APPLICATION OF QUANTUM FIELD THEORY METHODS TO THE PROBLEM OF DEGENERATION OF HOMOGENEOUS TURBULENCE

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The problem of homogeneous turbulence in an incompressible viscous fluid is considered on the basis of the Hopf equation (in variational derivatives) for the characteristic functional. Owing to the analogy between this equation and the Schrödinger equation for a vector Bose field with a strong interaction, the mathematical theory of quantum field theory can be applied to the problem. The solution is obtained in the form of a continual integral. An analysis of the solution shows that for infinite Reynolds numbers of the initial state the law of turbulence degeneration is independent of the form of the initial-state probability distribution.

CORRELATION functions of the form $\langle v_i(\mathbf{x}_1, t) \ldots v_l(\mathbf{x}_n, t)\rangle$ are frequently used in statistical descriptions of the turbulent motion of a liquid. If a differential equation for the time variation of such a correlation function is derived from the Navier-Stokes equation of motion [by multiplying the Navier-Stokes equation written for $v_i(\mathbf{x}_1, t)$ by $v_j(\mathbf{x}_2, t) \ldots v_e(\mathbf{x}_n, t)$] and subsequently averaged, then the nonlinear term in the equation of motion yields the correlation function of order $(n+1)$, and the resultant equation is not closed. This results in a system of coupled equations, similar to the chain of equations for Fock's functional. In practice, such a system is always solved by using various supplementary hypotheses, which allow us to close the system of equations and find an approximate solution. This method has resulted in considerable progress, in that Kolmogorov, Obukhov, Heisenberg and others obtained for very large Reynolds numbers second-order correlation functions for a certain distance interval (inertial interval) in which the form of these functions is universal, i.e., independent of the way the turbulence is generated.

Great interest attaches, however, to a more rigorous approach to this problem, in which Kolmogorov's solution is an asymptotic form of the exact solution of the problem. This rigorous approach to turbulent motion of fluids was suggested by Hopf,[1] who formulated for the characteristic functional a variational differential equation equivalent to the entire infinite chain of equations for the correlation functions. Rosen[2] used Hopf's equation to analyze the degeneration of turbulent motion (in the absence of permanent energy sources viscosity causes the turbulence to attenuate with time). This paper does employ certain mathematical procedures of quantum field theory, but unfortunately contains an error, a matter to which we shall return. The analogy with quantum field theory was noted by Wyld[3] in a discussion of the theory of stationary turbulence. This paper establishes a direct connection between a solution expanded in powers of the external force and the series in powers of the coupling constant used in quantum field theory.

In the present paper we trace an even closer analogy between turbulence theory and quantum field theory. The Hopf equation for the characteristic functional is found to be analogous to the Schrödinger equation for a nonlinear vector Bose field with strong interaction and an interaction Hamiltonian corresponding to the fusion of two bosons into one. This problem can be solved by a field-theoretical method such as the ordering of the S matrix (after Hori) with subsequent expression of the solution in the form of a continual integral. The kernel of this integral, which can be called the Green's functional for the Hopf equation, can be written in a closed although complicated form. The same Green's functional enables us to write the solution of the equation in the presence of external forces, too (this solution is not presented in the article).

The exact solution obtained in this paper for the problem of degeneration of turbulence is too complicated to yield specific conclusions directly. The task of extracting such conclusions from the solution will apparently turn out to be no less complicated than the work already performed.

