INVESTIGATION OF THE FOUR-FERMION INTERACTION BY MEANS OF UNITARITY AND ANALYTICITY

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Equations that follow from unitarity and analyticity are given for four-fermion contact interactions. The asymptotic behavior of the fermion-fermion and antifermion-fermion scattering amplitudes is investigated at high energies by means of these equations. It is shown that if a power law is assumed for the energy dependence of the amplitudes, no contradictions appear in the equations, in contrast to perturbation theory. A connection between the orders of growth n_j of the various amplitudes and the upper and lower bounds on the quantities n_j has been established.

1. INTRODUCTION

LIKE any other nonrenormalizable interaction, the four-fermion interaction leads in perturbation theory to divergent expressions such that the order of divergence for large momenta increases with the order of the diagram. The use of unitarity and analyticity yields equations [1-4] for the scattering amplitudes of arbitrary particles (more precisely, for the partial wave amplitudes or for the spectral functions). Such equations have been obtained for some renormalizable interactions ($\pi - \pi$, $K - \pi$, $\pi - N$, $N - N$, etc) and it is evident that the iteration of these equations with respect to the coupling constant leads to the usual perturbation theory series. No such equations have been written down for nonrenormalizable theories. It is evident that the iteration of such equations would lead to expressions that diverge for large energies, so that the number of subtractions to be introduced in each order would increase.

In the present paper we analyze the behavior of the amplitudes for four-fermion interactions at large energies by means of a method proposed by Gribov [5]. This calls for the knowledge of equations which express the spectral functions of the amplitudes in terms of the absorptive parts of these amplitudes. We show that, in contrast to perturbation theory, no contradictions appear in our equations at large energies. Namely, if we assume that the amplitudes increase at infinity as some power of the energy, then the equations are satisfied and a connection exists between the orders of growth of the five independent (scalar, vector, etc) amplitudes.

The very form of the equations and the condition that the partial wave amplitudes be bounded imply rigid limits between which the growth indices n_j must be situated: $0 \le n_S, n_P \le 1$ and $-1 \le n_V$, n_T, $n_A \le 0$, $A_j(s, t) \sim s^{n_j}$. If the scattering cross section is constant, then $n_S = n_P = 1$, $n_V = n_T = 0$, and $-1 \le n_A \le 0$. Moreover, the equations imply that the fermion-antifermion scattering amplitudes have the same behavior at large energies as the amplitudes for fermion-fermion scattering.

For the sake of simplicity we assume here that the masses of all fermions are equal and that parity is conserved.

2. DEFINITION OF THE INVARIANT AMPLITUDES AND THE UNITARITY RELATION

We define the transition amplitude $D_{\beta\alpha}$ in the same manner as has been done for the nucleon-nucleon scattering amplitude in the paper by Amati, Leader, and Vitale [6]

$$S_{\beta\alpha} = \delta_{\beta\alpha} + i\delta^{(4)}(p_\beta - p_\alpha) D_{\beta\alpha}. \tag{1}$$

The function $D_{\beta\alpha}$ is related to the covariant Feynman amplitude through the relation

$$D_{\alpha\beta} = (m^2/4\pi^2E^2) M_{\alpha\beta}, \tag{2}$$

where E is the energy of the particle in the center of mass system (c.m.s.). The cross section is expressed in terms of $M_{\beta\alpha}$

$$d\sigma/d\Omega = | m^2 M_{\beta\alpha} / 4\pi E |^2. \tag{3}$$

The unitarity relation for the amplitudes $M_{\beta\alpha}$ has in the two-particle approximation the form

$$i\,(M^{+}_{\beta\alpha} - M_{\beta\alpha}) = \frac{m^2 p}{2\pi E}\int \frac{d\Omega}{4\pi} M^{+}_{\beta\gamma} M_{\gamma\alpha}, \quad (4)$$

where p is the c.m.s. momentum.

