INTEGRAL REPRESENTATIONS IN PERTURBATION THEORY

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Submitted to JETP editor April 12, 1962

J. Exptl. Theoret. Phys. (U.S.S.R.) **43**, 1371-1377 (October, 1962)

For integral representations obtained on the basis of perturbation theory a study is made of the boundaries of the region where the spectral function becomes zero for the case of equal masses.

1. INTRODUCTION

In the integral representations of amplitudes which are used in the construction of a theory on the basis of unitarity and causality, analytical properties express the causality in interaction processes. Unfortunately these representations —for example, the representations of Mandelstam [1] and of Jost, Lehmann, and Dyson [2] —cannot be extended to the case of a large number of particles. No proof of the Mandelstam representation has been found for the case in which perturbation theory is valid. The amplitudes (Green's functions) obtained from perturbation theory can be written in a form which gives a certain integral representation of these quantities. For the case in which perturbation theory is valid this way of writing the amplitudes is simply a restatement of the α-representation of the Feynman integrals, which involves loss of part of the information contained in the theory.

For the recovery of the perturbation-theory series it is necessary to use unitarity. If, on the other hand, perturbation theory does not apply, the writing of the amplitudes in the form of the representations in question is a condition on the amplitudes of the theory, owing to which their analytical properties are connected with the properties of Feynman diagrams. We consider such integral representations in the present paper. The structure of perturbation theory has recently been actively studied in a number of papers.[3-7] We shall use the technique developed in [5]. The case considered is that of scalar particles. The notations are the same as in [5]: indices i, k, and so on number the vertices of the diagram. A process involving N particles is characterized by the four-momenta $p_1, \ldots, p_N$, which are all regarded as entering the diagram, so that the conservation law for the external momenta is of the form

$$\sum_{i=1}^{N} p_i = 0. \tag{1}$$

The process is characterized by the invariants $p_i^2 = M_i^2$ and $p_i p_k$; we shall number the invariants $2p_i p_k$ $(i, k \neq N)$ in some order and denote them by s_l.

The vertex indices $1, \ldots, N$ refer to external vertices, the others to internal vertices of the diagram.

The integral represented by the Feynman diagram is proportional to [4]

$$F = \int_0^1 \ldots \int_0^1 \frac{\prod_{ik} d\alpha_{ik}\delta(1 - \Sigma\alpha_{ik}) L(\alpha_{ik})}{(\varphi - i\varepsilon)^{n-2l}}, \tag{2}$$

where n is the number of lines and l is the number of independent circuits of the diagram. According to [5] the function φ is identical with

$$f = \sum \alpha_{ik} (m_{ik}^2 - q_{ik}^2) \tag{3}$$

with

$$q_{ik} = \beta_{ik}(a_i - a_k), \qquad \beta_{ik} = 1/\alpha_{ik}, \tag{4}$$

$$\sum_k q_{ik} = p_i \tag{5}$$

and is of the form

$$\varphi = \sum \frac{m_{ik}^2}{\beta_{ik}} - \sum_{i,\, k \neq N} A_{ik} p_i p_k. \tag{6}$$

The method of [5] can be applied to find the A_{ik} in the following way: by means of the reduction formulas

$$\beta_{ik}^{(n)} = \frac{\beta_{is}^{(n-1)}\beta_{ks}^{(n-1)}}{\sum_l \beta_{il}^{(n-1)}} + \beta_{ik}^{(n-1)}, \qquad \beta_{ik}^{(0)} = \beta_{ik} \tag{7}$$

one eliminates all of the vertices of the diagram except the vertices i, k, N. We get ($\tilde{\beta}$ denotes the parameters remaining as the result of all the eliminations)

$$A_{ik} = \tilde{\beta}_{ik}/\Delta, \qquad A_{ii} = (\tilde{\beta}_{iN} + \tilde{\beta}_{ik})/\Delta,$$

$$\Delta = \tilde{\beta}_{kN}\tilde{\beta}_{iN} + \tilde{\beta}_{kN}\tilde{\beta}_{ik} + \tilde{\beta}_{iN}\tilde{\beta}_{ik}; \tag{8}$$

consequently, for $\beta > 0$

$$A_{ii} > A_{ik} > 0. \tag{9}$$

By means of the relations

$$\sum_{l=1}^{\nu} s_l = 2M_N^2 - \sum_{i=1}^{N} M_i^2 \qquad (\nu = (N-1)(N-2)/2) \tag{10}$$

we eliminate from Eq. (6) the invariant s_{l_0} with the smallest coefficient for given values of β, and reduce φ to the form

$$\varphi = \sum \frac{m_{ik}^2}{\beta_{ik}} - \sum_{i=1}^{N} z_{0i}M_i^2 - \sum_{i \neq l_0}^{\nu} z_l s_l. \tag{11}$$

