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REGGE POLES AND LANDAU SINGULARITIES

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It is shown that an infinite number of Regge poles accumulates on the line $\mathrm{Re}\, l = -1/2$ at the energy $\sqrt{t} = \sqrt{t_i}$ of any two-particle threshold. This result signifies in particular that the invariant scattering amplitude of two spinless particles cannot decrease faster than $1/\sqrt{s}$ with increasing s at a t corresponding to a two-particle threshold. This statement is a consequence of unitarity and analyticity. The proof of the accumulation of the poles requires that the range of the interaction be finite at a given energy. It is shown that an interaction which is consistent with unitarity and analyticity cannot be repulsive at all distances in the nonrelativistic limit.

IT is known that the scattering amplitude 1) as a function of the energy t and the momentum transfer s has singularities at the threshold values of these variables and on the Landau curves.[1,2] The Landau singularities have the remarkable property that the threshold value of one variable, say t, forms a line in the $s-t$ plane toward which an infinite number of Landau curves move when the other variable s approaches infinity. On the other hand, it has become clear lately that the asymptotic behavior of the scattering amplitude for large s is determined by the analytic properties of the partial wave amplitudes $f_l(t)$ as a function of the angular momentum l in the channel where $\sqrt{t}$ is the energy.[3] The question now arises how the increase in the density of the Landau curves at threshold values of t manifests itself in the behavior of the singularities of the amplitude as a function of l. It will be shown below that at the energy $t = t_i$ which corresponds to some arbitrary two-particle threshold an infinite number of poles approach the line $\mathrm{Re}\, l = -1/2$. We will also give arguments which indicate that a similar accumulation of an infinite number of poles will occur at an energy corresponding to the threshold of the creation of n particles at the line $\mathrm{Re}\, l = -1/2 - 3/2\,(n-2)$.

This result in particular indicates that the invariant scattering amplitude at a value t corresponding to an arbitrary two-particle threshold cannot decrease faster than $s^{-1/2}$ with increasing s. This result concerning the decrease of the scattering amplitude is a consequence only of unitarity and analyticity. In the demonstration of the accumulation of the poles one needs to make the assumption that the range of the interaction is finite at the given energy.

Despite this clustering of poles it is possible to find the asymptotic behavior of the scattering amplitude for t close to $4\mu^2$ (μ is the mass of a π meson). It turns out that the contribution from the poles accumulating at the line $\mathrm{Re}\, l = -1/2$ is oscillatory even for $t < 4\mu^2$. The oscillating behavior of the absorptive part of the amplitudes which correspond in the s-channel to elastic scattering contradicts the unitarity conditions in the s-channel since it follows from this condition that the absorptive parts of the amplitudes are positive in the interval $0 \le t < 4\mu^2$.[4] From this it necessarily follows that the partial wave amplitude in the t-channel in these cases has to have at least one pole on the real axis to the right of the line $\mathrm{Re}\, l = -1/2$ for $4\mu^2 - t$ small and positive. One may consider this result to be a purely theoretical argument indicating the necessity of the existence of the vacuum pole. From the point of view of nonrelativistic quantum mechanics the appearance of a pole for $\mathrm{Re}\, l > -1/2$ means that the potential cannot be everywhere repulsive ($\int u r\, dr\, \varphi^2 > 0$[5]). The above assertion indicates that the interaction must in this sense be attractive in order to be compatible with unitarity and analyticity.

We now consider the amplitudes for the partial waves, $f_l(t)$, for the scattering of spinless iden-

1)We use the word "scattering amplitude" for the sake of brevity. All our considerations apply to any invariant amplitude describing the transition of a two-particle state into a two particle state.

