QUANTUM FIELD THEORY IN CONSTANT CURVATURE p-SPACE

Yu. A. GOL'FAND

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

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A quantum field theory formalism in constant-curvature p-space is proposed, by which one can calculate the matrix elements of any process. The theory is formulated in elliptic p-space; this corresponds to the "Euclidean" formulation of the usual theory. All expressions encountered in the theory are finite. The Schwinger equation for the Green's function is generalized. Some features of the theory connected with non-commutativity of the displacement operator in constant curvature space are considered.

INTRODUCTION

THE difficulties of contemporary (local) field theory are apparently connected with the fact that the law governing the interaction between particles ceases to be valid at large values of the momentum. To construct a consistent theory it is necessary to change the law of interaction in suitable fashion. Since addition of the momenta of the interacting particles occurs in each act of elementary interaction (i.e., absorption or emission of a meson or a quantum), it is necessary first to modify the law governing the addition of momenta. To generalize the momentum-addition law it is necessary to have a guiding geometrical principle. One such principle calls for changing from the pseudo-Euclidean p-space to a corresponding space of constant curvature. The radius of curvature μ_0 of this space plays the role of the cutoff parameter. This method of generalizing the field theory was proposed earlier [1].

In [1] we formulated some diagram-technique rules based on the new law of momentum addition. In the simplest cases these rules made it possible to construct uniquely the corresponding matrix element. However, for more complicated diagrams (higher approximations for the vertex parts and particularly the energy parts), the applications of the aforementioned rules have led to difficulties connected with the noncommutativity of the new law of momentum addition. To develop the theory further it became necessary to construct a suitable procedure which would lead to unique expressions for arbitrary matrix elements, similar to what is done in ordinary field theory. The present article is devoted to a solution of this problem.

In constructing the procedure for the theory we have incidentally refined and represented in clearer form certain basic aspects of the theory. Let us note the most important of these. Redefinition of p-space means physically a change in the laws governing particle interaction at large momenta. The result is a theory without divergences, while the theory of the "free" particles retains its previous form. Speaking more accurately, we assume for the Green's function of the noninteracting particles the same expression as in ordinary theory. This premise will be introduced for the time being in the form of a postulate, with the idea of deriving it later on from a more general principle.

The constant-curvature p-space introduced in [1] just like pseudo-Euclidean p-space, has an indeterminate metric. We shall henceforth call such a p-space pseudo-elliptic. An evaluation of the integrals over the momenta of the virtual particles in pseudo-elliptic space encounters a difficulty connected with the singularity of the volume element on the boundary of the "physical" region ($p^2 = 1$). To overcome this difficulty we formulated in [1] a special rule for circuiting around the singularity, with subsequent rotation of the contour of integration with respect to the variable p_0. Such a method of calculating the integrals denotes in fact a transition from a pseudo-elliptic p-space to a corresponding elliptic space with a positive-definite metric. We note that an analogous method of calculating the Feynman integrals with transition from pseudo-Euclidean p-space to Euclidean space is frequently useful in ordinary theory.

However, the introduction into the theory of an additional calculation rule (which in itself is logically unsatisfactory) makes it quite difficult to obtain a consistent formulation of the field procedure.

A more fruitful way is to construct the principles of the theory directly in elliptic p-space (see

Sec. 2). In the case of ordinary field theory, analogous ideas of "Euclidean" formulation of the theory were developed in the papers by Schwinger [2], Nacano [3] and Fradkin [4]. They have shown that the equations of field theory can be constructed in four-dimensional Euclidean coordinate or momentum space. All the intermediate derivations (integration over the virtual states) are carried out in the same space, and only in the final expressions is it necessary to carry out the corresponding analytic continuation from Euclidean quantities to pseudo-Euclidean ones. In the case of local field theory, the Euclidean formulation leads to the same results as the ordinary formulation.

The relation between the elliptic and pseudo-elliptic spaces generalizes the relation between the Euclidean and pseudo-Euclidean spaces. However, from among all the constant-curvature spaces, elliptic space is outstanding in that its volume is finite. Therefore, if elliptic p-space is chosen as the space of the "virtual" momenta (from the physical point of view this raises no objections whatever), then all the integrals will be convergent. In the matrix elements of the real processes it is necessary to carry out the corresponding analytic continuation over the momenta of the real particles entering into the reaction. (A separate paper will be devoted to questions connected with this analytic continuation.)

Thus, whereas in the case or ordinary theory the introduction of a Euclidean space of "virtual" momenta is merely a convenient technical tool, in our present case the corresponding elliptic p-space plays a principal role, since it makes possible the construction of a theory free of divergences.

