REGULARIZATION OF THE CLASSICAL EQUATIONS OF ELECTRODYNAMICS

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It is shown that by using a "mesonic" field to compensate the electromagnetic field one can obtain a relativistically invariant stable model of an extended charge without introducing new constants into the theory.

As is well known, the classical equations of electrodynamics cannot be regularized in a relativistically invariant way within the framework of the purely electromagnetic picture of the field. The classical model of an extended charge is unstable by Earnshaw's theorem, and the nonelectromagnetic forces admitted in early treatments involved action at a distance and therefore did not agree with the theory of relativity.

Theoretical physics is now acquainted with examples of relativistic fields in addition to the electromagnetic field, for example the vector meson field. As a boson field it also allows passage to the limit of a nonquantum theory. Therefore we can again raise the question of constructing a stable classical model of a charge, in which the electric forces of repulsion are balanced by "mesonic" forces of attraction. Calculations with a classical model are of course of only methodological interest and cannot be directly compared with experiment.

It is curious, however, that regularization on the basis of a "mesonic" field succeeds without introducing any new constants into the theory. It turns out that a stable model of an extended charge can be obtained only if we assume that the particles corresponding to the "mesonic" field are virtual, not real. Real particles do not give a stable equilibrium with the electromagnetic forces.

We shall start from the relativistically invariant Lagrangian

$$L = \int dv \left[\frac{1}{16\pi}(F_{ik}^2 - \varkappa^2\psi_i^2 - E_{ik}^2) + j_i(\varphi_i + \lambda\psi_i)\right]. \quad (1)$$

Here φ_i are the electromagnetic potentials, and ψ_i are the mesonic potentials; E_{ik} and F_{ik} are the suitably defined fields; and κ and λ are invariant constants. The sign of κ^2 is chosen to correspond to the fact that the "mesonic" field is a virtual one. It is essential that the electromagnetic and "mesonic" terms E_{ik}^2 and F_{ik}^2 appear with opposite signs, since the "mesonic" field must be an attractive one.

It is convenient to make the further calculations in the rest system of the charge as a whole. Then we get from Eq. (1) in the usual way the equations of electrostatics and "mesostatics":

$$\Delta\varphi = -4\pi\rho, \quad (2)$$

$$\Delta\psi + \varkappa^2\psi = 4\pi\rho\lambda, \quad (3)$$

$$\rho(\mathbf{E} + \lambda\mathbf{F}) = 0. \quad (4)$$

Here φ, ψ, and ρ are the time components of the potentials and the current density, and $\mathbf{E}$ and $\mathbf{F}$ are the electrostatic and "mesostatic" fields. The formula (4) expresses the condition for equilibrium of the forces. We must, of course, also require that the equilibrium be stable.

Equation (4) has two solutions: $\rho = 0$ and $\mathbf{E} = -\lambda\mathbf{F}$. We shall assume that the first solution holds for $r > r_0$, and the second for $r < r_0$, i.e., inside the region occupied by the charge, in which we accordingly have

$$\varphi = -\lambda\psi + C. \quad (5)$$

Substituting this in Eqs. (2) and (3), we find

$$4\pi\rho = -\lambda\varkappa^2\psi/(1-\lambda^2) \quad (6)$$

and

$$\Delta\psi = -\varkappa^2\psi/(1-\lambda^2) \text{ for } r \leqslant r_0, \quad (7)$$

$$\Delta\psi = -\varkappa^2\psi \text{ for } r > r_0. \quad (8)$$

Thus even outside the charge the "mesonic" field oscillates in space. Since, however, no upper limit is imposed on κ^2, we can suppose that the spatial period is as small as we please, and therefore the "mesonic" field is not observed. The other type of theory, with a "mesonic" field falling off rapidly in space, does not give stable equilibrium with the electrostatic forces.

The solutions (7) and (8) can always be matched at $r = r_0$. We must also satisfy the conditions that the electric potential and field be continuous: $\varphi(r_0) = e/r_0$, $E(r_0) = e/r_0^2$. This gives the value of the constant C of Eq. (5):

$$C = \frac{e}{r_0}\left[1 - \frac{\operatorname{tg} \nu\varkappa r_0}{\nu\varkappa r_0}\right]^{-1}, \tag{9*}$$

where $\nu = (1-\lambda^2)^{1/2}$.

The energy of the charge can be expressed in terms of C very simply; it is given by the expression

$$\mathscr{E} = \frac{1}{8\pi}\int \{E^2 - (F^2 - \varkappa^2\psi^2)\}\, dv. \tag{10}$$

Making the usual transformations on this expression, using the facts $\mathbf{E} = -\nabla\varphi$, $\mathbf{F} = -\nabla\psi$ and Eqs. (2) and (3), we get

$$\mathscr{E} = \frac{1}{2}\int \rho\,(\varphi + \lambda\psi)\, dv = \frac{C}{2}\int \rho\, dv = \frac{Ce}{2}. \tag{11}$$

We shall now show that $\mathscr{E}$ has a minimum at a certain value of r_0. For this purpose it is convenient to write

$$\nu = \frac{\pi}{2\varkappa r_0} + \varepsilon, \tag{12}$$

which gives

$$\mathscr{E} = \frac{e^2}{2r_0}\left[1 + \frac{\operatorname{ctg} \varepsilon\varkappa r_0}{\pi/2 + \varepsilon\varkappa r_0}\right]^{-1}. \tag{13†}$$

For $r_0 = 0$ the denominator of this expression has a finite value and a positive derivative, and for $\epsilon\kappa r_0 = \pi$ it goes to $-\infty$. Consequently, there is a definite value of $\epsilon\kappa r_0$ for which the denominator is a maximum and $\mathscr{E}$ is a minimum; that is, the equilibrium is stable. It occurs at $\epsilon\kappa r_0 = 2.3$, for which we have

$$\mathscr{E} = e^2\varkappa\varepsilon / 3.54; \tag{14}$$

$\mathscr{E}$ is to be equated to the rest energy mc^2 of the particle, so that we have

$$\varkappa\varepsilon = 3.54\, mc^2/e^2, \quad r_0 = 0.65 e^2/mc^2. \tag{15}$$

It can be shown that in a weak external field the quantity $m = Ce^2/2c^2$ is the proportionality constant between the force and the acceleration of the charge.

Thus the answer involves only the product $\kappa\epsilon$, and not the separate quantities. We can, for example, take ϵ as small as we wish, and let κ go to infinity. If ϵ goes to zero, then the coupling constant λ goes to infinity, and the product $\lambda\psi$ remains finite, as can easily be seen from the boundary conditions at $r = r_0$. Since, however, the "mesonic" field has an infinitely small spatial period, it has no effect on the external charge, which, according to Eq. (15), is also of finite dimensions.

*tg = tan.

†ctg = cot.

Translated by W. H. Furry