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MAGNETIC SOUND WAVES IN A CYLINDRICAL PLASMA PINCH WITH NONLINEARITY AND ABSORPTION TAKEN INTO ACCOUNT

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The propagation of converging waves in a cylindrical plasma pinch is analyzed on the basis of the equations of magnetohydrodynamics, taking nonlinearity and absorption into account. The analysis is carried out by simplifying the initial set of equations. This approximation is valid for small initial perturbations, small dissipation, and cylinder radii which are large compared with the wavelength. The formation and gradual ''disintegration'' of the fronts in a converging cylindrical magnetic-sound wave are studied and the dissipation energy is determined as a function of the radius of the cylindrical plasma pinch.

1. INTRODUCTION

A great deal of attention is given at the present time to the study of magnetic-sound waves in a cylindrical plasma pinch, for example, in connection with problems of the heating of a plasma by a high-frequency electromagnetic field. If two longitudinal magnetic fields are applied to a cylindrical plasma pinch—a static field H_0 and a variable field h, then the ponderomotive force $\mathbf{j} \times \mathbf{H}_0/c$ ($\mathbf{j}$ = annular current density) arising during the growth of h compresses the plasma. In this case, a magnetic-sound wave arises, transferring the perturbation inside the plasma. The mechanism of transfer of the perturbation is the same here as in ordinary sound. The method of heating the plasma was first suggested by Frank-Kamenetskiĭ. He also gave a theoretical description of the phenomenon of magnetic-sound resonance in a cylindrical plasma pinch within the framework of the linear approximation.[1] However, the nonlinear effects are significant, as experimental investigations of magnetic-sound oscillations in a cylindrical plasma pinch have shown,[2] even for small amplitudes of the high-frequency magnetic field. Therefore, the study of the nonlinear problem of the propagation of cylindrical magnetic-sound waves, although not directly connected with the phenomena of magnetic-sound resonance, has significant interest, for example, in clarifying the specific nonlinear absorption of energy and its distribution along the radius of the cylindrical plasma pinch.

In the present work, cylindrical magnetic-sound waves of finite amplitude are considered on the basis of an approximate method.[3] The nonlinearity of the medium and the dissipation of energy in it are considered to be small. In this case, the set of equations of magnetohydrodynamics can be reduced to a single nonlinear partial differential equation of second order, the solution of which allows us to consider the simultaneous effect of nonlinear and dissipative effects. The solutions carried out in second approximation allow us to follow the spatial scales of distortion of the wave, to study the mechanism of formation and ''disintegration'' of shock waves, and to study their fronts. These solutions make it possible to consider the energy dissipation in the region of shock wave formation, with account of the finite thickness of the shock fronts.

2. FORMULATION OF THE PROBLEM AND DERIVATION OF THE APPROXIMATE EQUATION

Let us consider a cylindrical plasma pinch located in a longitudinal constant magnetic field of intensity H_0. A variable magnetic field h, coinciding in direction with H_0, leads, as was noted above, to the generation of converging magnetic-sound waves during the growth phase of the field. By working with this model, it is possible in what follows simply to consider as given some perturbation of the radial component of the velocity $v = v(t)$ on the surface of the plasma cylinder r_0.

The initial set of equations are the equations of magnetohydrodynamics. The problem is to find for these equations cylindrically symmetric solutions that describe the propagation of converging magnetic-sound waves.

In the case of weak nonlinearity and weak dissipation in a cylindrical plasma pinch region which satisfies the condition $kr \gg 1$, where r is the running radius of the cylinder and k is the wave number, the magnetohydrodynamic equations are greatly simplified. Actually, if we neglect the dissipation terms of the equations of magnetohydrodynamics generally, then, following Kaplan and Stanyukovich,[4], we can reduce the problem of magnetic-sound wave propagation to the following equations:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v) = 0, \tag{1}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial}{\partial r}\left(P + \frac{b^2}{8\pi}\rho^2\right). \tag{2}$$

Here ρ is the density, P the pressure, $b = H/\rho = \text{const}$. Equation (2) differs from the equation of motion of ordinary gas dynamics only by the change in the equation of state

$$P^*(\rho) = P(\rho) + \frac{b^2}{8\pi}\rho^2, \tag{3}$$

such that the role of the sound velocity is played by the quantity $c_0 = \sqrt{u_0^2 + H^2/4\pi\rho}$, where $u_0 = (\partial P/\partial\rho)^{1/2}$ is the ordinary sound velocity.

