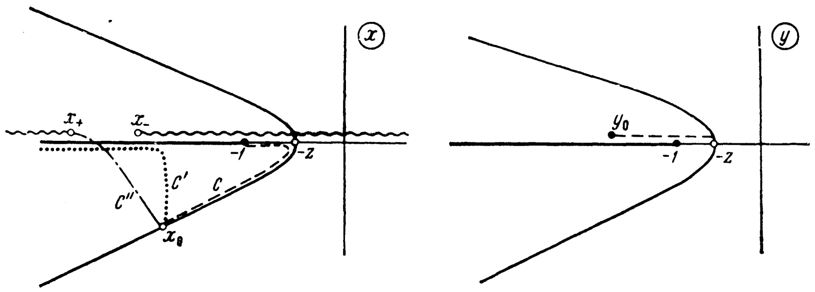

FIG. 3

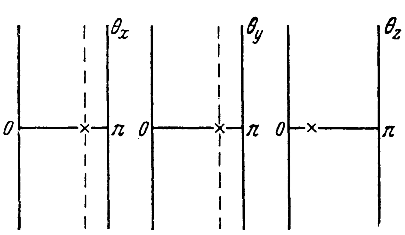

FIG. 2

respond to the hyperbolae shown in Fig. 3. The singularity occurs only for x and y lying to the left of the hyperbolae. In addition it is seen from Eq. (7) that $\operatorname{Im}\theta_x = -\operatorname{Im}\theta_y$ and, consequently, $\operatorname{Im}\theta_x \operatorname{Im}\theta_y < 0$ at the singular point.

For $y = y_0$ lying on the upper side of the cut (Fig. 3), Eq. (7) gives an x_0 lying on the lower branch of the hyperbola. When the point y_0 moves along the path shown in Fig. 3 by a dashed line, the corresponding x_0 determining the singularity moves along the path C shown in the left part of Fig. 3.

As is known,[9] the dispersion relation in x, Eq. (4), must be modified when the anomalous singularity given by Eq. (7) is taken into account. If the cut due to the anomalous singularity x_0 is directed along the path C of Fig. 3, as is usual to do, then one obtains

$$F(x, y_0, z) = \frac{1}{\pi}\int_{-\infty}^{-1} \frac{A(x', y_0, z)\,dx'}{x' - x} + \frac{1}{\pi}\int_{\uparrow C} \frac{dx'}{x' - x}\,\frac{2\pi i}{2\sqrt{\Lambda(x', y_0, z)}}. \tag{8}$$

The indeterminacy connected with the choice of the argument of Λ appearing in Eq. (8) is only apparent; it is compensated by the sign of the change in the phase of the logarithm in $A(x', y_0, z)$ in going around the point x_0. We found it convenient to choose the argument of Λ as shown in Fig. 3. The points x_+ and x_- are the zeros of the function $\Lambda(x, y, z)$:

$$x_+ = 1 - (\sqrt{1-y} + \sqrt{1-z})^2,$$

$$x_- = 1 - (\sqrt{1-y} - \sqrt{1-z})^2. \tag{9}$$

If we set, in accordance with Eq. (5), $A(x, y, z) = \ln W/2\sqrt{\Lambda}$, then arg W equals zero for $x < x_+$. However, in going from x_+ to x_- along the path of integration in Eq. (8) the arg W changes from zero to -2π. Therefore near $x' = x_-$ the function $A(x', y_0, z)$ becomes infinite:

$$A(x', y_0, z) \sim -\pi i/\sqrt{(x' - x_+)(x' - x_-)}, \quad x' \sim x_-.$$

The function $F(x, y_0, z)$ also becomes singular at $x = x_-$. This singularity is not one of the singularities obtained by the Landau method and written out above.[2)] Let us clarify the nature of this singularity. Since for $y_0 = \bar{y}_0 \pm i\epsilon$ we have $x_- = \bar{x}_- \pm i\epsilon'$ the contour of integration in Eq. (8) may become pinched between the singularities of $A(x', y, z)$ and $1/(x' - x)$ when $y = y_0 \pm i\epsilon$ and $x = x_- \mp i\epsilon$. Therefore the singularity of the function $F(x, y, z)$ arises only in the limit $y = y_0 \pm i\epsilon$ and $x = x_- \mp i\epsilon$. For physical values of x and y, when $x = \bar{x} - i\epsilon$, $y = \bar{y} - i\epsilon$, $F(x, y, z)$ has no singularity.