tical particles. For l in the complex plane to the right of all singularities of f_l there holds[6]

$$f_l(t) = \frac{4}{\pi} \int_{4\mu^2}^{\infty} Q_l\left(1 + \frac{2s}{t - 4\mu^2}\right) A_1(s, t) \frac{ds}{t - 4\mu^2}, \qquad (1)$$

where $A_1(s, t)$ is the absorptive part of the amplitude in channel s; $Q_l(x)$ is the Legendre function of the second kind. For $t \to 4\mu^2$ we have from this

$$f_l(t) = \frac{\Gamma(l+1)}{\sqrt{\pi} 2^{2l} \Gamma(l + 3/2)} (t - 4\mu^2)^l \int_{4\mu^2}^{\infty} \frac{A_1(s, t)}{s^{l+1}} ds,$$
$$l \neq -(2n + 3)/2, \qquad n = 0, 1, \ldots \qquad (2)$$

For such $t = 4\mu^2 + \delta$ the partial wave $f_l(t)$ obeys the unitarity condition

$$\frac{1}{2i}[f_l(t) - f^*_{l^*}(t)] = \frac{k}{\omega} f_l(t) f^*_{l^*}(t). \qquad (3)$$

It follows from this relation that $\varphi_l(t) = (k/\omega) f_l(t)$ for real l has absolute magnitude less than one. On the other hand according to (2) $\varphi_l(t)$ is proportional to $(t - 4\mu^2)^{l+1/2}$.

If (2) would hold for such l which lie to the left of the line $\mathrm{Re}\, l = -1/2$, then for sufficiently small $t - 4\mu^2$ the quantity $\varphi_l(t)$ could become arbitrarily large. Therefore the integral (2) must lose meaning for $\mathrm{Re}\, l \geq -1/2$, i.e., $A_1(s, t)$ for $t = 4\mu^2$ cannot decrease with increasing s faster than $1/\sqrt{s}$. Thus $\varphi_l(t)$ for $t \to 4\mu^2$ must have singularities not further to the left than $\mathrm{Re}\, l = -1/2$. We now assume that for Re $\geq -1/2$ there are only a finite number of poles at the points λ_n. Then $\varphi_l(t)$ can be written in the form

$$\varphi_l(t) = \tilde{\varphi}_l(t) + \sum_n \frac{r_n(t)}{l - \lambda_n(t)} e^{-\delta(l - \lambda_n)}, \qquad (4)$$

where $r_n(t)$ is the residue of the pole ($r_n \sim (t - 4\mu^2)^{\lambda_n + 1/2}$ [4]), δ is an arbitrary number greater than zero. Then $\tilde{\varphi}_l(t)$ does not have singularities for $\mathrm{Re}\, l \geq -1/2$ and, owing to the factors $e^{-\delta(l-\lambda)}$ introduced in (4), decreases exponentially as $\mathrm{Re}\, l \to \infty$. Furthermore $\varphi_l(t)$ can be written in the form (2) if one replaces A_1 by $A_1 = A_1 - \Delta A_1$, where ΔA_1 is the contribution from the considered poles. The integral containing ΔA_1 due to the lower integration limit $4\mu^2$ will be proportional to $t - 4\mu^2$. Since the sum

$$\sum r_n e^{-\delta(l - \lambda_n)} / (l - \lambda_n)$$

is finite for $t \to 4\mu^2$, $\tilde{\varphi}_l(t)$ will also be finite for real l because of the unitarity condition. On the other hand, $\tilde{\varphi}_l$ has the form

$$\tilde{\varphi}_l = \varphi_l - \sum \frac{c_n (t - 4\mu^2)^{\lambda_n + 1/2}}{l - \lambda_n} e^{-\delta(l - \lambda_n)}$$
$$= \chi(l)(t - 4\mu^2)^{l+1/2} - \sum c_n \frac{(t - 4\mu^2)^{\lambda_n + 1/2}}{l - \lambda_n} e^{-\delta(l - \lambda_n)},$$

where $\chi(l)$ is an analytic function of l, which can be represented for $\mathrm{Re}\, l$ larger than the largest $\mathrm{Re}\, \lambda_n$ as

$$\chi(l) = \frac{\Gamma(l+1)}{\sqrt{\pi}\, 2^{2l} \Gamma(l + 3/2)} \int_{4\mu^2}^{\infty} \frac{A_1(s, 4\mu^2)\, ds}{s^{l+1}}.$$