The velocity of propagation of the perturbation c_0 in the general case depends on both the acoustic pressure P_g and the magnetic pressure P_H. Two regions of sound propagation are possible, depending on the relation between P_g and P_H. These regions are limited by the number of collisions or by the frequency. In the case $P_H \gg P_g$, only charged particles take part in the motion. By the density ρ is meant the density of charged particles. For a large number of collisions, the neutral particles are carried along by the motion, and ρ denotes the total gas density (for further details, see [1]). Without neglecting the dissipation terms of the equations of magnetohydrodynamics, it is possible, in a way similar to that previously described[3] for plane magnetic-sound waves to combine the equations under the condition that the absorption over one wavelength is small. Then Eq. (2) takes the form

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial}{\partial r}\left(P + \frac{b^2}{8\pi}\rho^2\right) + \frac{\delta_1}{\rho}\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v^2}{r^2}\right). \tag{4}$$

Here the coefficient

$$\delta_1 = \left\{\frac{H^2}{4\pi\rho}\left(\frac{c^2\rho}{4\pi\sigma}\right) + u_0^2\left[\frac{\gamma-1}{\gamma}\frac{\varkappa}{c_v} + \frac{c_0^2}{u_0^2}\left(\frac{4}{3}\eta + \zeta\right)\right]\right\} \tag{4a}$$

takes simultaneous account of the effect of the bulk and shear viscosities (ζ, η), thermal conductivity κ, magnetic viscosity $\beta = c^2\rho/4\pi\sigma$, where σ is the conductivity of the medium; $\gamma = c_p/c_v$, where c_p and c_v are the specific heats at constant pressure and volume, respectively.

By considering small velocities v, small derivations of the density ρ' and of the magnetic field intensity h from their equilibrium values ρ_0 and H_0, it can be assumed that v/c_0, ρ'/ρ_0 and $h/H_0 \ll 1$ or, introducing the small parameter μ explicitly, we have

$$\frac{v}{c_0},\ \frac{\rho'}{\rho_0},\ \frac{h}{H_0} \sim \mu. \tag{5}$$

The dissipation coefficients are also assumed to be small quantities of first order in smallness, i.e.,

$$\eta,\ \zeta,\ \varkappa,\ \beta \sim \mu. \tag{6}$$

The quantity $1/kr$ is also assumed to be a small quantity of the first order of smallness:

$$1/kr \sim \mu. \tag{7}$$

Taking the conditions (5) — (7) into account, one can describe Eqs. (1) and (4) with accuracy up to small quantities of second order of smallness inclusive. The equation of state (3) in this case must be replaced by an approximate equation of state, also with accuracy up to small quantities of second order, inclusive:

$$P = P_0 + u_0^2\rho' + \frac{H_0^2}{4\pi\rho_0}\rho' + (\gamma - 1)\,u_0^2\frac{\rho'^2}{2\rho_0} - \frac{H_0^2}{4\pi\rho_0}\frac{\rho'^2}{2\rho_0}. \tag{8}$$

The foregoing analysis of the order of smallness of the quantities makes it possible to assume that the set of Eqs. (1), (4), (8) has (in second approximation) a solution in the form of a traveling wave, but the profile of this wave changes slowly with the distance r.[3] In this connection, it is natural to transform to a new "traveling" set of coordinates $\tau = t + r/c_0$ and $r' = \mu r$, where the introduction of the small parameter μ denotes a slow change of the profile of the wave in its propagation in the medium. It is convenient to make this transition after reduction of Eqs. (1), (4) and (8) to a single equation. This is easily achieved, either by transition to Lagrangian coordinates or, in a fashion similar to that used for plane waves.[3] In the new coordinates r' and τ, the problem reduces to the investigation of a single equation, all the terms of which are small quantities of second order:

$$\frac{\partial v}{\partial r} + \frac{v}{2r} - \alpha v\frac{\partial v}{\partial \tau} = \delta\frac{\partial^2 v}{\partial \tau^2}, \tag{9}$$

where

$$\alpha = \frac{1}{2c_0^2}\left\{(\gamma + 1) + \frac{(2-\gamma)}{c_0^2}\frac{H_0^2}{4\pi\rho_0}\right\}, \qquad \delta = \delta_1/2\rho_0 c_0^3. \tag{9a}$$

In Eq. (9) and below, the prime on r is omitted.