As can be seen from Fig. 3 the smallest value of y starting with which a singularity develops at $x = x_-$ is determined by the equation $x_-(y) = -z$, or $y = -1 - 2\sqrt{1 - z^2}$. Besides, for complex y the values x_+ and x_- move away from the real axis and therefore the singularity at $x = x_-$, or equivalently $\Lambda(x, y, z) = 0$, develops only for real x and y (when z is real).

This singularity may be removed from the sheet under consideration by appropriately choosing the cut from the anomalous singularity x_0 in the x plane. Indeed, if one chooses the contour C″ in Fig. 3 then the change in arg W in going from x_+ to x_- around x_0 will be equal to zero, $\arg W(x = x_+) = \arg W(x = x_-) = 0$, and the function A has no singularities anywhere along the path of integration.

When z moves off the real axis, $z = z + i\delta$, the point x_- moves downwards and lies in the lower half plane. In that case the contour of integration

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2)The possibility of such a singularity appearing in some one sheet of the function F has been mentioned by Cutkosky.[10]

in Eq. (8) should be changed so as to pass the point $x' = x_-$ from below, which gives rise to the appearance of the additional integral of the anomalous kind.

The physical significance of the singularity $x = x_-$ is easily understood when x, y, z are expressed in terms of the $p_i^2 = M_i^2$. The condition $x = x_-$ turns out to be equivalent to $M_2 = M_1 + M_3$, i.e., the threshold for the decay of the particle M_2 into M_1 and M_3.

4. Let us first set z real $(-1 < z < 1)$ and write the Weil representation for the function $F(x, y, z)$ in the variables x and y. To that end we must determine the form of all the hypersurfaces—cuts of the function F.

The first two hypersurfaces are the usual cuts starting from the threshold values $x = -1$, $y = -1$:

$$x = -1 - r, \quad 0 \leqslant r < \infty; \qquad y = -1 - r', \quad 0 \leqslant r' < \infty. \tag{10}$$

As was remarked in Sec. 2, the hypersurfaces connected with the singularity (6) may be constructed either as

$$k(x, y, z) = r, \tag{11}$$

or as

$$k[x(r), y(r), z(r)] = 0. \tag{12}$$

In the latter case the cut goes through points corresponding to singularities of the function F for values of the internal masses varying from the given ones up to infinitely large ones.

For equal internal masses $x = (2m^2 - p_1^2)/2m^2$ and if we set $m^2 = m_0^2 + r$; $0 \le r < \infty$, then we obtain

$$x(r) = (x + r')/(1 + r), \quad x = (2m_0^2 - p_1^2)/2m_0^2, \quad 0 \leqslant r < \infty.$$

After substitution of these expressions for $x(r)$, $y(r)$, and $z(r)$ we obtain the following equation for the cut:

$$k(x, y, z) + r\Lambda(x, y, z) = 0 \tag{13}$$

[Λ is defined in Eq. (5)]. If the internal masses are not equal we arrive by the same method at the hypersurface of the form

$$K(p_1^2, p_2^2, p_3^2) + r\Lambda(p_1^2, p_2^2, p_3^2) = 0.$$

For fixed y and z the cut, Eq. (11), in the x plane is shown in Fig. 3 as the dotted line C′. For the same y and z the cut, Eq. (13), connects the singular point x_0 with the point x_+, lying on the cut, along the path C″ (see Fig. 3).

As we know from the preceding section, with the cut introduced in this way there are no singularities at the points $x = x_-$ and $x = x_+$; consequently, in view of the symmetry of the expression (13), the surface $\Lambda(x, y, z) = 0$ is not singular. The superiority of Eq. (13) consists in the fact that on this hypersurface lie the singularities of a whole class of diagrams, obtainable from the triangle of Fig. 1 by simultaneous replacement of all internal lines by double lines, triple lines, etc.

We may now make use of the Weil formula, Eq. (1). For the hypersurfaces

$$\begin{aligned} &S_1: \ x = -1 - r, \ P_1 = 1, \quad Q_1 = 0; \\ &S_2: \ y = -1 - r', \ P_2 = 0, \quad Q_2 = 1; \\ &S_a: \ k(x', y', z') + r'\Lambda(x', y', z') = 0, \end{aligned} \tag{13'}$$

with P_3 and Q_3 determined by the equations

$$\begin{aligned} P_3 &= (1 + r')(x' + x) - (z + r')(y' + y) + 2r'(1 - z), \\ Q_3 &= (1 + r')(y' + y) - (z + r')(x' + x) + 2r'(1 - z), \end{aligned} \tag{14}$$

where r′ should be expressed in terms of x′, y′, z′ with the help of Eq. (13′).