The last remarks concerns the principle of correspondence with the ordinary theory. The theory developed here satisfies the correspondence principle in the sense that its results go over into the ordinary results at small momenta ($p_\mu \ll \mu_0$, where μ_0 is the radius of curvature of the p-space). This circumstance, however, is directly self-evident only in those cases when the ordinary theory leads to convergent results. The most interesting singularities of the new theory arise where the ordinary theory gives rise to infinities (see Section 5).

1. ELLIPTICAL p-SPACE

The general theory of constant-curvature space is developed in the book by Klein [5] on the basis of projective geometry. The geometrical information on the pseudo-elliptical space needed for the development of the theory was briefly given in [1]. These questions were considered in greater detail in the paper by Kadyshevskiĭ [6] in which, in particular, the interesting possibility is pointed out of introducing two different pseudo-elliptic metrics ($\varepsilon = \pm 1$) in p-space. Bearing in mind the formulation of the theory in elliptic p-space (Sec. 2), we present here briefly the necessary information concerning this space.

A point in four-dimensional elliptic p-space is defined by four real coordinates $p_\mu = (p_1, p_2, p_3, p_4)$.

The quantity p_μ can be regarded as a four-vector with respect to the four-rotation group. Accordingly, we define the scalar product of two vectors p and q by means of the equality

$$pq = \sum_{\mu=1}^{4} p_\mu q_\mu. \qquad (1.1)$$

As in [1] we choose as the measurement unit the quantities c, ħ, and the p-space radius of curvature μ_0. The distance between two points in elliptic p-space is defined by the relation

$$s(p, q) = \arccos\left|\frac{1+pq}{\sqrt{1+p^2}\sqrt{1+q^2}}\right|, \quad 0 \leqslant s \leqslant \pi/2. \qquad (1.2)$$

It is clear from (1.2) that the metric of elliptic p-space is positive definite. With the aid of (1.2) we obtain the differential metric form

$$ds^2 = (1+p^2)^{-2}\{(1+p^2)\,dp^2 - (pdp)^2\} \qquad (1.3)$$

and the volume element

$$d\Omega_p = (1+p^2)^{-5/2}\,d^4p. \qquad (1.4)$$

Integrating (1.4) over all of space, we find the total volume of the elliptic p-space, which turns out to be

$$\Omega = \int \frac{d^4p}{(1+p^2)^{5/2}} = \frac{4}{3}\pi^2. \qquad (1.5)$$

The finite nature of the space volume Ω plays the decisive role for the convergence of the integrals with respect to the virtual momenta.

Transformations that conserve the metric (1.2) are called motions of elliptic p-space. Obviously, four-rotations which conserve the scalar product (1.1) are motions. There exist also motions which are not rotations and are described by bilinear transformations of the vector p. These are the so-called displacement transformations. A displacement of a vector p by a vector k is defined by the relation [1)]

[1)]Previously the displacement of a vector p by a vector k was denoted by the symbol p(+)k (see [1,6]). We avoid this method of notation in the present article, as well as the unfortunate terms "addition" and "sum" of vectors, used in [1] to denote displacements. We note also that a misprint has crept into formula (2.3) of [1], which defines the displacement. Actually, this formula has the form

$$q = p(+)k = \left[p\sqrt{1-k^2} + k\left(1+\frac{pk}{1+\sqrt{1-k^2}}\right)\right] \Big/ (1+pk).$$

$$q = d_0(k)\,p = \left[p\sqrt{1+k^2} + k\left(1 - \frac{pk}{1+\sqrt{1+k^2}}\right)\right]/(1-pk). \tag{1.6}$$

If both vectors p and k are small (p_μ, $k_\mu \ll 1$) then, accurate to terms of third order of smallness, the displacement (1.6) coincides with ordinary addition of vectors, $q \approx p + k$. Thus, the displacement operation (1.6) generalizes the ordinary vector addition. In Sec. 2 we shall use the displacement (1.6) to generalize the law of addition of momenta at the nodes of the Feynman diagrams. It must be emphasized, however, that the displacement properties (1.6) are generally speaking quite different from the properties of ordinary addition. In particular, the displacement operation is not commutative with respect to the vectors p and q. It is curious to note that the square of the vector q, defined by Eq. (1.6), depends in commutative fashion on the vectors p and k. This is clear from the relation

$$1/(1+q^2) = (1-pk)^2/(1+p^2)(1+k^2). \tag{1.7}$$

We note also that displacement by a vector—k is an operation which is the inverse with respect to displacement by a vector k, something that can be symbolically expressed by the equality

$$d_0(-k)\,d_0(k) = 1. \tag{1.8}$$

Two displacements $d_0(k_1)$ and $d_0(k_2)$ do not, generally speaking, commute with each other.