If the number of the poles λ_n is finite then the function $\chi(l)$ can be continued into the half plane $\mathrm{Re}\, l < -1/2$ since the line $\mathrm{Re}\, l = -1/2$ is not everywhere covered densely with singularities. In this half plane $\tilde{\varphi}_l \to \infty$ when $t \to 4\mu^2$. This contradicts the finiteness of $\tilde{\varphi}_l$. From this we conclude that also $\tilde{A}_1(s, 4\mu^2)$ cannot decay faster than $1/\sqrt{s}$, and consequently $\tilde{\varphi}_l(t)$ must have singularities which do not lie to the left of the line $\mathrm{Re}\, l = -1/2$, in contradiction of its definition. This indicates that $\varphi_l(t)$ cannot have a finite number of singularities for $\mathrm{Re}\, l \geq -1/2$ and $t \to 4\mu^2$. It is easy to show that this conclusion does not change if $\varphi_l(t)$ in addition has also a finite number of branch points for $\mathrm{Re}\, l \geq -1/2$.

This way we find that $\varphi_l(t)$ for $t \to 4\mu^2$ must have an infinite number of singularities in the vicinity of the line $\mathrm{Re}\, l = -1/2$.

It is easy to see that a similar situation arises at any two-particle threshold. To that end it suffices to consider the amplitude of an arbitrary reaction in which these two particles appear either in the initial or in the final state since all amplitudes which are connected by the unitarity condition have common poles. The invariant amplitude f_{ab} of the transition of two particles into two is proportional to $k_a^l k_b^l$ where k_a and k_b are the relative momenta in the initial and final state respectively. The amplitude which in analogy to φ is restricted in absolute magnitude is

$$\varphi_{ab} = k_a^{1/2} f_{ab} k_b^{1/2} \sim (k_a k_b)^{l+1/2},$$

and consequently at the corresponding threshold t ($k_a \to 0$ or $k_b \to 0$) we meet the same conditions. In this connection it is of interest to note the following. In a theory which excludes electromagnetic interactions there must exist atomic nuclei of arbitrarily large mass of the order of mA due to the saturation of nuclear forces. Therefore two-particle thresholds must exist at arbitrarily large t. This shows that the invariant amplitude A_1 at any arbitrary $t \approx (2mA)^2$ cannot decrease faster than $1/\sqrt{s}$.

We now consider the vicinity of many-particle thresholds. For that end one may consider the partial amplitude for transforming two particles into n particles with angular momentum l which enters the unitarity condition of the elastic amplitude. Such an amplitude in the vicinity of the production threshold is proportional to

$$\Big[t-\Big(\sum_{i=1}^{n} m_i\Big)^2\Big]^{l/2}.$$

The amplitude analogous to φ_{ab} which is restricted as to absolute value equals $\sqrt{k_a}\, f_{ab}\sqrt{\Gamma_b}$, where Γ_b is the phase space for n particles which is proportional to

$$\Big[t-\Big(\sum_{i}^{n} m_i\Big)^2\Big]^{(3n-5)/2}.$$

Therefore the restricted amplitude is proportional to

$$\Big[t-\Big(\sum_{i=1}^{n} m_i\Big)^2\Big]^{\frac{1}{2}\left(l+\frac{3n-5}{2}\right)}.$$

We believe that such a dependence will lead to the appearance of an infinite number of singularities on the line $\mathrm{Re}\, l = -(3n-5)/2$ for $t \to (\Sigma m_i)^2$, and that this will reflect the accumulation of the Landau curves for $t = (\Sigma m_i)^2$ and $s \to \infty$. In this connection it seems to us that the hypothesis of Predazzi and Regge[7] concerning the symmetry of the l-plane with respect to $\mathrm{Re}\, l = -\frac{1}{2}$, a hypothesis which does not allow this behavior, can apparently not be realized when one considers inelastic processes, since one cannot see how the accumulation of Landau curves in the l-plane at the many-particle thresholds could appear.