The Weil formula now takes the form

$$\begin{aligned} (2\pi i)^2 F(x, y, z) &= \int_{S_1} \frac{dx'}{x' - x} \int_{S_2} \frac{dy'}{y' - y} F(x', y', z) \\ &+ \int_{S_1} \frac{dx'}{x' - x} \int_{S_a} \frac{dy' Q_3(x', y'; x, y) F(x', y', z)}{k(x, y, z) + r'\Lambda(x, y, z)} \\ &+ \int_{S_2} \frac{dy'}{y' - y} \int_{S_a} \frac{dx' P_3(x', y'; x, y) F(x', y', z)}{k(x, y, z) + r'\Lambda(x, y, z)}. \end{aligned} \tag{15}$$

The first integral in Eq. (15) involves the double jump of the function F across the cuts S_1 and S_2. It is seen from expression (8) that this jump vanishes for our choice of the anomalous cut, Eq. (13). The second integral in Eq. (15) involves the jump of $F(x, y, z)$ across the anomalous cut, which is easily obtained from Eq. (8). In summary the Bergmann-Weil representation takes the form

$$\begin{aligned} F(x, y, z) &= \frac{1}{2\pi i} \int_{S_1} \frac{dx'}{x' - x} \int_{S_a} \frac{dy' Q_3(x', y'; x, y)}{k(x, y, z) + r'\Lambda(x, y, z)} \frac{1}{2\sqrt{\Lambda(x', y', z)}} \\ &+ \frac{1}{2\pi i} \int_{S_2} \frac{dy'}{y' - y} \int_{S_a} \frac{dx' P_3(x', y'; x, y)}{k(x, y, z) + r'\Lambda(x, y, z)} \frac{1}{2\sqrt{\Lambda(x', y', z)}}. \end{aligned} \tag{16}$$

The integration here is carried out in such a way that, for example, when y′ takes on values along the upper side of the cut S_2 the integration over x′ proceeds from the point x_0 to x_+ along the path C″, x_0 moving along the lower branch of the hyperbola as y′ increases. For y′ on the lower side of the cut S_2 the point x_0 moves along the upper branch of the hyperbola (Fig. 3).

Let us see now what should be the form of the Bergmann-Weil representation in the three variables x, y, z. In that case we must add to the hypersurfaces S_1 and S_2 the hypersurface S_3: $z = -1 - r''$, $0 \le r'' < \infty$, and by S_a understand the totality of hypersurfaces, that pass into each other:

$$S_a = \begin{cases} k(x, y, z) + r\Lambda(x, y, z) = 0, & 0 \leqslant r < \infty \\ \Lambda(x, y, z) = r', & 0 \leqslant r' < \infty \end{cases} .$$

The trace of this last hypersurface in the x plane is shown in Fig. 3 as a wavy line.

We shall not write out here the Bergmann-Weil representation in three variables. We only note that the integration proceeds over the intersections $S_1 \cap S_2 \cap S_3$, $S_1 \cap S_2 \cap S_a$, $S_1 \cap S_3 \cap S_a$, $S_2 \cap S_3 \cap S_a$. From the explicit form of the hypersurfaces S_i it can be seen that on all these intersections x, y, and z take on real values lying between minus and plus infinity, i.e., the region of integration is purely real.

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APPENDIX 1

ANALYTIC PROPERTIES OF THE WEIL INTEGRAL

We consider for simplicity the case when the function $f(z_1, z_2)$ has three cuts:

$$\begin{aligned} &S_1\colon z_1 = -1 - r, \quad 0 \leqslant r < \infty, \\ &S_2\colon z_2 = -1 - r', \quad 0 \leqslant r' < \infty, \\ &S_3\colon Z(z_1, z_2) = r'', \quad 0 \leqslant r'' < \infty. \end{aligned}$$

For every z_2 the hypersurface S_3 has two branches in the z_1 plane, of which one $S_3^{(1)}$ corresponds to the singularity of the function $f(z_1, z_2)$, whereas on the other branch $S_3^{(2)}$ the function $f(z_1, z_2)$ has neither a jump nor a singularity.

The Weil formula has the following form:

$$\begin{aligned} (2\pi i)^2 f(z_1, z_2) = &\int_{S_1} \frac{dz_1'}{z_1' - z_1} \int_{S_2} \frac{dz_2'}{z_2' - z_2} f(z_1', z_2') \\ &+ \int_{S_1} \frac{dz_1'}{z_1' - z_1} \int_{S_3^{(1)}} \frac{dz_2' Q_3 f(z_1', z_2')}{Z(z_1', z_2') - Z(z_1, z_2)} \\ &+ \int_{S_2} \frac{dz_2'}{z_2' - z_2} \int_{S_3^{(1)}} \frac{dz_1' P_3 f(z_1', z_2')}{Z(z_1', z_2') - Z(z_1, z_2)} . \end{aligned} \tag{A.1}$$