Let us consider a scalar function of the vector p, i.e., f(p). If the vector p is subjected to a displacement by a constant vector k, then the function f(p) goes over into a different function g(p). The function g(p) is by definition the result of the action of the displacement operator on the function f(p). In accord with the foregoing, the displacement operator $d_0(k)$ is defined (for a scalar function) by the equation

$$g(p) \equiv \hat{d}_0(k)\,f(p) = f(d_0(-k)\,p). \tag{1.9}$$

Starting from (1.9), we can readily find the operators of an infinitesimally small displacement (see also [6]). These operators are the analog of the Snyder "coordinate" operator [7] for the case of an elliptic p-space metric. In explicit form the operators of the "coordinates" are expressed by the formulas

$$\hat{x}_\mu = i\,\{\partial/\partial p_\mu + p_\mu(p\partial/\partial p)\}. \tag{1.10}$$

The displacement transformations together with the four-rotation of p-space form a group which is isomorphous to the rotation group in five-dimensional space. This can be readily verified in the following fashion. We introduce the five-momentum operators M where

$$M_{\mu\nu} = -i\,(p_\mu\partial/\partial p_\nu - p_\nu\partial/\partial p_\mu), \qquad \mu,\nu = 1,\ldots,4 \tag{1.11}$$

are operators of infinitesimally small four-rotation, and

$$M_{\mu 5} = -M_{5\mu} = \hat{x}_\mu. \tag{1.12}$$

The momentum operators (1.11) and (1.12) satisfy the following commutation relations:

$$[M_{\alpha\beta}, M_{\beta\gamma}] = -iM_{\gamma\alpha}, \qquad \alpha,\beta,\gamma = 1,\ldots,5,$$
$$\alpha \neq \beta, \qquad \beta \neq \gamma. \tag{1.13}$$

All the remaining commutators vanish. From the form of the commutation relations (1.13) it follows that the momentum operators $M_{\alpha\beta}$ are operators of infinitesimally small transformations for a certain representation of a group of five-dimensional rotations.

The connection between the displacements and five-dimensional rotations makes it possible to define the displacement operations for spinor functions of the vector p. According to general spinor theory (see, for example, [8]) a four-component spinor defines a spinor representation of a group of five-dimensional rotation. To construct this representation we define the matrices Γ_α ($\alpha = 1, \ldots, 5$):

$$\Gamma_\mu = \gamma_5\gamma_\mu \quad (\mu = 1,\ldots,4); \qquad \Gamma_5 = \gamma_5, \tag{1.14}$$

where γ_m (m = 1, 2, 3) are ordinary Dirac matrices, $\gamma_4 = -i\gamma_0$, and $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. The matrices Γ_α satisfy the relations

$$\Gamma_\alpha\Gamma_\beta + \Gamma_\beta\Gamma_\alpha = 2\delta_{\alpha\beta}. \tag{1.15}$$

With the aid of the Γ_α matrices we define the momentum operators

$$M_{\alpha\beta} = (-i/4)\,(\Gamma_\alpha\Gamma_\beta - \Gamma_\beta\Gamma_\alpha), \tag{1.16}$$

which satisfy the commutation relations (1.13). In particular, according to (1.12), the operator of infinitesimally small displacement has for spinors the form

$$\hat{x}_{s\mu} = \tfrac{1}{2}\,i\gamma_\mu. \tag{1.17}$$

Knowing the infinitesimally small displacement operator (1.17), we can define an operator corresponding to a finite displacement (1.6). This operator can be represented in the form

$$\hat{d}_s(k) = [(1-\hat{k})/(1+\hat{k})]^{1/4}, \tag{1.18}$$

where $\hat{k} = k_\mu\gamma_\mu$. The operator (1.18) defines a displacement for a constant spinor ψ. For a spinor

$\psi(p)$, which depends on the vector p, the displacement is defined by the product of the operators (1.9) and (1.18):

$$\hat{d}(k) = \hat{d}_0(k)\hat{d}_s(k). \quad (1.19)$$

There exists a simple connection between the elliptical geometry of p-space considered here and various pseudo-elliptic geometries, since all these geometries become identical in the region of complex values of p_μ. Starting from any relation of the elliptic geometry, it is possible to obtain the previously considered corresponding relations of the pseudo-elliptic geometry [1] by putting $p_4 = \pm ip_0$, where p_0 is a real number. It is important here that the expressions that are invariant with respect to four-rotation of p-space in elliptic geometry go over into Lorentz-invariant expressions of pseudo-elliptic geometry. This circumstance guarantees relativistic invariance of the theory in the physical region.

