PARALLEL TRANSFER OF SPIN IN LOBACHEVSKIĬ SPACE

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Reactions and decays can be described in such a way that in going from one system to another the spin is transformed not by means of a Lorentz transformation, but by parallel transfer on the surface of a hyperboloid (imaginary sphere) in velocity space. The spin then always remains a three-dimensional vector and the entire theory of spin effects (composition of spins and change of spins in scattering) reduces to the ordinary nonrelativistic theory. The only correction is that which comes from the relativistic spin-orbit coupling (Thomas precession), and it can be calculated from simple geometrical considerations.

1. INTRODUCTION

THERE are many papers (see [1-5]) devoted to the exposition of relativistic transformations of spin. In these papers the spin, or more exactly its average value, the polarization of the particle, is usually regarded as a pseudovector with four components which satisfies the condition that it is orthogonal to the four-velocity of the particle itself. In the rest system of the particle the spin is a three-dimensional axial vector, the fourth component being identically zero in this coordinate system. In any other reference system the components of the spin are given by a Lorentz transformation; the expressions for the components are

$$\omega \to \frac{(\omega \mathbf{v})\,\mathbf{v}}{1+\gamma} + \omega, \qquad \omega_0 \to \omega \mathbf{v}. \tag{1.1}$$

Here $\gamma = \epsilon/m$; $\mathbf{v} = \mathbf{p}/m$; ϵ, $\mathbf{p}$ is the four-momentum of the particle. The formulas (1.1) are of course also valid for the transformations of the classical angular-velocity vector.

The four-vector so obtained is obviously still orthogonal to the four-velocity, so that as before it essentially has only three (and indeed three spacelike) independent components. Although this sort of definition of spin as a four-vector is indeed correct, it is unsatisfactory from the physical point of view. When we specify the three quantities in a chosen coordinate system (in our case, in the rest system), then to determine the spin in an arbitrary system we must set up a convention about its transformation properties. When we assume that ω is a pseudovector we arrive at Eq. (1.1).

One can regard the spin as an antisymmetric tensor, only three components of which remain nonvanishing in the rest system, and then one arrives at a different definition of the spin. Finally, we can regard the spin components in the rest system as components of a higher-rank tensor, and in this way also arrive at a logically closed scheme. At the same time it is obvious that all of these definitions must be physically equivalent. The transformation (1.1) depends on the velocity of the particle and includes the orbital motion in the definition of the spin; with a different definition of the spin the velocity will appear in the formulas in a different way, and this will lead to different transformation properties.

Furthermore, the definition of the spin in the rest system makes necessary a separate treatment of particles with zero mass, and although this does not lead to new results, it goes outside the framework of a unified system. Finally, the development of the theory of spin on the basis of Eq. (1.1) leads to cumbersome and far from obvious rules for the composition of the spins of several particles (the determination of these rules makes up the main contents of most of the published papers).

It can also be added that the problem of distinguishing between spin and orbit (for the classification of states) has somehow not been as trivial in relativistic theory as it was in the nonrelativistic theory. At the same time it is clear that the change to a uniformly moving coordinate system [1)] cannot introduce anything physically new in the properties of the spin, beyond simple kinematical corrections, and therefore it is natural to try to find a different scheme for describing spin which

[1)]It is well known that for motion in a curved path there is a kinematical coupling between spin and orbit (Thomas procession), and we shall return to this matter later.

would be as close as possible to the ordinary nonrelativistic theory.

The idea of such a description is based on the fact that the spin actually transforms not by a representation of the full Lorentz group, but by a representation of a smaller group which is isomorphic to the three-dimensional rotation group. It follows at once from this that in any coordinate system the spin vector can be represented by a three-vector. For this purpose one needs only to make a transformation such that the fourth component ω_0 becomes zero.

This can be done very simply if we use the beautiful apparatus of Lobachevskian geometry in velocity space.[2)] The spin transformation we need is nothing but parallel transfer of the spin vector along a geodesic line on the surface of the velocity hyperboloid (with pseudoeuclidean metric).

After the transformation has been reduced to a parallel transfer, the theory of relativistic spin is quite simple. In fact, if spins can be parallel-transferred to any system (and in this it makes no difference whether or not the mass is zero), they can be compounded by the rules of ordinary nonrelativistic theory. Furthermore the separation of spin from orbit occurs in a most natural way.