At first sight it seems that, as usual, a singularity of the function $f(z_1, z_2)$ due to the second integral arises when both denominators vanish at the boundary of the region of integration, for example for $Z(z_1, z_2) = 0$, i.e., for z_2 lying on $S_3^{(1)}$ as well as for z_2 lying on $S_3^{(2)}$, in spite of the fact that the integration over z_2' proceeds over the region $S_3^{(1)}$. However, from the equality $Z(z_1', z_2') - Z(z_1, z_2) = (z_1' - z_1)P_3 + (z_2' - z_2)Q_3$ it is obvious that for $z_1' = z_1$ there appears under the second integral sign the usual term of the form $f(z_1', z_2')/(z_2' - z_2)$, which can produce singularities only when $z_2 = z_2' \in S_3^{(1)}$.

Let us now choose $z_1 = z_1^0$ and z_2 such that they lie on the hypersurfaces S_i, and let us in the z_1 plane draw a circle of a small radius C_ρ centered at z_1^0. The entire analyticity domain D may be broken up into two parts $D = D_1 + D_2$; the domain D_1 consists of all z_2 and z_1 inside C_ρ, the domain D_2 consists of all z_2 and z_1 outside C_ρ.

In view of the additivity of the Weil formula the Weil integral over the boundary of the domain D equals the sum of the integrals over the boundaries of the domains D_1 and D_2. The second integral vanishes since the point z_1^0 lies outside the domain D_2. In this way our function $f(z_1^0, z_2)$ reduces to the sum of two integrals [it is easy to verify that the third integral in Eq. (A.1) vanishes]:

$$\begin{aligned} (2\pi i)^2 f(z_1^0, z_2) = &\int_{C_\rho} \frac{dz_1'}{z_1' - z_1^0} \int_{S_2} \frac{dz_2'}{z_2' - z_2} f(z_1', z_2') \\ &+ \int_{C_\rho} \frac{dz_1'}{z_1' - z_1} \int_{S_3^{(1)}} \frac{dz_2' Q_3 f(z_1', z_2)}{Z' - Z} . \end{aligned}$$

Letting the radius ρ go to zero we obtain a one-dimensional dispersion relation in z_2:

$$2\pi i f(z_1^0, z_2) = \int_{S_2} \frac{dz_2'}{z_2' - z_2} f(z_1^0, z_2') + \int_{S_3^{(1)}(z_1^0)} \frac{dz_2'}{z_2' - z_2} f(z_1^0, z_2'),$$

which has singularities for z_2 only from $S_3^{(1)}$.

In a similar way the Weil representation in n variables may be reduced to a representation in $(n-1)$ variables.

APPENDIX 2

THE MASS DERIVATIVE OF THE FUNCTION F

The function

$$\sum_{i=1}^{3} \frac{\partial}{\partial m_i^2} F(p_1^2, p_2^2, p_3^2)$$

has been calculated by Kallen and Wightman in [7] and is given in Eq. (A.46). In terms of the variables x, y, z the function

$$\Phi(x, y, z) \equiv \sum_i \frac{\partial}{\partial m_i^2} F(p_1^2)$$

has a simple form:

$$\Phi(x, y, z) = \frac{\text{const}}{k(x, y, z)}\left\{\frac{(1-x)(1+z-x-1)}{\sqrt{x^2-1}}\theta(x)\right.$$
$$\left. + \text{cyclic permutation}\right\}.$$

When $k(x, y, z) = 0$ the expression in the braces is proportional to $\theta(x) + \theta(y) + \theta(z)$, and therefore a singularity develops when that sum equals 2π and there is no singularity when that sum vanishes.

APPENDIX 3

ABSORPTIVE PART OF THE FUNCTION F FOR UNEQUAL MASSES

The absorptive part of the function $F(p_1^2, p_2^2, p_3^2)$ may be calculated as the jump over p_1^2 of the integral (2) or from the unitarity relation:

$$\mathrm{Im}_1 F(p_1^2, p_2^2, p_3^2) = -\frac{1}{2\sqrt{\Lambda(p_1^2, p_2^2, p_3^2)}} \ln \frac{I_+}{I_-}.$$

Here

$$\Lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc,$$
$$I_\pm = p_1^4 + p_1^2(2m_1^2 - m_2^2 - m_3^2 - p_2^2 - p_3^2)$$
$$+ (m_3^2 - m_2^2)(p_2^2 - p_3^2)$$
$$\pm\sqrt{\Lambda(p_1^2, m_2^2, m_3^2)\,\Lambda(p_1^2, p_2^2, p_3^2)}.$$

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387