Assuming the perturbation of the velocity $v = v_0 \sin(\omega t + kr)$ as given on the surface of the cylinder, we rewrite the latter in the new coordinates r' and τ: for

$$r = r_0, \qquad v = v_0 \sin \omega \tau. \tag{10}$$

The problem now consists of finding for arbitrary value of r a solution of Eq. (9) with boundary condition (10), which is periodic in τ and describes traveling converging cylindrical waves.

3. SOLUTION OF THE APPROXIMATE EQUATION

The solution of (9) and (10) can be carried out in steps. Considering the case of large values of the magnetohydrodynamic analog of the Reynolds' number

$$\mathrm{Re} = \alpha v_0/2\omega\delta \approx \alpha v v_\tau/\delta v_{\tau\tau} \gg 1, \tag{11}$$

which is the case of greatest practical interest, we can neglect the dissipative term $\delta v_{\tau\tau}$ in Eq. (9) for the first stage in the propagation of a magnetic-sound wave of finite amplitude. The exact solution of Eq. (9), without the viscous term and with the boundary condition (10), has the form

$$\omega\tau = \arcsin \Phi - 2Z_0 (1 - \sqrt{r/r_0})\, \Phi, \tag{12}$$

where

$$\Phi = v \sqrt{r}/v_0 \sqrt{r_0}, \qquad Z_0 = \alpha\omega v_0 r_0. \tag{12a}$$

Equation (12) can be analyzed graphically. If the value of Φ is plotted along the abscissa and $\omega\tau$ along the ordinate, it is easy to see that the wave profile, which is represented, in accord with (12), as the sum of two functions, the arcsine and a straight line with slope $2Z_0(1 - \sqrt{r/r_0})$, becomes distorted as the wave progresses. When the slope is less than unity, the distortion of the profile of the magnetic-sound wave is slight. If the slope of the straight line is of the order of unity, the distortion is great, and if the slope is larger than unity, then the function becomes many-valued. In this case, the solution (12) is no longer valid. The function $\Phi(\tau)$, without account of dissipative processes, becomes discontinuous, and the value r_1 at which the discontinuity is formed can be determined from the condition that the slope be equal to unity:

$$2Z_0 (1 - \sqrt{r_1/r_0}) = 1 \text{ or } 2Z_0 (1 - \sqrt{Z_1/Z_0}) = 1, \tag{13}$$

where $Z_1 = \alpha\omega v_0 r_1$ has the meaning of a dimensionless coordinate of the discontinuity.

The condition (13) is not always satisfied in a converging cylindrical wave, but only for sufficiently large values of the dimensionless coordinate $Z_0 = \alpha\omega v_0 r_0$, which is proportional to the running radius of the plasma cylinder and to the amplitude of the initial perturbation. Thus, for $Z_0 = 0.5$, the discontinuous solution, in accord with (13), is formed at the point $r = 0$; for $Z_0 < 0.5$, it is not formed at all, i.e., for $Z_0 < 0.5$ the cylindrical magnetic-sound wave converges without developing into a "discontinuous" shock wave.

We note that consideration of the point $r = 0$ on the basis of Eq. (12) is generally incorrect [Eq. (9) was obtained under the assumption that $kr \gg 1$]. One must consider a certain small coaxial cylindrical region kr which satisfies the condition $kr \gg 1$. In converging to this cylinder, it is all the more true that the cylindrical magnetic-sound wave is not converted to a shock wave. As the magnetic-sound wave converges to the axis of the cylindrical plasma pinch, the phenomenon of "inversion" of the magnetic field may be observed, since the variable component of the magnetic field h carried along the wave increases [in the given approximation as $h = h_0 (r_0/r)^{1/2}$]; at the point determined by the condition

$$r/r_0 = (h/H_0)^2, \tag{14}$$

it reaches a value of the order of H_0. Upon further convergence of the wave, h can become larger than H_0. The ponderomotive forces $\mathbf{j} \times \mathbf{h}/c$ and $\mathbf{j} \times \mathbf{H}_0/c$, which act in opposite directions, are quantities of first order in the region determined by the relation (14).