In order to investigate in more detail the structure of the singularities on the line $\mathrm{Re}\, l = -\frac{1}{2}$ for $t \to 4\mu^2$, we use the energy-independent boundary condition for the wave function, which applies in the nonrelativistic region $t \to 4\mu^2$ for interactions that decrease exponentially with the distance. This boundary condition usually is applied for integer l. We shall continue it analytically in l. We write the wave function in the noninteracting region in the form

$$\psi_\nu = j_\nu(kr) + i\varphi_\nu(k)\, h_\nu^{(1)}(kr), \tag{5}$$

where

$$j_\nu(x) = \sqrt{\frac{\pi x}{2}}\, J_\nu(x), \quad h_\nu^{(1)}(x) = \sqrt{\frac{\pi x}{2}}\, H_\nu^{(1)}(x),$$

$\nu = l + \frac{1}{2}$; $J_\nu(x)$ and $H_\nu^{(1)}(x)$ are Bessel and Hankel functions respectively. Denoting $(r\psi_\nu'/\psi_\nu)|_{r=R}$ by χ_ν we find, using the relation

$$h_\nu^{(1)} = \frac{i}{\sin\pi\nu}\left(e^{-i\pi\nu} j_\nu - j_{-\nu}\right), \tag{6}$$

that

$$\varphi_\nu = -\Lambda \sin\pi\nu \,/\, (1 - e^{-i\pi\nu}\Lambda), \tag{7}$$

$$\Lambda = [j_\nu'(kR) - \chi_\nu j_\nu(kR)] \,/\, [j_{-\nu}'(kR) - \chi_\nu j_{-\nu}(kR)]. \tag{8}$$

For $k \to 0$ ($\nu \neq 0, \pm 1, \pm 2, \ldots$) we have

$$\Lambda = (kR)^{2\nu}\frac{\chi_\nu - \nu}{\chi_\nu + \nu}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}, \tag{9}$$

χ_ν does not depend on the energy. It follows from this that for small k the quantity Λ oscillates around the line $\mathrm{Re}\,\nu = 0$ and changes fast in magnitude as it departs insignificantly from the line. Therefore φ_ν must have an infinite number of poles.

We consider the region $\nu \ll 1$ more accurately. If one expands the factor of $(kR)^{2\nu}$ into a power series of ν then the term linear in ν can be excluded by a suitable choice of R. Then

$$\Lambda = x^{2\nu}(1 + \gamma\nu^2), \tag{10}$$

where $x = ka$, and a is the chosen radius. In order to remove $e^{-i\pi\nu}$ in the denominator of (7) we consider the region $t < 4\mu^2$. Then the equation for the positions of the poles has the form

$$x^{-2\nu} = 1 + \gamma\nu^2, \qquad x = \varkappa a, \qquad \varkappa = \tfrac{1}{2}\sqrt{4\mu^2 - t},$$
$$-\nu\tau = \gamma\nu^2 + 2i\pi n, \quad \tau = \ln x^2, \quad n = 0, \pm 1, \ldots \tag{11}$$

From here we have

$$\nu_n \approx -2i\pi n/\tau + 4\pi^2 n^2\gamma/\tau^3. \tag{12}$$

This way we have obtained an infinite number of complex conjugate poles. Since $\tau \to -\infty$ then for $\gamma > 0$ the poles lie to the left of the line $\mathrm{Re}\,\nu = 0$. We note that in nonrelativistic quantum mechanics $\gamma \geq 0$ for arbitrary potentials since there no complex poles can exist for $\mathrm{Re}\,\nu > 0$ and $t < 4\mu^2$.[8] The case $\gamma = 0$ corresponds to a potential which for small distances grows faster than r^{-2}.

In the following we shall assume that $\gamma > 0$. The equations (7) and (12) allow the evaluation of the asymptotic behavior of $A_1(s,t)$ for small $t - 4\mu^2$ and large s. $A_1(s,t)$ contains contributions from a finite number of poles lying to the right of the line $\mathrm{Re}\,\nu = 0$ with which we shall not be concerned. Denoting the contribution of the poles which lie to the left of the line $\mathrm{Re}\,\nu = 0$ by $\widetilde{A}_1(s,t)$ we obtain

$$\widetilde{A}_1(s,t) = -\frac{1}{2i}\sqrt{\frac{4\mu^2}{s}}\int_{-i\infty}^{i\infty}\frac{d\nu\, e^{\xi\nu}\,\nu\,(1+\gamma\nu^2)}{1 - e^{\tau\nu}(1+\gamma\nu^2)}, \tag{13}$$

where $\xi = \ln(sa^2)$. We employed (13) in an approximate expression for φ_ν which applies for $\nu \ll 1$.