The connection between two forms of pseudo-elliptic geometry, considered by Kadyshevskiĭ [6], is realized by analytic continuation to the values $p_\mu = iq_\mu$, where q_μ is a real vector.

2. DEVELOPMENT OF THE PROCEDURE

If the p-space is a constant-curvature space, then the coordinate operators (1.10) do not commute with one another. This circumstance, already noted by Snyder [7], makes it necessary to formulate the theory directly in the p-representation. In order to display the method of generalizing the theory in clearest fashion, it is convenient to use the following leading arguments.

In general field theory with local interaction, the scattering matrix has the form

$$S = Te^{i\Lambda}, \quad (2.1)$$

where

$$\Lambda = \int L(x)\,dx. \quad (2.2)$$

Let us consider (for the sake of being definite) the case of pseudo-scalar meson theory. The interaction Lagrangian in formula (2.2) has the form

$$L(x) = g\bar{\psi}(x)\gamma_5\varphi(x)\psi(x). \quad (2.3)$$

We change over to the p-representation by expanding the fields in Fourier integrals

$$\psi(x) = \frac{1}{(2\pi)^2}\int\psi(p)e^{-ipx}dp, \qquad \bar{\psi}(x) = \frac{1}{(2\pi)^2}\int\bar{\psi}(p)e^{ipx}dp,$$
$$\varphi(x) = \frac{1}{(2\pi)^2}\int\varphi(k)e^{-ikx}dk. \quad (2.4)$$

Taking (2.3) and (2.4) into account, we can represent the quantity Λ in the form

$$\Lambda = \frac{g}{(2\pi)^2}\int\varphi(k)\,dk\int\bar{\psi}(p)\gamma_5 e^{-ik\hat{x}}\psi(p)\,dp. \quad (2.5)$$

In formula (2.5), $\hat{x}_\mu = i\partial/\partial p_\mu$ is the coordinate operator, which is an operator of infinitesimally small displacement in p-space. The quantity $e^{-ik\hat{x}}$ is a finite-displacement operator, inasmuch as

$$e^{-ik\hat{x}}\psi(p) = \psi(p-k). \quad (2.6)$$

The quantity Λ, written down in the form (2.5), can be readily generalized to the case of elliptic p-space. For this purpose it is sufficient to replace the Euclidean displacement operator (2.6) by the displacement operator defined in (1.9), and extend the integration over all of the elliptic space, taking into account the form (1.4) of the volume element of this space.

In formulating the field theory in elliptic p-space, it is very convenient to make use of the formalism of so-called causal operators or quasi-fields, introduced from different points of view in the papers of Novozhilov [9], Coester [10], and the author [11]. The meaning of the quasi-fields lies in the fact that the algebra of their natural multiplication duplicates the algebra of T-products of ordinary fields. Because of this, the main properties of the T-product turn out to be independent of the x-representation of the operators and can be readily transferred to any other representation.

All the vectors p, k, . . . , encountered from now on belong to the elliptic p-space. We define the quasi-field operators in elliptic p-space by means of the equations

$$\psi(p) = (m+\hat{p})^{-1/2}\{a(p) + b^+(-p)\}, \quad (2.7)$$
$$\bar{\psi}(p) = \{a^+(p) + b(-p)\}(m-\hat{p})^{-1/2}\gamma_5,$$
$$\varphi(k) = (\mu^2 + k^2)^{-1/2}\{c(k) + c^+(-k)\}. \quad (2.8)$$

The quantities ψ and $\bar{\psi}$ are spinors, while φ is a pseudo-scalar. The amplitudes a, b, and c are by definition annihilation operators, and the corresponding amplitudes a^+, b^+, and c^+ are creation operators.

The commutation relations between the amplitudes have the following form (we write out only the nonvanishing brackets).

$$[a(p), a^+(q)]_+ = -[b(p), b^+(q)]_+ = \gamma_5\delta(p,q), \quad (2.9)$$
$$[c(k), c^+(l)] = \delta(k,l). \quad (2.10)$$

The δ-functions are defined by the equation in the right halves of the commutation relations

$$f(p) = \int\delta(p,q)f(q)\,d\Omega_q \quad (2.11)$$

for an arbitrary function f(p).

The commutation relations (2.9) and (2.10) define a certain natural metric in the Hilbert space of the states. This metric is not positive definite. In the natural metric the creation operator a^+ is conjugate to the annihilation operator a, etc. The quasi-field operators satisfy the following conjugation relations:

$$\bar{\psi}(p) = \psi^+(p)\gamma_5, \tag{2.12}$$

$$\varphi^+(k) = \varphi(-k). \tag{2.13}$$

From relation (2.13) it follows that $\varphi(k)$ is a real field.