There exist ν regions of variation of the β, in each of which it is one of the s_l which has the smallest coefficient. Breaking up the integral in Eq. (2) into integrals over these ν regions and changing from the variables α_{ik} to variables y, x_i,

$$y = \frac{1}{\sum z_i}\left(\sum \frac{m_{ik}^2}{\beta_{ik}} - \sum z_{0i}M_i^2\right), \qquad x_i = z_i/\sum z_i, \tag{12}$$

we reduce Eq. (2) to the form

$$F = \int dy \int_0^1 dx_1 \int_0^1 \ldots \int_0^1 dx_\nu \times \frac{\sum^{\nu} \delta(x_l)\,\rho(y, x_1, \ldots, x_\nu)\,\delta(1 - \sum x_l)}{y - \sum x_l s_l - i\varepsilon}. \tag{13}$$

We note that the Jacobian for the change from α_{ik} to y, x_i has a singularity, and that the equations for the position of the singularity are identical with the equations of Landau [3] for the singular curves of the diagram.

In the case in which the Feynman integral is divergent we must consider the regularized value of this integral. If the representation (13) is assumed valid for the entire amplitude (and not only for the terms of the series in the coupling constant), then the convergence of the integral in Eq. (13) is associated with a necessary number of subtractions, which must be prescribed as an independent parameter in the theory.

An important characteristic of the representation (13) is the lower limit of the spectrum, i.e., the value $y_0 = \min y$, for which

$$\rho(y, x_1, \ldots, x_\nu) \neq 0. \tag{14}$$

It is easy to see that if there is a region U in the space of the invariants such that for $s_i \in U$ and for all β_{ik}

$$\varphi(\beta, s_i, m_{ik}) \geqslant 0, \tag{15}$$

then

$$\min y \geqslant \max \min s_l; \tag{16}$$

$\max \min s_l$ means that we take the largest value of the smallest of the $\nu - 1$ invariants in the region U.

The region $\varphi \geq 0$ is the region of analyticity for the given diagram. To find the intersection of the regions of analyticity for all the diagrams for a given process we use the technique of the theory of majorization of the Feynman diagram. [4,7] We note that in the region of Euclidean behavior of the external momenta p_i, when we choose the q_{ik} as linear combinations of the p_i, we get from Eqs. (3)–(6)

$$f - \varphi = -k_{ik}p_i p_k \leqslant 0. \tag{17}$$

Therefore if for Euclidean p_i we can get the q_{ik} as linear combinations of the p_i which satisfy Eqs. (4) and (5) and

$$m_{ik}^2 - q_{ik}^2 \geqslant 0, \tag{18}$$

then s_i is in the region $\varphi \geq 0$.

We shall consider the simplest case with equal masses m_{ik} and M_i. In what follows it is convenient to go over to the invariants $(p_i + p_k)^2$; obviously there is no change in the representation (13) and the other formulas. The problem of finding the lower limit of the spectrum for $N \leq 5$ has been treated by Nakanishi [6] by a different method. For $N = 4$ the results of the present paper are identical with Nakanishi's results.

2. THE CONSTRUCTION OF MAJORIZING DIAGRAMS

When represented in the form (13) diagrams which are disconnected or have pole terms reduce to the case of a smaller number of external lines. It is convenient at once to exclude from consideration the trivial case of the diagram of the first order. The insertion of self-energy parts does not change our treatment, and therefore we can suppose that there are no self-energy insertions. For the transformation of the diagrams which we shall consider the following operations are required: a) elimination of a line, b) insertion of a line at a vertex, c) splitting of an external line.

Operation a) means that $q_{ik} = 0$ on the line to be eliminated; then if only two lines meet at one of the internal vertices they are replaced by a single line; b) changes the diagram into a diagram with the numbers of lines and vertices increased by unity, so that when the inserted line is contracted to a point the diagram returns to the original form; operation c) replaces one external line by two lines, so that the sum of their momenta equals that of the line that was split. If for given p_i the diagram obtained by the application of a), b), or c) satisfies Eqs. (5) and (18), then the original diagram also satisfies these conditions. By means of operation b) we can always convert a vertex at which n lines meet into two vertices, one with three lines and the other with $n-1$ lines converging to it. Thus we reduce a problem with an arbitrary interaction to a problem with three lines meeting at each vertex.

Starting from some external vertex of the diagram we can go through some sequence of internal lines to another external vertex without traversing any line twice. Then we start from a third vertex and pass over internal lines of the diagram until we come to a vertex through which we already passed in going from the first external line to the second, and so on. We eliminate the lines that are not traversed in doing this. We then get the diagram of Fig. 1, where the external vertices are denoted by crosses. By means of operation b) one can always arrange matters so that not more than two lines meet at an external vertex.