Corresponding to the five usual types of four-fermion interaction are five invariant amplitudes in terms of which one can expand $M_{\beta\alpha}$. There exist many possibilities for choosing the Fermi amplitudes or their linear combinations F_i (cf. for example the paper by Goldberger et al [7]). We use the Fermi amplitudes proper. Then

$$M_{\beta\alpha} = \sum_j A^j (s, t, \bar{t}) \langle 1' | O_j | 1\rangle \langle 2' | O_j | 2\rangle,$$

$$j = S, V, T, A, P, \quad (5)$$

and, for instance,

$$\langle 1' | O_T | 1\rangle \langle 2' | O_T | 2\rangle = \tfrac{1}{2} (\bar{u}'\sigma_{\mu\nu} u)(\bar{v}'\sigma_{\mu\nu} v).$$

The amplitudes A^i and F_i are linear combinations of each other and therefore all relations for the A^i can be easily rewritten in terms of the F_i.

The amplitude $M_{\beta\alpha}$ describes both the scattering of a fermion on a fermion and the scattering of a fermion on an antifermion. For the first process:

$$p_1 + n_1 \to p_2 + n_2,$$

$$M_{\beta\alpha} = \sum_j A^j (s, t, \bar{t}) (\bar{u}(p_2) O_j u(p_1)) (\bar{u}(n_2) O_j u(n_1)),$$

$$s = -(p_1 + n_1)^2, \quad t = -(n_1 - n_2)^2,$$

$$\bar{t} = -(n_1 - p_2)^2. \quad (6)$$

For fermion-antifermion scattering (the third reaction channel):

$$n_1 + n_2 \to p_1 + p_2,$$

$$\widetilde{M}_{\beta\alpha} = \sum_j A^j (s, t, \bar{t}) (\bar{u}(p_2) O_j v(-p_1)) (\bar{v}(-n_2) O_j u(n_1)),$$

$$t = -(n_1 + n_2)^2, \quad s = -(n_1 - p_1)^2,$$

$$\bar{t} = -(n_1 - p_2)^2. \quad (7)$$

The unitarity relation for the third channel has the form

$$\sum_j \operatorname{Im} A^j (s, t, \bar{t}) [\bar{u}(p_2) O_j v(-p_1)] [\bar{v}(-n_2) O_j u(n_1)]$$

$$= -\frac{1}{4\pi}\sqrt{\frac{t-4m^2}{t}}\int \frac{d\Omega}{4\pi}\sum_{l,k} [\bar{u}(p_2) O_l v(-p_1)]$$

$$\times [\bar{v}(-n_2) O_k u(n_1)]$$

$$\times A^{l*}(s', t', \bar{t}') A^k (s'', t'', \bar{t}'') I_{lk},$$

$$I_{lk} = \tfrac{1}{4}\operatorname{Sp}[O_l (m - i\hat{k}_2) O_k (m + i\hat{k}_1)]. \quad (8)$$

One can use the following dispersion relation for the A^j

$$A^j (s, t, \bar{t}) = \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{A_1^j (t, \sigma)\, d\sigma}{\sigma - s} + \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{A_2^j (\sigma, t)\, d\sigma}{\sigma - \bar{t}}. \quad (9)$$

Substituting Eq. (9) into the right hand side of Eq. (8) we obtain an expression of the form

$$\Omega_{lk} = \int \frac{d\Omega}{4\pi} \frac{[\bar{u}(p_2) O_l v(-p_1)][(\bar{v}(-n_2) O_k u(n_1)]}{(\sigma' - s')(\sigma'' - s'')} I_{lk}. \quad (10)$$