If we carry out a parallel transfer along a closed path, then by the properties of Lobachevskiĭ space we return to the original system with a rotated coordinate system (rotated in the direction opposite to that of the path). It is not hard to verify that in this case the result agrees with that of the product of successive Lorentz transformations, and therefore in calculations of physical effects the results obtained by parallel transfer and by means of Lorentz transformations are the same. The conclusion is that for quantities which transform according to the small group (polarization tensors, multipole moments of a system of charges, form-factors [9]) one should make the change from one system to another by using parallel transfer, and not the usual Lorentz transformation.

We note that in a recently published paper Wick [10] has developed similar ideas, using a classification according to helicities for the construction of three-particle wave functions. The choice of the axis of quantization along the relative velocity of two systems is the simplest case of parallel transfer of a vector which makes zero angle with the geodesic. In this work the choice of convenient parameters at once leads to a great simplification of the problem.

2)The use of the geometry of Lobachevskiĭ for the description of kinematics is due to Sommerfeld and Klein.[6,7] Recently these ideas have been revived in papers by Chernikov.[8]

2. PARALLEL TRANSFER OF A VECTOR AND A SPINOR

We obtain all of the essential results by considering the transformation of a spinor and a vector. Let us begin with the spinor. We consider a spinor which describes a free particle moving with the velocity v relative to some system. (Here and hereafter v will denote the four-velocity, so that $v = p/m$ and $v_0^2 - \mathbf{v}^2 = 1$. 3)) This spinor (for a state with positive energy) is of the form

$$u = \begin{pmatrix} \varphi \\ \frac{\sigma \mathbf{v}}{1+\gamma}\varphi \end{pmatrix}, \tag{2.1}$$

if φ denotes a Pauli two-component spinor and γ is the Lorentz factor.

The transformation which reduces the spinor to the form

$$u_0 = \begin{pmatrix} \varphi \\ 0 \end{pmatrix}, \tag{2.2}$$

is the inverse of the Lorentz transformation which takes the spinor into the rest system R of the particle:

$$L^{-1} = \exp\left\{-\frac{i}{2|\mathbf{v}|}\gamma_4\boldsymbol{\gamma}\,\mathbf{v}\,\mathrm{Arth}\,|\mathbf{v}|\,\gamma\right\} = \left(\frac{\gamma+1}{2}\right)^{1/2}\left(1-\frac{\alpha\mathbf{v}}{1+\gamma}\right), \tag{2.3*}$$

$$L^{-1}u = u_0. \tag{2.4}$$

We recall that L is nonunitary and that

$$L^{-1} = \beta L^{+}\beta. \tag{2.5}$$

There is another transformation that has the same property—the Foldy-Wouthuysen transformation [11]:

$$F = \exp\left\{-\frac{i}{2|\mathbf{v}|}\boldsymbol{\gamma}\mathbf{v}\,\mathrm{arctg}\,|\mathbf{v}|\right\} = \left(\frac{\gamma+1}{2\gamma}\right)^{1/2}\left(1+\frac{\beta\alpha\mathbf{v}}{1+\gamma}\right), \tag{2.6†}$$

$$Fu = u_0 \tag{2.7}$$

(up to normalization). This transformation is unitary:

$$FF^{+} = 1. \tag{2.8}$$

It leaves the charge density $\psi^{+}\varphi$ invariant, whereas L leaves the scalar $\bar{\psi}\varphi$ invariant. The product of the two transformations

$$FL^{-1} = \beta\gamma^{1/2} \tag{2.9}$$

3)It is convenient to use the four-velocity, because then the formulas do not involve the mass of the particle.

*Arth = $\tanh^{-1}$.

†arctg = $\tan^{-1}$.

changes only the normalization of the spinor (for a state with positive energy).

To understand the physical meaning of these formulas it is convenient to use Lobachevskiĭ space. Let us consider a four-dimensional space, and let a point in it represent a four-velocity. Owing to the condition

$$v^2 = 1 \tag{2.10}$$

all of the points will lie on one of the sheets of a hyperboloid of two sheets. As is well known, the group of motions of the three-dimensional surface of such a hyperboloid is isomorphic to the Lorentz group. The geometry on this surface corresponds to the geometry of velocity vectors in special relativity theory.