Putting $Z_0 \gg 1$ in what follows, one can assume that a discontinuity is formed at some point Z_1 in accordance with the relation (13), and beginning with the values

$$2Z_0 (1 - \sqrt{Z_2/Z_0}) = \pi/2 - 1, \tag{15}$$

the wave acquires a sawtooth form. The amplitude of the jump in such a wave varies as

$$v \sqrt{r} = \pi v_0 \sqrt{r_0}/2\,[1 + 2Z_0(1 - \sqrt{r/r_0})]. \tag{16}$$

The structure of the resulting discontinuities can be investigated, on the basis of the solution of the auxiliary problem of the propagation of a solitary shock, either by numerical integration of Eq. (9), or by finding a quasi-stationary solution. We shall consider the latter method.

Making the substitutions $x = 2\sqrt{r/r_0}$ and $w = v\sqrt{r/r_0}$ in Eq. (9), we get

$$\partial w/\partial x - \alpha w \partial w/\partial \tau = (\delta x/2r_0)\, \partial^2 w/\partial \tau^2. \tag{17}$$

The quasi-stationary solution of Eq. (17), which is valid if $\partial w/\partial x$ is small compared with the remaining terms of (17), has the form

$$w = w_0 \operatorname{th}\left[\frac{\alpha w_0 \tau}{\delta x/r_0}\right]. \tag{18*}$$

*th = tanh.

In this case, the term $\partial w/\partial x$ can be neglected when $2r_0(\alpha w_0)^2/2\delta \gg 1$, which can be rewritten in terms of the nondimensional parameters as

$$2Z_0 \mathrm{Re} \gg 1. \tag{19}$$

It is easy to see that in the case $Z_0 > 0.5$, when the formation of shock waves is possible, and condition (11) is satisfied, the relation (19) is always valid, i.e., use of the quasi-stationary solution (18) is legitimate.

Turning now to the problem of the propagation of the wave, which is given in the form (10) on the surface of a cylindrical plasma pinch, one can describe the configuration of the wave in the region $-\pi \leq \omega\tau \leq \pi$ by the following relation

$$v\sqrt{r} = \frac{\pi v_0 \sqrt{r_0}}{2\left[1 + 2Z_0\left(1 - \sqrt{r/r_0}\right)\right]}\left(-\omega\tau + \pi \operatorname{th}\frac{\omega\tau}{\Delta}\right), \tag{20}$$

where Δ has the meaning of the dimensionless thickness of the shock front:

$$\Delta = \frac{\sqrt{r/r_0}}{\pi^2 \mathrm{Re}}\left[1 + 2Z_0\left(1 - \sqrt{r/r_0}\right)\right]. \tag{21}$$

Relations (20) and (21) were obtained by matching the quasi-stationary solution with the sawtooth solution. They are valid in the region $r \leq r_2 \approx r_1$ (15), which it is possible to establish by direct substitution of (20) in (9). Equation (9) is approximately satisfied in this case, while the necessary condition for approximate satisfaction of (9) coincides exactly with the condition of applicability of the quasi-stationary solution (19).

4. THE STRUCTURE OF A SHOCK WAVE AND ENERGY FLOW INTO THE MEDIUM

The width of a shock wave, as follows from analysis of Eq. (21), does not remain stationary, for two reasons. In the first place, a "diffusion" of the shock front takes place, brought about by the decrease in its amplitude because of absorption, so that Δ increases in proportion to $1 + 2Z_0(1 - \sqrt{r/r_0})$. This diffusion mechanism is completely analogous to the corresponding mechanism which acts in the case of plane waves, where the growth in the wave front takes place according to a linear law.[3] In the second place, Δ decreases because of convergence (the factor $\sqrt{r/r_0} < 1$).