With the aid of the commutation relations (2.9) and (2.10) we can readily establish the following properties of quasi-fields:

1. The operators $\psi(p)$ and $\bar{\psi}(q)$ anticommute for any pair of points in p-space. The operators $\varphi(k)$ commute with one another and with the operators (2.7) for any pair of points of p-space.

2. The vacuum-averaged products of the quasi-field operators (2.7) and (2.8) are defined by the equations

$$\langle\psi(p)\bar{\psi}(q)\rangle_0 = (m+\hat{p})^{-1}\delta(p,q),$$

$$\langle\varphi(k)\varphi(l)\rangle_0 = (\mu^2+k^2)^{-1}\delta(k,-l). \tag{2.14}$$

All the remaining averages of the products of the two operators (2.7) and (2.8) vanish. In the case of elliptic p-space, relations (2.14) are the analog of the well known expressions for the vacuum-averaged T-product of the ordinary field operators.

Let us introduce a notation which makes it possible to write down the formulas that follow in more compact form. Let $\hat{B}$ be an operator acting on the spinor function $\psi(p)$, defined in all of p-space. By definition

$$\langle\bar{\psi}|\hat{B}|\psi\rangle = \int\bar{\psi}(p)\langle p|\hat{B}|q\rangle\psi(q)\,d\Omega_p d\Omega_q. \tag{2.15}$$

The quantity $\langle p \mid \hat{B} \mid q\rangle$ in formula (2.15) is the matrix element of the operator $\hat{B}$ in the p-representation. In particular, it follows from (2.11) that the function $\delta(p, q)$ is a matrix element of the unit operator. Using this fact, we can readily obtain different relations analogous to the properties of ordinary δ-functions for the functions $\delta(p, q)$. The quantity ψ in (2.15) need not be a c-number. Later on we shall use ψ in expressions such as (2.15) to denote a quantized operator of the spinor quasi-field (2.7).

We define an operator

$$\hat{\varphi} = \frac{1}{(2\pi)^2}\int d\Omega_k\varphi(k)\hat{d}(k), \tag{2.16}$$

where $\varphi(k)$ is the Bose-particle quasi-field operator (2.8), and $\hat{d}(k)$ is the displacement operator (1.19). Comparing relations (2.16) and (2.4) we see that the operator $\hat{\varphi}$ is a certain analog of the field $\varphi(x)$ in ordinary theory. With the aid of (2.15) and (2.16) we can represent the quantity Λ (2.5), generalized to the case of elliptic p-space, in the form

$$\Lambda = g\langle\bar{\psi}|\gamma_5\hat{\varphi}|\psi\rangle. \tag{2.17}$$

Following [11], we base our theory on the operator

$$\sigma = e^{i\Lambda}. \tag{2.18}$$

With the aid of the conjugation relations (2.12) and (2.13), and using the unitarity of the displacement operator (1.19), we can readily show that the operator Λ (2.17) is Hermitian in the natural metric,

$$\Lambda = \Lambda^+,$$

and consequently the operator σ is unitary in the same metric, i.e., it satisfies the relation

$$\sigma\sigma^+ = \sigma^+\sigma = 1. \tag{2.19}$$

The meaning of the operator σ becomes clear if we use Wick's theorem and expand the exponential function in (2.18) in normal products of the operators ψ, $\bar{\psi}$, and φ. This expression can be represented in the form

$$\sigma = \sum_{(n,\nu)}\int K^{(n,\nu)}(p_1,\ldots,p_n;\ q_1,\ldots,q_n;\ k_1,\ldots,k_\nu)$$
$$\times N\{\bar{\psi}(p_1)\ldots\bar{\psi}(p_n)\psi(q_1)\ldots\psi(q_n)\varphi(k_1)\ldots\varphi(k_\nu)\}$$
$$\times\Pi d\Omega_p\Pi d\Omega_q\Pi d\Omega_k. \tag{2.20}$$

The coefficient functions of the expansion (2.20), $K^{(n,\nu)}$ are generalized Feynman amplitudes, which have 2n external fermion lines and ν external boson lines. The amplitude $K^{(n,\nu)}$ is defined in elliptic p-space. In order to find the amplitudes of the real physical processes, it is necessary to go over by suitable analytic continuation to the values of the momenta belonging to the pseudo-elliptic space.