We note two formulas. The scalar product of two velocity vectors—the length a_{12} of the geodesic (hyperbola) between two points on the surface—is given by

$$v_1 v_2 = \operatorname{ch} a_{12} = \gamma, \tag{2.11}^*$$

where γ is the Lorentz factor corresponding to the relative three-velocity $\mathbf{v}_{12}$.

We shall also need the formula for the area of a triangle on the hyperboloid. It is expressed in terms of the hyperbolic defect (A, B, C are the angles of the triangle):

$$\Omega = \pi - A - B - C; \tag{2.12}$$

one of the formulas for the defect is

$$\cos\varphi = \frac{1 + \operatorname{ch} a + \operatorname{ch} b + \operatorname{ch} c}{4\operatorname{ch}(a/2)\operatorname{ch}(b/2)\operatorname{ch}(c/2)}\,. \tag{2.13}$$

The spin vector of a particle is normal to its velocity,[4)] i.e., to the radius vector; therefore it lies in the (three-dimensional) hyperplane which touches the hyperboloid at the point R_1 corresponding to the velocity. In the new coordinate system—let us call it R_2—whose velocity corresponds to another point of the hyperboloid, the space axes lie in a new tangent hyperplane (and the timelike axis is normal to the hyperboloid). The projection of the spin on the new axes leads to the appearance of the fourth component, Eq. (1.1). The two lower components of the spinor are of this same nature.

It is not hard to see that without departing from the new system R_2 we can turn the spin vector so that all three spatial components lie on the tangential hyperplane in the system R_2. The possibility of such a transformation follows from the spacelike character of the spin vector. The transformation (2.6) indeed serves as such a rotation transformation for the spinor.

To illustrate the transformation (26) we recall that the transition from the spinor (2.2) to the spinor (2.1) is a hyperbolic rotation in the plane $(v_0, \mathbf{v})$ by the angle $\alpha = \cosh^{-1}\gamma$. This gives to the spacelike unit vector (0, 1) the components $(\sinh\alpha, \cosh\alpha) = (v\gamma, \gamma)$. If we now make an ordinary rotation through the angle $\beta = \arccos(1/\gamma)$ in the same plane, the spatial component again becomes equal to unity. This gives the geometrical meaning of the angles in Eqs. (2.3) and (2.6)[5)].

In connection with the Foldy-Wouthuysen transformation we may note another unitary transformation considered in a paper by Cini and Touschek[13] (cf. also [14]). In the plane $(v_0, \mathbf{v})$ one can make a rotation by any angle; these rotations generate a group of transformations (different from the Lorentz group). The Cini-Touschek transformation corresponds to the rotation which makes the spatial component of the vector zero. According to what we have said above, this is a rotation (in the opposite direction, as compared with the Foldy-Wouthuysen rotation) by the angle $\sin^{-1}(1/\gamma) = \tan^{-1}(1/|\mathbf{v}|)$.

The transformation is written:

$$C = \exp\left\{\frac{i}{2}\frac{\beta\boldsymbol{\alpha}\mathbf{v}}{|\mathbf{v}|}\operatorname{arctg}\frac{1}{|\mathbf{v}|}\right\} = \left(\frac{1+|\mathbf{v}|}{2}\right)^{1/2}\left[1 - \beta\boldsymbol{\alpha}\mathbf{v}\left(\frac{1}{|\mathbf{v}|} - 1\right)\right]. \tag{2.14}$$

If we further introduce a rotation by the angle $\pi/2$,

$$U = \exp\left(\frac{i\pi}{4}\frac{\boldsymbol{\alpha}\mathbf{v}}{|\mathbf{v}|}\right) = \frac{1}{\sqrt{2}}\left(1 + \frac{\beta\boldsymbol{\alpha}\mathbf{v}}{|\mathbf{v}|}\right), \tag{2.15}$$

then from what has been said it is clear that

$$C = UF. \tag{2.16}$$

The Lorentz transformation can be written in a more general form. Namely, the Lorentz transformation which takes a spinor from the system with the velocity v to that with the velocity u is written in the form

$$L^{-1} = \left(\frac{1+uv}{2}\right)^{1/2}\left[1 + i\frac{\sigma_{\alpha\beta}u_\alpha v_\beta}{1+uv}\right]. \tag{2.17}$$

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*ch = cosh.