By plotting the dependence of the front thickness on the dimensionless coordinate Z, which is proportional to r (Fig. 1), it is not difficult to show that Δ, which is equal to the quantity $\frac{1}{2}\pi\mathrm{Re}$ at the point $Z_2 \approx Z_1$, increases at first, in accord with (21), reaching a maximum at the point $Z_{max} = (1 + 2Z_0)r_0/16Z_0$, which is equal to

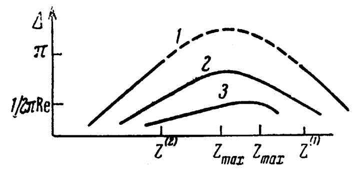

FIG. 1. Dimensionless front thickness Δ of a converging magnetic-sound wave as a function of the distance, measured from the location of the initial disturbance. Curves 1, 2, 3 correspond to cylindrical plasma pinches of radii r_{01}, r_{02}, r_{03}, such that $r_{01} \gg r_{02} \gg r_{03}$.

$(1 + 2Z_0)^2/8\pi^2 Z_0 \mathrm{Re}$. Thereafter, it decays as the wave is propagated in the medium, and approaches zero as $r \to 0$. Thus, a shock wave is always formed in a converging cylindrical magnetic-sound wave if only the initial radius of the cylindrical column is chosen sufficiently large or the initial amplitude of the variable magnetic field h is not infinitely small, i.e., it is necessary that Z_0 be much larger than unity, otherwise the magnetic sound wave converges before discontinuities are formed.

Curves 1, 2, and 3 in Fig. 1 correspond to different values of the dimensionless coordinate Z_0. In the case $Z_0 \gg \mathrm{Re}$ (curve 1), an interesting phenomenon is observed: a shock front is formed twice in the converging magnetic sound wave. Actually, the maximum dimensionless shock front thickness Δ_{max} can be shown formally to be greater than or equal to π, in accord with Eq. (21). Equation (20) is valid only in the limits $\Delta \leq \pi$. Curve 1 does not have physical meaning in the region $[Z^{(1)}, Z^{(2)}]$ where the points $Z^{(1)}$ and $Z^{(2)}$ correspond to the value $\Delta = \pi$ as noted in Fig. 1. In this case, the amplitude of the magnetic-sound wave calculated by Eq. (20) is shown to be $\sim v_0/2\pi^2\mathrm{Re}$, i.e., it is a small quantity of second order, in accord with (11). This means that the propagation of converging cylindrical waves in the region $[Z^{(1)}, Z^{(2)}]$ of the cylindrical plasma pinch can be described by the formulas of the linear theory of magnetic sound, although the increase in the wave amplitude as a consequence of convergence does not exceed the decrease in the amplitude resulting from dissipation. This takes place in the region $Z < Z^{(2)}$, where the thickness of the shock front decreases rapidly, approaching zero.

The maximum width of the front is reached at the point $Z_m \geq Z_0/4$, so that for $Z_0 \gg \mathrm{Re}$ one can assume $Z_{max} = Z_0/4$, and the coordinate of maximum thickness of the front is shifted to the right upon decrease in Z_0 (curves 2, 3, Fig. 1). With decreasing Z_0, the maximum achieved shock front thickness Δ_{max} after formation of the discontinuity is simultaneously decreased. Thus, in the case Z_0

$= R_1$, the maximum thickness of the front reaches a value of the order of 0.05, which is significantly larger than $\Delta_{min} = \frac{1}{2}\pi Re$ (curve 2, Fig. 1). For Z_0 equal, say, to 5, the difference between Δ_{max} and Δ at the point Z_2 is unimportant. Simultaneously, the coordinate Z_{max} approaches the dimensionless coordinate of discontinuity formation $Z_1 \approx Z_2$ (curve 3, Fig. 1).

It is interesting to consider the amplitude of the converging cylindrical magnetic-sound wave as a function of the nondimensional coordinate Z. The amplitude increases up to the value $Z_2 \approx Z_1$, at which point a periodic shock wave is formed which brings about a change in the amplitude as a result of the strong dissipation, while the increase in amplitude because of convergence does not outweigh the dissipation losses. The discussions that have been given are illustrated by a curve (Fig. 2) constructed in correspondence with Eq. (20) for the case $Z_0 \gg Re$. The dashed curve represents the amplitude obtained from the linear theory.

