EXPANSION OF THE SCATTERING AMPLITUDE IN RELATIVISTIC SPHERICAL FUNCTIONS

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The expansion of the scattering amplitude in relativistic spherical functions is found. The unitarity relation is found for the relativistic partial amplitudes for a given four-dimensional angular momentum. The analytic properties of the partial amplitudes as functions of the four-dimensional angular momenta are discussed.

RECENTLY it has become clear that the study of the analytic properties of the scattering amplitude as a function of the angular momentum is important.[1] At high energies, the expansion of the amplitude in three-dimensional spherical functions can be used effectively if the momentum transfer in the scattering channel and, consequently, the energy in the annihilation channel, are small.[2-4] If the momentum transfer is comparable to the energy, the study of the asymptotic behavior of amplitudes using this expansion becomes difficult. The problem then arises of expanding the amplitude in eigenfunctions of the four-dimensional angular momentum, or more precisely, expanding it in irreducible representations of the homogeneous Lorentz group.

For simplicity we shall here consider the amplitude for scattering of spinless particles with equal masses κ. The generalization to the case of particles with spin and with different masses does not involve any difficulties.

From the existence of dispersion relations in the momentum transfer for the scattering amplitude U(t, s) (where t, s are the usual Mandelstam variables) it follows that the invariant integral

$$N = \int \left| \frac{U(t, s)}{(t-a)^n} \right|^2 \frac{d^3p}{\varepsilon} < \infty, \qquad a > 0, \qquad n \geqslant 0, \tag{1}$$

(where p and ϵ are the momentum and energy of the scattered particle) converges for any fixed s. If we write p^0 and p for the 4-momenta of the incident and scattered particles, and introduce the quantity

$$Z = p^0 p/\varkappa^2, \tag{2}$$

the integral (1) can be rewritten in the form

$$N = 4\pi\varkappa^2 \int_1^\infty |f(Z, s)|^2 \sqrt{Z^2 - 1}\, dZ, \tag{1a}$$

$$f(Z, s) = U(t, s)/(t-a)^n. \tag{3}$$

We note that, since $d^3p/\kappa^2\epsilon$ is the element of surface of the unit sphere, if we assume p^0 fixed we can consider (1) as an integral over the surface of a four-dimensional sphere of a function of the 4-vector p. Our problem is completely analogous to that of finding the expansion of the scattering amplitude in three-dimensional spherical functions, and reduces to expanding a function of the timelike 4-vector p, satisfying condition (1), in eigenfunctions of the four-dimensional angular momentum operator $\hat{F}$:

$$\hat{F} = \tfrac{1}{2} M_{\mu\nu} M_{\mu\nu}, \qquad \mu, \nu = 1, 2, 3, 4, \tag{4}$$

where $M_{\mu\nu}$, in the case of spinless particles, has the form

$$M_{\mu\nu} = -i\,(p_\mu \partial/\partial p_\nu - p_\nu \partial/\partial p_\mu). \tag{5}$$

This expansion was obtained by the author in 1955 by using the theory of the representations of the Lorentz group which was developed by Gel'fand and Naĭmark (cf. [6]). We emphasize that, since the scattering amplitude satisfies condition (1), i.e., is absolutely square integrable over the four-dimensional sphere, it can be expanded only in the irreducible unitary representations of the Lorentz group, which are infinite-dimensional because of the noncompactness of the group (infinite group volume).

The irreducible representations of the Lorentz group are usually characterized by a pair of numbers (j_1, j_2). For the case of unitary representations it is convenient to introduce two new numbers m and ρ:

$$m = 2\,(j_2 - j_1), \qquad \rho = -2i\,(j_1 + j_2 + 1). \tag{6}$$

These numbers determine the eigenvalues of $\hat{F}$

and the invariant pseudoscalar operator $\hat{G} = -i\epsilon_{\mu\nu\lambda\sigma}M_{\mu\nu}M_{\lambda\sigma}$ which commutes with it:

$$F = -[1-(m^2-\rho^2)/4], \qquad G = -m\rho. \tag{7}$$

If the representation is unitary, m is an integer and ρ an arbitrary real number. It is obvious that the pairs (m, ρ) and $(-m, -\rho)$ give equivalent representations. We note that for our case of spinless particles the operator $\hat{G} = 0$, since we cannot construct a pseudoscalar from the two 4-vectors p and $\partial/\partial p$. For this reason and because of the timelike nature of the 4-vector p, m = 0 (which is not the case for particles with spin). According to [5], the following formulas are valid:

$$f(p, s) = \left(\frac{1}{4\pi}\right)^{3/2}\int C(\rho, s, \mathbf{n})\left(\frac{\varepsilon - \mathbf{pn}}{\varkappa}\right)^{-1+i\rho/2}\rho^2 d\rho, \tag{8a}$$

$$C(\rho, s; \mathbf{n}) = \left(\frac{1}{4\pi}\right)^{3/2}\int f(p, s)\left(\frac{\varepsilon - \mathbf{pn}}{\varkappa}\right)^{-1-i\rho/2}\frac{d^3p}{\varepsilon}. \tag{8b}$$

$$\int |C|^2 \rho^2 d\rho\, d\omega_n = \int |f|^2 \frac{d^3p}{\varepsilon}. \tag{9}$$

In formulas (8) the limits of the ρ integration are 0 and ∞; $d\omega_n$ is the element of solid angle in the space of the three-dimensional unit vector $\mathbf{n}$.