4)The metric on the hyperboloid is pseudoeuclidean. On a sphere of radius i the spin vector is normal to the radius in the ordinary sense.

5)We have defined two arguments: the angle β and the hyperbolic angle α. The relation between them is $\tanh(\alpha/2) = \tan(\beta/2)$. In hypergeometrical geometry β is called the Gudermannian of α: $\beta = \operatorname{gd}\alpha$; $\operatorname{gd}\alpha$ is connected with the Lobachevskiĭ parallelism angle $\Pi(\alpha)$ by the relation $\operatorname{gd}\alpha = \pi/2 - \Pi(\alpha)$ (see [12]).

For a vector the transformation of parallel transfer between the points u and v can be written in the still simpler form

$$F_{\alpha\beta} = \delta_{\alpha\beta} - 2V_\alpha V_\beta/V^2, \qquad V_\alpha = u_\alpha + v_\alpha. \tag{2.18}$$

It has the following properties:

$$F^2 = 1, \qquad Fu = -v, \qquad Fv = -u. \tag{2.19}$$

It is easily verified that F takes a vector orthogonal to v into a vector orthogonal to u. In the same notation the Lorentz transformation has the form (cf. [17])

$$L_{\alpha\beta} = \delta_{\alpha\beta} - 2V_\alpha V_\beta/V^2 + 2u_\alpha v_\beta. \tag{2.20}$$

It is not hard to verify that in the system u = (1, 0) the formula (2.20) gives the usual Lorentz matrix.

It is interesting to see what the transformation (2.18) becomes for a photon. In order to go to the limit of the speed of light, it is convenient to introduce q and p —the momenta of the photon and the coordinate system. Then, using the fact that eq = 0, we easily find

$$F_{\alpha\beta} = \delta_{\alpha\beta} - p_\alpha q_\beta/pq, \tag{2.21}$$

and for the transformation of the (two-component) polarization vector e we have

$$e \to e - (ep)\, q/(pq). \tag{2.22}$$

This is the gauge transformation considered in [15].

Thus in the case of a proton the parallel transfer reduces to a gauge transformation, which leaves the free photon a two-component quantity in an arbitrary coordinate system. We note that Eq. (2.21) leaves the vector q itself invariant:

$$Fq = q. \tag{2.23}$$

This is due to the fact that in the noneuclidean metric q is orthogonal to itself, qq = 0.

3. SCATTERING OF PARTICLES

The picture of parallel transfer makes the transformation of the spins in the scattering of particles an obvious one. In the center-of-mass system the scattering is described by the ordinary matrix which is used in nonrelativistic theory. Here the spin operators act on the wave functions (or the elements of the density matrix) which have been taken by parallel transfer from the rest system to the center-of-mass system. Formally this means that the scattering matrix and the density matrix transform according to the formulas

$$S \to FSF^+, \tag{3.1}$$

$$\rho \to F^+\rho F, \tag{3.2}$$

and after this they take the nonrelativistic form (perhaps with a different normalization).

In this operation the "rest system" of a photon or neutrino—a point at infinity in the Lobachevskiĭ space—does not differ in any way from points at finite distances, and the spins of these particles do not require special treatment. Therefore the formulas for the scattering in the center-of-mass system can differ from the corresponding nonrelativistic formulas only in normalization.