So far the entire construction was carried out for the case of a pseudo-scalar interaction between a Fermi field and a Bose field. It is easy to extend all the foregoing arguments also to the case of other interaction variants. For this purpose it is merely necessary to define the operator Λ in suitable fashion. For example, for the case of a direct four-fermion interaction we can introduce the current operator

$$J(k) = \langle\bar{\psi}\mid\hat{O}\hat{d}(k)\mid\psi\rangle,$$

where $\hat{d}(k)$ is the shift operator (1.19), and define the operator Λ by means of the relation

$$\Lambda = \int d\Omega_k J(k) J(-k).$$

3. REPRESENTATION OF AMPLITUDES IN OPERATOR FORM

In calculating the amplitudes $K^{(n,\nu)}$ of expansion (2.20), the ordinary matrix notation is less convenient than the operator form with notation of the type (2.15). The advantages of the operator form are connected with the noncommutativity of the displacement operators (1.19) and is particularly clearly manifest in the higher approximations of perturbation theory (see Sec. 4 and 5). In the present section we shall illustrate briefly the gist of the operator method using simplest examples with expressions of order g^2.

The second-order term of the operator σ (2.18) is

$$-\tfrac{1}{2}\Lambda^2 = -\tfrac{1}{2}g^2\langle\bar{\psi}\,|\,\gamma_5\hat{\varphi}\,|\,\psi\rangle^2. \qquad (3.1)$$

Applying Wick's theorem to (3.1) and taking expression (2.14) for pairing of operators into account, we represent (3.1) in the form of a sum of terms, corresponding to different Feynman diagrams. Let us give the explicit form of the normal products, corresponding to the diagrams in Figs. 1a, b, c, and d

$$-\frac{g^2}{(2\pi)^4} N\langle\bar{\psi}\,|\,\gamma_5\hat{d}(k_2)(m+\hat{p})^{-1}\gamma_5\hat{d}(k_1)\,|\,\psi\rangle, \qquad (3.2)$$

$$-\frac{g^2}{2(2\pi)^4}\int\frac{d\Omega_k}{\mu^2+k^2} N\{\langle\bar{\psi}\,|\,\gamma_5\hat{d}(-k)\,|\,\psi\rangle\langle\bar{\psi}\,|\gamma_5\hat{d}(k)\,|\,\psi\rangle\}, \qquad (3.3)$$

$$-\frac{g^2}{(2\pi)^4} N\langle\bar{\psi}\left|\int\frac{d\Omega_k}{\mu^2+k^2}\gamma_5\hat{d}(-k)(m+\hat{p})^{-1}\gamma_5\hat{d}(k)\right|\psi\rangle \qquad (3.4)$$

$$\frac{g^2}{2(2\pi)^4}\,\mathrm{Tr}\,\{\gamma_5\hat{d}(k_2)(m+\hat{p})^{-1}\gamma_5\hat{d}(k_1)(m+\hat{p})^{-1}\}. \qquad (3.5)$$

We see that to each diagram of Fig. 1 there corresponds a certain operator acting on the spinor functions in p-space. This operator is made up in

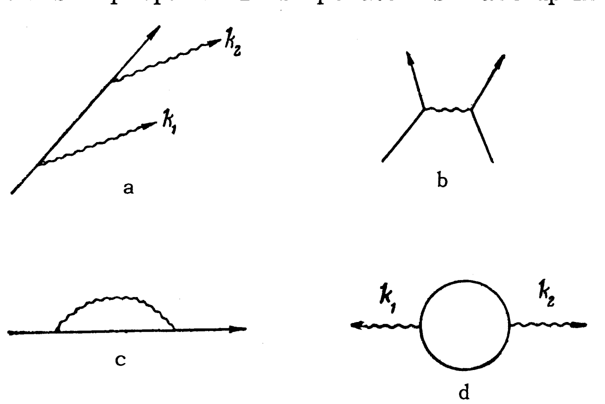

FIG. 1

simple fashion of the operators γ_5, displacement operator $\hat{d}(k)$, and the propagation operators $(m+\hat{p})^{-1}$. The amplitude is expressed in terms of the corresponding operator by a relation of the type (2.15). The symbol N in formula (3.2)—(3.4) denotes that the product of the operators ψ and $\bar{\psi}$ (2.7) is replaced in the final state by their normal product. The symbol Tr in formula (3.5) denotes the total trace of the operator in the brackets (summation over the spinor indices and integration over p-space).

We can construct analogously the amplitudes corresponding to any higher-order diagram. Each amplitude is expressed with the aid of a relation of the type (2.15) through a certain operator. Thus, the operator method considered here generalizes in natural fashion the Feynman diagram technique to the case of field theory in elliptic p-space.