As will become evident later in (13) for large ξ only the small values of ν are important. By closing the integration contour to the left and calculating the subtractions we obtain, omitting terms of the order $1/\tau^2$ and $1/\sqrt{\tau\xi}$ relative to unity,

$$\widetilde{A}_1 = 4\pi^2 \sqrt{\frac{4\mu^2}{s}} \frac{1}{\tau^2} \sum_{n=1}^{\infty} \exp\left(\frac{4\pi^2 n^2 \gamma}{\tau^3}\xi\right) n \sin 2\pi n \frac{\xi}{\tau}. \quad (14)$$

We now investigate the case $\xi \gg \tau^3$. Then in the sum (14) only one term is important and thus

$$\widetilde{A}_1 = 4\pi^2 \sqrt{\frac{4\mu^2}{s}} \frac{1}{\tau^2} \exp\left(\frac{4\pi^2\gamma\xi}{\tau^3}\right) \sin 2\pi \frac{\xi}{\tau}. \quad (15)$$

This way we have obtained an oscillatory behavior of $\widetilde{A}_1$. The consequences of this result have already been discussed above.

We have noted in this discussion that the interaction which in nonrelativistic quantum mechanics corresponds to repulsion is inconsistent with analyticity and unitarity. Simultaneously, an arbitrarily small arbitrary attractive potential is consistent with analyticity and unitarity since, as can be shown, it leads to a pole on the real axis at $l > -\frac{1}{2}$ for $t = 4\mu^2$ [9] With decreasing interaction strength at $t = 4\mu^2$ the pole moves towards $l = -\frac{1}{2}$. It is also of interest to remark that the discussed complex conjugate poles move towards $-\infty$ in the l-plane as the interaction strengths approaches zero. One can derive this result by calculating the quantity γ which enters in (10) by perturbation theory, which is permissible for a weak potential. Then it is easy to show that $\gamma = g^{-4}$ where $g^2 = -\int ur\,dr$. From this follows that the asymptotic expression for the scattering amplitude (15) has an identically vanishing expansion in powers of g^2.

Up till now we have considered the complex poles for $\mathrm{Re}\,\nu > 0$. The question arises as to how the formula (7) contains the possibility of the appearance of poles on the real axis if $\Lambda = \alpha_\nu(4\mu^2 - t)^\nu$ equals zero for $\mathrm{Re}\,\nu > 0$ and infinity for $\mathrm{Re}\,\nu < 0$. It is clear that the position of such poles, φ_ν, for $t = 4\mu^2$ coincides with the poles of α_ν for $\nu > 0$ and with the zeros of α_ν for $\nu < 0$. It is easy to derive from these considerations an expression for $\nu_n(t)$ in the vicinity of $t = 4\mu^2$. If one does this for $\nu_n(4\mu^2) > 0$ then one obtains a result in agreement with [4]. By putting $\alpha_\nu = \rho_n(\nu - \beta_n)$ and inserting it in (7) one can easily show for the poles with $\nu_n(4\mu^2) = \beta_n < 0$ that $\varphi_\nu(t)$ has a pole at

$$\nu_n(t) = \beta_n + \rho_n^{-1} \frac{(4\mu^2 - t)^{-\beta_n}}{\sin \pi\nu_n}. \quad (16)$$

If $-\beta_n$ equals an integer then

$$\nu_n(t) = \beta_n + \rho_n^{-1}(4\mu^2 - t)^{-\beta_n} \ln \frac{4\mu^2 - t}{4\mu^2}. \quad (17)$$

Comparing (16) and (17) with [4] we see that the character of the departure from the real axis is given completely by the modulus of ν. We remark that according to (17) a departure from the real axis at $\nu = 0$ is possible only if $\rho_n^{-1} = 0$ which then means that the calculation collapses to zero.

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