Now we use the fact that f(p, s) actually depends on the scalar product $p^0 \cdot p$. Then the partial amplitudes $C(\rho, s; \mathbf{n})$ should be independent of n. Performing the integration in (8a) and (8b) over $d\omega_n$, we find

$$\Phi(\chi, s) = \sqrt{\frac{2}{\pi}}\int_0^\infty c(\rho', s)\sin\chi\rho'\, d\rho', \tag{10a}$$

$$c(\rho', s) = \sqrt{\frac{2}{\pi}}\int_0^\infty \Phi(\chi, s)\sin\chi\rho'\, d\chi. \tag{10b}$$

Here

$$Z = \operatorname{ch}\chi, \qquad \rho' = \rho/2, \qquad c(\rho', s) = \sqrt{2}\rho C(\rho, s), \tag{11*}$$

$$\Phi(\chi, s) = \sqrt{Z^2-1}\, f(Z, s). \tag{12}$$

Thus the expansion of the scattering amplitude in 4-dimensional spherical functions reduces to a Fourier transformation. The nontrivial point of the formulas (10) is that the variable ρ' determines the value of the four-dimensional angular momentum. The variable ρ' is a relativistic invariant. Consequently the partial amplitudes $c(\rho', s)$ are also invariant (just as, in the three-dimensional case, the partial amplitudes $f(l, s)$ are invariant under three-dimensional rotations). Proceeding exactly as before, we can take out the s dependence from the partial amplitudes. We get [1)]

*ch = cosh.

1)Formulas (13) were also given in the paper of Dolginov and Toptygin.[9]

$$\Phi(\chi, \varphi) = \frac{2}{\pi}\int_0^\infty\int_0^\infty c(\rho', \rho_1')\sin\chi\rho'\sin\varphi\rho_1'\, d\rho' d\rho_1', \tag{13a}$$

$$c(\rho', \rho_1') = \frac{2}{\pi}\int_0^\infty\int_0^\infty \Phi(\chi, \varphi)\sin\chi\rho'\sin\varphi\rho_1'\, d\chi d\varphi. \tag{13b}$$

In these formulas

$$\operatorname{ch}\varphi = Y = s/2\varkappa^2 - 1, \tag{14}$$

$$\Phi(\chi, \varphi) = \sqrt{Z^2-1}\,\sqrt{Y^2-1}\, f(Z, Y). \tag{15}$$

Formulas (13) completely solve our problem. The point of introducing the invariant partial amplitudes $c(\rho', \rho_1')$ is to separate out of the amplitude all the "kinematic parts" which are due to the properties of the Lorentz group. We may say that $c(\rho', \rho_1')$ depends exclusively on the dynamics of the process. There is thus reason to hope that the study of the analytic properties of the partial amplitudes $c(\rho', \rho_1')$ will be important for the theory of strong interactions, although there is as yet no supporting evidence for this.

We note that the asymptotic behavior [4] $t^{\alpha(s)}$ for $t \to \infty$ corresponds, as one sees easily from (10), to a pole of $c(\rho', s)$ in the complex ρ' plane.

As already stated, the extension of these formulas to the scattering of particles with spin is no problem. Expansions analogous to (8), for particles with spin, have been published in [7,8]. From them one can easily get the generalization of formulas (13) to the scattering amplitude for particles with spin.

For particles with spin, the quantum number m will be different from zero and the expansion in relativistic spherical functions will contain, in addition to the Fourier transformation, a summation over m (from -2σ to $+2\sigma$, where σ is the spin of the particle) and the angle functions for the symmetric top.

Using the addition theorem for the relativistic spherical functions [2)]

$$\int (Z_1 - \sqrt{Z_1^2-1}\,\mathbf{v}_1\mathbf{n})^{-1+i\rho/2}(Z_2 - \sqrt{Z_2^2-1}\,\mathbf{v}_2\mathbf{n})^{-1-i\rho/2}d\omega_n = \frac{8\pi}{\rho\sqrt{Z^2-1}}\sin\frac{1}{2}\rho\chi, \tag{16}$$

where

$$Z = Z_1Z_2 - \mathbf{v}_1\mathbf{v}_2\sqrt{Z_1^2-1}\sqrt{Z_2^2-1}, \quad \mathbf{v}_1^2 = \mathbf{v}_2^2 = 1, \tag{16a}$$

can be gotten by using formulas (8a), (8b) and the following unitarity relation for the two-particle intermediate states:

2)The derivation of these formulas will be given in a more detailed paper.

$$\operatorname{Im} C(\rho, s) = 32\varkappa^4 [\pi s (s - 4\varkappa^2)]^{-1/2} C(\rho, s) \frac{1}{\rho} \quad \frac{\eta\rho}{2}$$
$$\times \int_0^\infty C^*(\mu, s) \mu \sin \frac{\mu\eta}{2} d\mu. \tag{17}$$

Here

$$\eta = \ln\left\{\sqrt{\frac{s}{4\varkappa^2}} - \sqrt{\frac{s - 4\varkappa^2}{4\varkappa^2}}\right\}. \tag{17a}$$

In Eq. (17), the quantities $C(\rho, s)$ are the partial amplitudes of the matrix $U(s, t)$, which is related to the T-matrix by the formula

$$T_{ab} = (2\pi)^4 \frac{U_{ab}}{4\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4}} \delta(p_a - p_b), \tag{17b}$$

where the ϵ_i are the energies of the colliding and scattered particles. From (17) it follows, in particular, that the ratio

$$\operatorname{Im} C(\rho, s) / \operatorname{Re} C(\rho, s) = Q(s) \tag{17c}$$

is independent of ρ and is a function only of the variable s. If $C(\rho, s)$ is a meromorphic function of ρ in the upper halfplane, relation (17) takes a particularly simple form. Possibly the study of the unitarity relation (17) in the annihilation channel for small s will give useful information about the properties of $C(\rho, s)$.

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