4. THE SCHWINGER EQUATIONS

The Schwinger equations [12] for the Green's functions can be extended to the case of field theory in p-space of constant curvature. These equations are of interest, since (as was shown by Fradkin [13]) they are directly related to the dynamic equations of the field theory for the Heisenberg operators. In the derivation of the equations we use the method developed in the book by Bogolyubov and Shirkov [14]. This derivation is based only on the properties of the operator σ (2.18) and does not involve any new dynamic principle. We derive the Schwinger equations in p-representation in elliptic p-space (the Schwinger equations in Euclidean p-space were derived for ordinary field theory in Fradkin's dissertation [4]).

Let us introduce an auxiliary classical field $\zeta(k)$ and consider the operator

$$\sigma[\zeta] = e^{i\Lambda}\exp\left\{\int\varphi(k)\,\zeta(k)\,d\Omega_k\right\}. \qquad (4.1)$$

When $\zeta \equiv 0$ the expression (3.1) goes over into the operator σ defined by (2.18). For the sake of brevity we no longer write out the functional argument ζ in explicit form. From the form of the equations it is clear which operators should be regarded as functionals of the auxiliary field ζ. In calculating the functional derivatives with respect to ζ we make use of the relation

$$\delta\zeta(k)/\delta\zeta(l) = \delta(k, l), \qquad (4.2)$$

which defines the "normalization" of the functional derivatives.

The Green's operator $\hat{G}$ acting on the spinor functions in p-space is defined by means of its matrix element

$$\langle p|\hat{G}|q\rangle = \langle\psi(p)\sigma\bar{\psi}(q)\rangle_0/\langle\sigma\rangle_0, \quad (4.3)$$

where the operator σ is given by relation (4.1), and the index zero denotes vacuum averaging. Taking (2.16) and (2.14) into account, we represent the quantity Λ in the form

$$\Lambda = \frac{g}{(2\pi)^2}\int d\Omega_k \varphi(k)\langle\bar{\psi}|\gamma_5\hat{d}(k)|\psi\rangle. \quad (4.4)$$

Further derivation of the equations is carried out in the same fashion as in [14]. Using the generalized Wick's theorem and taking the definition (4.3) and relations (4.1) and (4.4), as well as the pairing expressions (2.14), into account we obtain after simple calculations the first Schwinger equation. This equation, written in operator form, has the form

$$\left\{m + \hat{p} - \frac{ig}{(2\pi)^2}\int d\Omega_k \gamma_5 \hat{d}(k)\left[\Phi(k) + \frac{\delta}{\delta\zeta(k)}\right]\right\}\hat{G} = 1. \quad (4.5)$$

The quantity $\Phi(k)$ in (4.5) is defined by

$$\Phi(k) = \frac{1}{\langle\sigma\rangle_0}\frac{\delta\langle\sigma\rangle_0}{\delta\zeta(k)} = \frac{\langle\varphi(k)\sigma\rangle_0}{\langle\sigma\rangle_0}. \quad (4.6)$$

Analogously we obtain the second Schwinger equation for $\Phi(k)$:

$$(\mu^2 + k^2)\Phi(k) + \frac{ig}{(2\pi)^2}\mathrm{Tr}\{\gamma_5\hat{d}(-k)\hat{G}\} = \zeta(-k). \quad (4.7)$$

The Schwinger equations (4.5) and (4.7) can be represented in a different form, which is more convenient for applications, if we assume $\Phi(k)$ to be the independent functional variable. Let us define the Green's function of a boson by the relation

$$D(k, l) = \delta\Phi(k)/\delta\zeta(l). \quad (4.8)$$

Taking (4.6) into account, we can represent the function $D(k, l)$ in the form

$$D(k, l) = \frac{\langle\varphi(k)\varphi(l)\sigma\rangle_0}{\langle\sigma\rangle_0} - \frac{\langle\varphi(k)\sigma\rangle_0}{\langle\sigma\rangle_0}\frac{\langle\varphi(l)\sigma\rangle_0}{\langle\sigma\rangle_0}. \quad (4.9)$$

From (4.9) it follows, in particular, that $D(k, l)$ is a symmetrical function of its arguments.

Making a change of functional variables and taking (4.2) and (4.8) into account, we obtain a system of equations for the fermion Green's operator $\hat{G}$ and for the boson Green's function D:

$$\left\{m + \hat{p} - \frac{ig}{(2\pi)^2}\int d\Omega_k\gamma_5\hat{d}(k)\left[\Phi(k) + \int d\Omega_l D(k, l)\frac{\delta}{\delta\Phi(l)}\right]\right\}\hat{G} = 1,$$

$$(\mu^2 + k^2)D(k, l) + \frac{ig}{(2\pi)^2}\int d\Omega_n D(n, l)\,\mathrm{Tr}\left\{\gamma_5\hat{d}(-k)\frac{\delta\hat{G}}{\delta\Phi(n)}\right\} = \delta(k, -l). \quad (4.10)$$

In (4.10), the quantities $\hat{G}$ and D are functionals of the independent variable $\Phi(k)$.