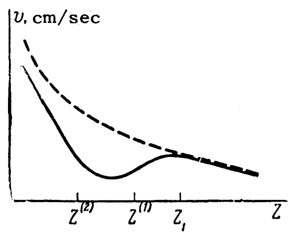

FIG. 2. Dependence of the amplitude of the velocity (or of the variable component of the magnetic field intensity h) of a converging cylindrical magnetic-sound wave on the distance from the location of the initial disturbance, with account of the nonlinearity and dissipation. The dashed curve corresponds to the linear theory, without absorption.

The problem of the energy flow in a cylindrical plasma pinch is of great importance. The energy density of the wave, as is well known, is determined by the expression $E = (\rho_0 v^2)/2$ with accuracy up to small quantities of second order. The expression for E, in the case of converging cylindrical magnetic-sound waves, even without consideration of dissipation and nonlinearity, does not remain constant, but increases in proportion to r_0/r, i.e., $E_r = E_0(r_0/r)$, where E_0 is the initial energy density of the wave. With account of dissipative and nonlinear effects, the energy density E'_r can be calculated on the basis of Eq. (20), which determines the amplitude of the velocity of converging magnetic-sound waves in the region $Z \leq Z_2 \approx Z_1$, where the nonlinear and dissipative effects are significant. Obviously, the difference $E_r - E'_r$ also determines the loss (absorption) of energy in the r section of the cylindrical plasma pinch. E'_r is computed by elementary means if the region of integration is divided into three intervals $[-\pi, -\epsilon]$, $[-\epsilon, +\epsilon]$ and $[+\epsilon, +\pi]$, replacing $\tanh(\omega\tau/\Delta)$ in each of these intervals by $-\pi$, $\omega\tau/\Delta$, and π, respectively. Then the energy absorption is determined by the formula

$$E_r - E'_r = E_r\left[1 - \frac{\pi(\pi-\Delta)}{4\left[1+2Z_0(1-\sqrt{r/r_0})\right]^2}\right]. \qquad (22)$$

The expression in the curly brackets in Eq. (22) depends essentially on the running radius of the cylindrical plasma pinch. For such small values of r at which the wave assumes a sawtooth form because of convergence, the usual limiting damping law for shock waves holds. The velocity amplitude v decreases in proportion to $1/[1 + 2Z_0(1 - \sqrt{r/r_0})]$. When the finite shock thickness Δ in each r section of the cylindrical plasma pinch is taken into account, the energy absorption is greater than the energy absorption obtained in accord with the limiting law of decrease in the amplitude of the shock wave Δ, so that the relative energy absorption $(E_r - E'_r)/E_r$ is maximum in the region $\Delta = \Delta_{max}$.

Thus, although the dissipation of energy is also proportional to the energy density of the wave E_r, this proportionality cannot be described by the absorption coefficient $\delta\omega^2 = \text{const}$, in contrast to the linear approximation.

CONCLUSION

The problem of the propagation of converging magnetic-sound waves of finite amplitude in a cylindrical plasma pinch has been considered in connection with the analysis of the mechanism of energy absorption in heating of the plasma at the expense of the high frequency electromagnetic field.

The equations of magnetohydrodynamics in second approximation reduce to Eq. (9) or, in terms of Eq. (17), to the problem of the propagation of plane waves in a medium with a linearly decreasing viscosity. The solution of Eq. (9) has made it possible to separate the entire region of wave propagation into three parts. In the first of these $(r_0 - r_1)$ the dissipation processes are negligibly small, while the nonlinear effects lead to the transformation of the sinusoidal wave into a wave of nearly sawtooth form. In the second, the front thickness of the shock wave being transformed increases because of the dissipation. In the third, the increase in the amplitude of the wave from convergence decreases the growth of the shock front generated by the dissipative effects. The wave again takes on a sawtooth form.

In accordance with the solutions obtained, the density of energy absorption in the r cross section of a cylindrical plasma pinch is determined by Eq. (22). Its relative value is maximum in the region of maximum thickness of the wave front.

Equation (9) can also be used for the analysis of a diverging cylindrical wave.

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