In Eq. (10) it is necessary to carry out the integration and then expand the matrix element of the resulting expression between spinor eigenstates in terms of the five Fermi invariants. It is then necessary to continue (8) in s (or $\bar{t}$) and calculate the discontinuity in this variable. As a result of these operations, one obtains the equations

$$A_{32}^j = -\frac{1}{4\pi}\sqrt{\frac{t-4m^2}{t}}\frac{1}{\pi^2}\iint \frac{dz_1\, dz_2}{\sqrt{\bar{z}^2 + z_1^2 + z_2^2 - 2\bar{z} z_1 z_2 - 1}}$$

$$\times \{\Lambda^j_{lk}(-z_1, z_2) \operatorname{Re}(A_1^{l*} A_1^k) + \Lambda^j_{lk}(z_1, -z_2)$$

$$\times \operatorname{Re}(A_2^{l*}(z_1) A_2^k(z_2))\}, \quad (11)$$

$$A_{31}^j = -\frac{1}{4\pi}\sqrt{\frac{t-4m^2}{t}}\frac{1}{\pi^2}\iint \frac{dz_1\, dz_2}{\sqrt{z^2 + z_1^2 + z_2^2 - 2z z_1 z_2 - 1}}$$

$$\times \{\Lambda^j_{lk}(z_1, z_2) \operatorname{Re}(A_1^{l*}(z_1) A_2^k(z_2)) + \Lambda^j_{lk}(-z_1, -z_2)$$

$$\times \operatorname{Re}[A_2^{l*}(z_1) A_2^k(z_2)]\};$$

$$z = 1 + 2s/(t - 4m^2), \quad \bar{z} = 1 + 2\bar{t}/(t - 4m^2). \quad (12)$$

We give here the coefficients Λ^j_{ik} with the strongest asymptotic behavior as $z \to \infty$ and z_1, $z_2 \to \infty$:

$$\Lambda^S_{VT}, \Lambda^P_{VT}, \Lambda^P_{TT} \sim z; \quad \Lambda^S_{VA}, \Lambda^S_{TA}, \Lambda^P_{VA}, \Lambda^P_{TA}, \Lambda^P_{TP} \sim z_1;$$
$$\Lambda^S_{SV}, \Lambda^P_{SV}, \Lambda^S_{ST} \sim z_2; \quad \Lambda^S_{VV}, \Lambda^S_{TT}, \Lambda^P_{VV} \sim (z_1^2 + z_2^2)/z.$$

The other coefficients either remain constant or decrease as z, z_1, $z_2 \to \infty$.

3. ASYMPTOTIC BEHAVIOR OF THE AMPLITUDES

The integrals in (11) and (12) are of the form

$$\rho(z, t) = \sqrt{\frac{t-4m^2}{t}}\iint_{z_0 z_0} \frac{dz_1\, dz_2 F(z_1, z_2, t)}{\sqrt{z^2 + z_1^2 + z_2^2 + 2z z_1 z_2 - 1}}. \quad (13)$$

Here $z_0 = 1 + 8m^2/(t - 4m^2)$; the upper limit of integration corresponds to the vanishing of the expression under the square root. An expression of this kind has been investigated in Gribov's paper [5]; the difference in our case is that we deal with a system of equations and that the coefficients Λ^j_{ik} depend on z, z_1, and z_2.

By means of a simple change of variables the integral (13) reduces to the form

$$\rho(z, t) = \sqrt{\frac{t-4m^2}{t}}\int_{\sqrt{\frac{2z_0^2}{z+1}}}^{1} \frac{r\,dr}{\sqrt{1-r^2}}$$

$$\times \int_{-\lambda+\varphi_0}^{\lambda-\varphi_0} F(r \operatorname{ch}(\varphi + \lambda), r \operatorname{ch}(\varphi - \lambda), t)\, d\varphi; \quad (14)^*$$

*ch = cosh; Arch = cosh⁻¹.

$$z_1 = r\,\mathrm{ch}\,(\varphi + \lambda), \qquad z_2 = r\,\mathrm{ch}\,(\varphi - \lambda),$$
$$\lambda = \mathrm{Arch}\sqrt{(z+1)/2}, \qquad \varphi_0 = \mathrm{Arch}\,(z_0/r)\,.$$