Let us first consider the cases $N = 3$ and $N = 4$ (Fig. 2). For $N = 3$ and equal masses, p_1, p_2, and p_3 are always Euclidean, and

$$(p_1+p_2)^2=(p_1+p_3)^2=(p_2+p_3)^2=p^2=M^2. \qquad (19)$$

For this case the conditions (5) and (18) are satisfied for $m = M$.

For $N = 4$ we choose p_1 and p_2 to coincide with the corresponding vectors for $N = 3$. In this case

$$(p_1+p_2)^2+(p_1+p_3)^2+(p_2+p_3)^2=4M^2. \qquad (20)$$

We set

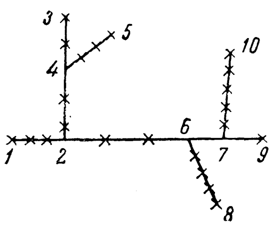

FIG. 1.

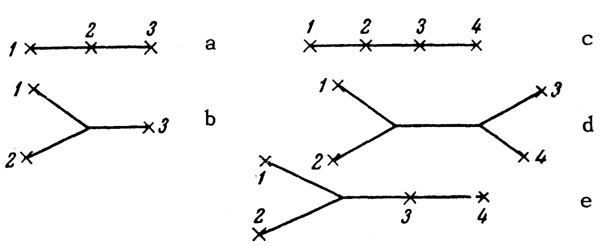

FIG. 2

$$(p_1+p_3)^2=M^2. \qquad (21)$$

For $m = M$ the conditions (5) and (18) are satisfied in this case also. Let us return to the diagram of Fig. 1. We consider a part of it included between an external vertex from which there comes only one line and the first vertex where more than two lines meet (the tail of the vertex); for example, the part between vertices 1 and 2. By eliminating some of the lines of the tail, we form from it several diagrams of the type of Fig. 2, a; here one or two external vertices can remain in the diagram. Proceeding in this way with all of the tails, we can continue the separation of the diagrams of Fig. 2, combining the remnants of tails that meet at one vertex.

The diagram of Fig. 1 is thus broken up into diagrams of Fig. 2, and possibly one diagram with $N = 2$. We orient all diagrams with $N = 3$, choosing corresponding vectors in different diagrams to be equal, and do the same with the diagrams with $N = 4$, so that the vectors corresponding to p_2 and p_3 in Fig. 2 are equal to two of the three vectors of the diagram with $N = 3$, while the other two vectors must satisfy the conditions (20) and (21). If there is a diagram with $N = 2$ we choose one of the vectors equal to one of those already present and exclude the other from the set of vectors from which the invariants are formed. Our set of vectors is Euclidean, the conditions (5) and (18) are satisfied, and it is not hard to show that the smallest invariant is

$$\min s_l = M^2. \qquad (22)$$

It follows from this that in the case of equal masses we have for any N

$$y_0 \geqslant M^2. \qquad (23)$$

For more accurate estimates we must construct more complicated majorizing diagrams. By setting $p_l = 0$, $l = 3, \ldots, N-1$ we get a three-point diagram, and by means of operations a) and b) we can always reduce this to the form of Fig. 3. When we then reintroduce the fourth vertex, the fifth vertex, and so on, and part of the internal lines of the diagram, we can construct diagrams which do not come apart when one internal line is eliminated. To obtain the internal momenta it is convenient to

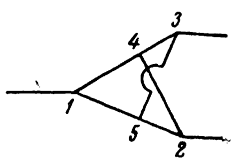

FIG. 3

use the condition

$$q_{ik} = a_i - a_k. \tag{24}$$

As for the choice of the region in the space of the invariants, for N = 5 it is necessary to take into account nonlinear equations connecting the invariants. Therefore we shall confine ourselves to the cases N = 4 and N = 5.

3. FOUR- AND FIVE-POINT DIAGRAMS

For N = 4 the quantity (11) has been calculated in [5]. If we choose the invariants in the forms

$$s_1 = (p_1 + p_2)^2, \qquad s_2 = (p_1 + p_3)^2, \qquad s_3 = (p_2 + p_3)^2; \tag{25}$$

$$s_1 + s_2 + s_3 = \sum_{i=1}^{4} M_i^2 \tag{26}$$

[we choose the numbering of the external lines so that invariant s_3 is subject to elimination by means of Eq. (26)] then

$$\varphi = \sum \alpha_{ik} m_{ik}^2 - \sum z_{0i} M_i^2 - z_1 s_1 - z_2 s_2, \tag{27}$$

with

$$z_1 \geqslant 0, \qquad z_2 \geqslant 0, \qquad z_{0i} \geqslant 0, \qquad i = 1, \ldots, 4. \tag{28}$$