The equations in (4.10) can be represented in a different form by introducing the vertex operator

$$\hat{\Gamma}(k) = -g^{-1}\delta\hat{G}^{-1}/\delta\Phi(k). \quad (4.11)$$

From (4.11) we obtain

$$\delta G/\delta\Phi(k) = g\hat{G}\hat{\Gamma}(k)\hat{G}. \quad (4.12)$$

Substituting (4.12) in (4.10) and introducing the mass operator

$$\hat{M} = m - \frac{ig^2}{(2\pi)^2}\int d\Omega_k d\Omega_l D(k, l)\gamma_5\hat{d}(k)\hat{G}\hat{\Gamma}(l) \quad (4.13)$$

and the polarization operator

$$P(k, l) = ig^2(2\pi)^{-2}\,\mathrm{Tr}\{\gamma_5\hat{d}(-k)\hat{G}\hat{\Gamma}(l)\hat{G}\}, \quad (4.14)$$

we obtain the system of equations

$$\left\{\hat{M} + \hat{p} - \frac{ig}{(2\pi)^2}\int d\Omega_k\gamma_5\hat{d}(k)\Phi(k)\right\}\hat{G} = 1,$$

$$(\mu^2 + k^2)D(k, l) + \int P(k, n)D(n, l)\,d\Omega_n = \delta(k, -l). \quad (4.15)$$

From the first equation of (4.15) we can determine the connection between the vertex mass operators

$$\hat{\Gamma}(k) = i(2\pi)^{-2}\gamma_5\hat{d}(k) - g^{-1}\delta\hat{M}/\delta\Phi(k). \quad (4.16)$$

The equations in (4.10) or (4.15) represent different forms of the Schwinger equations, and are perfectly analogous to the corresponding equations of ordinary theory.

5. CONCLUDING REMARKS

As a result of the noncommutativity of the displacement operators (1.19), unique effects which have no analogs in ordinary theory, occur in the theory developed here. Some of these effects, connected with the deviation from the energy and momentum conservation laws in particle collisions, were considered in [1]. We shall dwell here briefly on questions connected with the determination of the state of a physical particle. Unlike ordinary theory, the mass operator (4.13) (with $\zeta = 0$) is not diagonal in the p-representation. To verify this let us consider the contribution to $\hat{M}$, corresponding to a fourth-order diagram (Fig. 2):

$$\hat{M}_4 = \frac{g^4}{(2\pi)^8}\int\frac{d\Omega_k}{\mu^2 + k^2}\frac{d\Omega_l}{\mu^2 + l^2}$$

$$\times\{\gamma_5\hat{d}(-k)(m + \hat{p})^{-1}\gamma_5\hat{d}(-l)(m + \hat{p})^{-1}\gamma_5\hat{d}(k)$$

$$\times(m + \hat{p})^{-1}\gamma_5\hat{d}(l)\}, \quad (5.1)$$

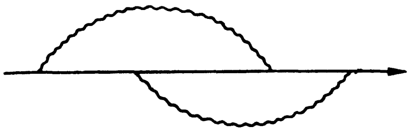

FIG. 2

inasmuch as the shift operators do not commute, we have

$$\hat{d}(-k)\,\hat{d}(-l)\,\hat{d}(k)\,\hat{d}(l) \neq 1,$$

from which follows the statement that the operator $\hat{M}$ is non-diagonal (we note that the second-order term in (3.4) contributes only to the diagonal part of the operator $\hat{M}$). The mass operator is thus an integral operator in p-representation. Analogous deductions can be drawn with respect to the polarization operator (4.14) and the vertex operator (4.16).

From the integral character of the operator $\hat{M}$ it follows that the states of a physical particle, specified by the equation

$$(\hat{M} + \hat{p})\,\psi = 0, \qquad (5.2)$$

cannot have a definite momentum. From relativistic invariance considerations we can conclude that the solution of Eq. (5.2) will be "smeared" over the masses. One should expect this smearing to be sufficiently small ($\Delta_m \sim M^2$ in dimensionless units, where M is the physical mass of the particle).

The integral nature of the mass operator M calls for an appreciable review of the entire problem of mass renormalization. Inasmuch as no divergent expressions are encountered in the theory, it is most tempting to relate the problem of renormalization with the problem of determining the values of the masses of elementary particles.

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