It is convenient to investigate the asymptotic behavior of $\rho(z, t)$ by making use of a simpler formula, which is derived from Eq. (14) when $z \gg 1$ and when the integrand does not decrease faster than z_1^{-1} or z_2^{-1}:

$$\rho(z, t) \sim \sqrt{\frac{t - 4m^2}{t}} \int\limits_0^1 \frac{r\,dr}{\sqrt{1-r^2}} \times \int\limits_{-1/2 \ln z}^{1/2 \ln z} F\left(\sqrt{\frac{z}{2}}\, r e^{\varphi}, \sqrt{\frac{z}{2}}\, r e^{-\varphi};\, t\right) d\varphi. \qquad (15)$$

From Eq. (15) it is immediately clear that if $F(z_1, z_2; t) \sim C_1(t)\, z_1^l z_2^m$ ($l, m \geq 1$),

$$\rho(z, t) \sim C_2(t) \begin{cases} z^l, & l > m \\ z^m, & m > l, \\ z^l \ln z, & l = m. \end{cases} \qquad (16)$$

If $l, m < -1$, nevertheless,

$$\rho(z, t) \sim C_3(t)\, z^{-1}. \qquad (17)$$

Furthermore, it is easy to see from (13) that if $F(z_1, z_2, t)$ does not change sign in the domain of integration, then

$$|\rho(z, t)| \geqslant C/z. \qquad (18)$$

Suppose now that $A^j(s, t, \bar{t})$ admits the following representation:

$$A_1^j(s, \bar{t}, t) \sim z^{n_j} f_j(t), \qquad (19)$$

$$A_2^j(s, \bar{t}, t) \sim \bar{z}^{\bar{n}_j} \bar{f}_j(t). \qquad (20)$$

Substituting (19) and (20) into (11) and (12), we can derive the asymptotic form of A^j_{31} and A^j_{32}. On the other hand, the order of growth of A^j_{31} must be less than or equal to the order of growth of A^j_1, i.e., n_j (the same is true for A^j_{32} and A^j_2, respectively).

Neglecting logarithmic behavior everywhere, we obtain finally a series of inequalities (the equalities arise from superpositions of several inequalities)

$$n_j = \bar{n}_j, \quad n_S = n_V + 1 = n_T + 1, \quad n_S \geqslant n_A \geqslant n_S - 2, \quad n_P \geqslant n_S. \qquad (21)$$

The first equality in (21) implies that the fermion-antifermion scattering cross section has the same behavior with increasing energy as the fermion-fermion cross section. However the coefficients of the corresponding powers of the energy can be different in the two cases.

We are now in a position to derive upper and lower bounds for the n_j. The lower bound can be found with the aid of the inequality (18). From (11) and (12) we obtain

$$n_S, n_P \geqslant 0; \qquad n_V, n_T, n_A \geqslant -1. \qquad (22)$$

Before determining the upper bound, we introduce helicity amplitudes [6,7]. There are five independent amplitudes ($\pm 1/2$ indicates the projection of spin on the corresponding momentum direction):

$$4\pi E\varphi_1 \equiv 4\pi E \langle +\tfrac{1}{2} +\tfrac{1}{2} |\varphi| +\tfrac{1}{2} +\tfrac{1}{2} \rangle = \tfrac{1}{2} m^2 (1+x)(A_S + A_V + A_T + A_A) + 2(p^2 A_V - E^2 A_A - m^2 A_T); \qquad (23)$$

$$4\pi E\varphi_2 \equiv 4\pi E \langle +\tfrac{1}{2} +\tfrac{1}{2} |\varphi| -\tfrac{1}{2} -\tfrac{1}{2} \rangle = 2(E^2 + p^2) A_T + 2m^2 A_A + \tfrac{1}{2}(x-1)\left[E^2 A_S + m^2 A_V + (E^2+p^2) A_T + m^2 A_B + p^2 A_P\right], \qquad (24)$$