In the case of equal external masses we take as the values of the invariants

$$s_1 = 2M^2, \qquad s_2 = 2M^2, \qquad s_3 = 0. \tag{29}$$

Let us construct the majorizing diagrams. They are all obtained from the diagram of Fig. 3 by bringing in a fourth external vertex and two internal lines connecting it with the diagram. Any closed nonselfintersecting circuit in the diagram of Fig. 3 consists of four internal lines. Let us single out that one of the paths of this kind whose lines have the lines from the fourth external vertex joined to them. We split each of the external lines of the diagram into two lines; we insert one internal line at each of the external vertices that lie on the circuit we have singled out, and replace the internal lines connecting the vertices of this circuit with the other vertices by external lines with the momenta

$$q_{il} = p_l/2, \tag{30}$$

where the index l refers to the external vertices that do not lie on the circuit. We get a diagram in the shape of a simple circuit with eight external vertices, with the momentum k_i coming to each,

$$k_i = \gamma_{il} p_l; \qquad \gamma_{il} = 0,\ 1/2; \qquad i = 1, \ldots, 8, \qquad l = 1, \ldots, 4, \tag{31}$$

(there is no summation over l). From Eqs. (5) and (24) we find q:

$$q_{lm} = \frac{1}{8} \sum_{L=1}^{8} L k_{l+L}, \qquad k_{8+L} = k_L. \tag{32}$$

From Eq. (32) we have

$$q_{lm}^2 \leqslant M^2/2. \tag{33}$$

The condition (30) does not impair this inequality. From this and Eq. (27) we find

$$\left(\sum \alpha_{ik} m^2 - 2 \sum z_{0i} m^2\right) \Big/ (z_1 + z_2) \geqslant 4m^2 \tag{34}$$

and in consequence of Eq. (28)

$$y = \left(\sum \alpha_{ik} m^2 - \sum z_{0i} m^2\right) \Big/ (z_1 + z_2) \geqslant 4m^2. \tag{35}$$

For N = 5 the majorizing diagrams are obtained by reinserting two vertices in the diagram of Fig. 3. We choose the region of the invariants

$$p_i p_k = -M^2/4; \qquad i \neq k \tag{36}$$

and proceeding as in the case N = 4 we get

$$q^2 \leqslant M^2/2. \tag{37}$$

It follows from Eqs. (36) and (37) that

$$\left(\sum \alpha_{ik} m^2 - 2 \sum z_{0i} m^2\right) \Big/ \sum z_i \geqslant 3m^2. \tag{38}$$

Let us consider the expression

$$\varphi' = \sum_{i=1}^{5} z_{0i} M^2 + 2 \sum_{i=1}^{5} z_i s_i. \tag{39}$$

Since the value of φ' is

$$f = \sum \alpha_{ik} q_{ik}^2 \tag{40}$$

for certain values q_{ik} we are in the Euclidean region, and φ' is not negative. The invariants s_i satisfy the equation

$$\sum_{i=1}^{6} s_i = M_5^2 + 2 \sum_{i=1}^{4} M_i^2 \tag{41}$$

[we suppose that the s_i are $(p_l + p_m)^2$; $l, m \neq 5$]. We take

$$s_6 = 4M^2, \qquad s_1 = s_2 = \ldots = s_5 = M^2. \tag{42}$$

Using the fact that $\varphi' \geq 0$, we get from Eq. (39)

$$\sum z_{0i} \Big/ \sum z_i \geqslant -1. \tag{43}$$

Combining Eqs. (43) and (38), we find

$$y_0 = \min \frac{\sum \alpha_{ik} m^2 - \sum z_{0i} m^2}{\sum z_i} \geqslant 2m^2. \tag{44}$$

The case N > 5 can be treated in an analogous way. In the case of arbitrary unequal masses it is necessary to use the conditions of stability of the particles, which greatly complicates the discussion. It is obvious that an increase of the internal masses does not decrease y_0. As for the external masses, one must figure out the signs of the quantities z_{0i}, which can be done by the method of [5], but requires cumbersome calculations when N is large. It follows from the results for N = 4 that a decrease of the external mass does not decrease y_0.

Note added in proof (September 15, 1962). When the present paper was in press the author learned of a new preprint by Nakanashi in which representations of just the same form are obtained, and problems of the uniqueness of such representations are discussed on the assumption that a limit on y_0 exists.

[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).
[2] F. J. Dyson, Phys. Rev. **110**, 1460 (1958).
[3] L. D. Landau, JETP **37**, 62 (1959), Soviet Phys. JETP **10**, 45 (1960).
[4] N. Nakanashi, Supplement to Progr. Theoret. Phys. **18**, 1 (1961).
[5] A. Z. Patashinskiĭ, JETP **42**, 812 (1962), Soviet Phys. JETP **15**, 567 (1962).
[6] N. Nakanashi, Preprint.
[7] Logunov, Todorov, and Chernikov, Preprint D-578, Dubna, 1960.

Translated by W. H. Furry
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