In order to see how the spin is rotated in a different system, let us consider the hyperbolic triangle of Fig. 1 (it has been used in the papers of Stapp[4] and Wick[10]). Its three vertices correspond to three coordinate systems—the laboratory (or target) system R_1, the center-of-mass system C, and the rest system of the scattered particle, R_2. The description of the scattering goes as follows. The spin is taken from R_1 to C by parallel transfer. In the system C it is rotated by a certain angle φ by means of the ordinary scattering matrix, and then it is transferred to the system R_2. There is one more parallel transfer, from the system R_2 to the system R_1,

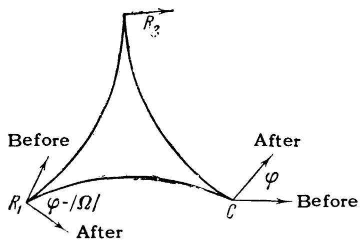

FIG. 1

and we get as the result the polarization vector after the scattering in the laboratory system. It is obvious that the angle of rotation ψ in this system differs from the angle φ in the system C by the hyperbolic defect Ω (for the arrangement of the systems shown in Fig. 1 it is measured counterclockwise, i.e., in the direction opposite to the scattering angle in the system R_1). The spin component normal to the plane of the triangle remains unchanged during the transfer. Thus we get without any calculations

$$\psi = \varphi - |\Omega| \tag{3.3}$$

(ψ is the angle of rotation in the system R_1, and φ is the angle in the system C).

The rotation is around the normal to the plane of scattering, which can be defined in a covariant way as the vector

$$N_\delta = \varepsilon_{\alpha\beta\gamma\delta} R_{1\alpha} R_{2\beta} C_\gamma, \tag{3.4}$$

where R_1, R_2, and C are the four-velocities of the three systems. In each of the three systems this vector has only the three spacelike components.

The hyperbolic triangle also allows us to calculate other geometrical characteristics directly. For example, for nucleon-nucleon scattering the sides R_2C (speed of the particle in the c.m.s.) and R_1R_2 (speed of the system C relative to the laboratory system) are equal. Therefore

$$\angle R_1 = \angle R_2. \qquad (3.5)$$

But the angle R_2 is the difference between the angles between the directions of the polarization and the momentum of the particle in the system C and in the laboratory system. On the other hand the angle R_1 is the scattering angle in the laboratory system. This is the "relativistic rotation" obtained by Stapp.[4] If the angle R_2 goes to zero, i.e., if the speed of the particle approaches the speed of light, we can see from the triangle that a particle of mass zero is longitudinally polarized in any coordinate system.6)

4. COMPOSITION OF SPINS

After the operation of transfer of a spin from one coordinate system to another has been defined, so that the spin remains a three-vector, the composition of spins practically reduces to the rules of nonrelativistic mechanics.

If we have a system of several particles, then their spins can always be taken into one of the systems by parallel transfer. After the transfer the axes of quantization for the different particles will in general not be parallel, and before compounding them we shall have to reduce them to the same direction by means of the usual coefficients $D^J_{MM'}$. Then the problem reduces to the composition of the angular momentum of particles without spin, and then we must add the total spin.

To determine the directions of the axes of quantization for the various particles, the simplest procedure is to make the following convention about the correspondence of the axes. One of the axes is defined along the relative three-velocity of the particle and the system C to which the spin is to be transferred.7) The second axis is defined as the normal N_γ to the plane which passes through the particle (sic), the system C, and some other chosen system (laboratory or c. of m.). The third axis is taken orthogonal (in the three-dimensional sense) to these two in each of the separate systems. This procedure indeed is a parallel transfer. Knowing the direction of the axis of quantization in these coordinates in the rest system of the particle, we thus have it defined in the system C also. The rest of the procedure is standard.

In his paper[10] Wick chooses the axes of quantization along the momenta, i.e., along our first chosen axis. Therefore Wick's method of helicity amplitudes realizes one of the possibilities of parallel transfer. The choice of the system C depends on the conditions of the problem, and the relations between the various schemes of composition can be determined from elementary geometrical relations.

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6)If we consider the hyperbolic triangle made up of the three-velocity **V** of the particle and its three-acceleration $\dot{\mathbf{V}}$, then the area of this triangle determines the Thomas precession of a top (cf. [16]):

$$\Omega = -\gamma(1+\gamma)^{-1}[\mathbf{V}\times\dot{\mathbf{V}}] \approx -\tfrac{1}{2}[\mathbf{V}\times\dot{\mathbf{V}}].$$

7)The system C need not be a c.m. system. A c.m. system is convenient, for instance, in the case of nucleon-nucleon scattering in which on account of charge symmetry the total spin is also conserved in the relativistic case. If the spins are composed in accordance with (1.1) then the triplet and singlet states are confused.

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