$$4\pi E\varphi_3 = 4\pi E \langle +\tfrac{1}{2} -\tfrac{1}{2} |\varphi| +\tfrac{1}{2} -\tfrac{1}{2} \rangle = \cos^2(\theta/2)\{m^2(A_S + A_T) + (E^2 + p^2)(A_V + A_A)\}, \qquad (25)$$

$$4\pi E\varphi_4 = 4\pi E \langle +\tfrac{1}{2} -\tfrac{1}{2} |\varphi| -\tfrac{1}{2} +\tfrac{1}{2} \rangle = \sin^2(\theta/2)\{A_S E^2 + m^2(A_V + A_A + A_T) - p^2 A_P\}, \qquad (26)$$

$$4\pi\varphi_5 = 4\pi \langle +\tfrac{1}{2} +\tfrac{1}{2} |\varphi| +\tfrac{1}{2} -\tfrac{1}{2} \rangle = -\tfrac{1}{2} m \sin\theta \{A_S + A_V + A_T + A_A\};$$
$$x = \cos\theta, \qquad s = 4(p^2 + m^2) = 4E^2. \qquad (27)$$

The φ_j are simply related to the cross section:

$$d\sigma_j / d\Omega = |\varphi_j|^2, \qquad \mathrm{Im}\,\varphi_1(\theta = 0) = p\sigma^T_{++}/4\pi,$$
$$\mathrm{Im}\,\varphi_3(0) = p\sigma^T_{+-}/4\pi; \qquad (28)$$

$$\tfrac{1}{2}\,\mathrm{Im}\,[\varphi_1(0) + \varphi_3(0)] = p\sigma^T/4\pi, \qquad (29)$$

where σ^T_{++} denotes, for instance, the total interaction cross section for fermions with their spins along the initial momenta and σ^T denotes the total cross section, averaged with respect to spins.

The S-wave amplitude $f_0^J = \exp(i\delta_J)\sin\delta_J$ can also be expressed in terms of the φ_j:

$$f_0^J = \frac{p}{2} \int\limits_{-1}^{1} (\varphi_1 - \varphi_2) P_J(x)\, dx = \frac{1}{2p} \int\limits_{4m^2 - s}^{0} (\varphi_1 - \varphi_2) P_J\left(1 + \frac{t}{2p^2}\right) dt. \qquad (30)$$

We put $J = 0$ in this expression and assume that in integrating with respect to t the essential contribution comes from a region in which $\varphi_1 - \varphi_2$

can be factored into a product of a function depending only on t and a function depending only on s. If, for example, the essential contribution in the integral (30), comes for any large s from a finite domain of values for t, we can always use for $\varphi_1 - \varphi_2$ a representation of the form (19)—(20). Then the boundedness of f_0^J leads to the conclusion that $\varphi_1 - \varphi_2$ does not increase faster than p for fixed t.

From (23) and (24) it follows then, that n_S, $n_P \leq 1$ and $n_A \leq 0$. This, together with (22), implies the following bounds:

$$0 \leqslant n_S, n_P \leqslant 1, \qquad -1 \leqslant n_V, n_T, n_A \leqslant 0.$$

The n_J are related here by the inequalities (21).

Diffraction scattering corresponds to the case when σ^T and $d\sigma_j/dt|_{t=0}$ become constant as $s \to \infty$. Equation (28) then allows us to arrive at the conclusion that in this case $n_S = n_P = 1$, $n_V = n_T = 0$, and $-1 \leq n_A \leq 0$.

The asymptotic expression for the amplitudes φ_j (for fixed t) is

$$\varphi_1, \varphi_3, \varphi_5 \sim E^{2n_S-1}, \qquad \varphi_2, \varphi_4 \sim E^{2n_P-1}. \tag{31}$$

From the definition of φ_2, φ_3, φ_5 it follows that these amplitudes describe spin flip scattering for one or both particles (in the small angle limit). One can see that as $E \to \infty$ φ_2 and φ_4 are of the same order as φ_1 and φ_3 and therefore the relative contribution of the spin flip processes remains constant in the high energy limit.

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