1. We consider the degeneration of turbulence in an unbounded volume of incompressible viscous liquid. Its evolution is described by the Navier-Stokes equations. If $v_i(\mathbf{x}, t)$ is the random velocity, then

$$\frac{\partial v_i}{\partial t} = -\frac{\partial v_i v_k}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\Delta v_i, \qquad \frac{\partial v_l}{\partial x_l} = 0. \tag{1}$$

We put

$$v_i(\mathbf{x}, t) = \int \exp(i\mathbf{kx})\, q_i(\mathbf{k}, t)\, d^3k \qquad (q_i^*(-\mathbf{k}, t) = q_i(\mathbf{k}, t)),$$

$$p(\mathbf{x}, t) = \int \exp(i\mathbf{kx})\, r(\mathbf{k}, t)\, d^3k \qquad (r^*(-\mathbf{k}, t) = r(\mathbf{k}, t)).$$

Then (1) becomes

$$\frac{\partial q_i(\mathbf{k}, t)}{\partial t} = -ik_l \int q_i(\mathbf{k}', t)\, q_l(\mathbf{k}-\mathbf{k}', t)\, d^3k' \tag{2}$$

$$-\frac{ik_i r}{\rho} - \nu k^2 q_i, \quad k_i q_i(\mathbf{k}, t) = 0. \tag{3}$$

Multiplying the first equation in (3) by k_i and using the second equation, we obtain

$$\frac{r}{\rho} = -\frac{k_j k_l}{k^2}\int q_j(\mathbf{k}', t)\, q_l(\mathbf{k}-\mathbf{k}', t)\, d^3k'.$$

Substituting this expression in (3) and introducing the notation*

$$B_{l,ij}(\mathbf{k}) = -ik_l(\delta_{ij} - k_i k_j/k^2),$$

$$Q_i[\mathbf{q}(\varkappa, t); \mathbf{k}] = B_{l,ij}(\mathbf{k})\int q_l(\varkappa, t)\, q_j(\mathbf{k}-\varkappa, t)\, d^3\varkappa,$$

we obtain

$$\partial q_i(\mathbf{k}, t)/\partial t = Q_i[\mathbf{q}(\varkappa, t); \mathbf{k}] - \nu k^2 q_i(\mathbf{k}, t). \tag{4}$$

We note that $B_{l,ij}(\mathbf{k})$ satisfies the conditions $k_i B_{l,ij}(\mathbf{k}) = 0$ and $B_{l,ij}(-\mathbf{k}) = B_{l,ij}(\mathbf{k})$. It is necessary to add to (4) the incompressibility equation $k_i q_i(\mathbf{k}, t) = 0$, which actually reduces the number of independent components of the velocity field to two. This complicates the solution somewhat. We therefore do not add the incompressibility equation to (4) and regard all three velocity components as independent, imposing this condition on the initial state. The incompressibility equation is then satisfied for all t. Indeed, multiplying (4) by k_j and taking the equality $k_i B_{l,ij}(\mathbf{k}) = 0$ into account, we obtain $\partial(k_i q_i)\partial t = -\nu k^2(k_i q_i)$, from which it follows that $k_i q_i(\mathbf{k}, t) = 0$ if $k_i q_i(\mathbf{k}, 0) = 0$.

Instead of the functions $q_i(\mathbf{k}, t)$ it is convenient to introduce new functions $g_i(\mathbf{k}, t)$:

$$q_i(\mathbf{k}, t) = \exp(-\nu k^2 t)\, g_i(\mathbf{k}, t), \qquad g_i^*(-\mathbf{k}, t) = g_i(\mathbf{k}, t). \tag{5}$$

Substituting (5) in (4) we obtain

$$\partial g_i(\mathbf{k}, t)/\partial t = \exp(\nu k^2 t)\, Q_i[\exp(-\nu\varkappa^2 t)\, \mathbf{g}(\varkappa, t); \mathbf{k}]. \tag{6}$$

We add to this equation the initial condition

$$g_i(\mathbf{k}, 0) = q_i^0(\mathbf{k}), \tag{7}$$

which satisfies the relation $k_i q_i^0(\mathbf{k}) = 0$.

2. We now turn to the statistical analysis. Let $q_i^0(\mathbf{k})$ be a random field. All its statistical characteristics can be specified with the aid of the characteristic functional

$$\Phi_0[\mathbf{p}(\varkappa)] = \langle Z[\mathbf{p}(\varkappa); \mathbf{q}^0(\varkappa)]\rangle_{\mathbf{q}^0}, \tag{8}$$

where

$$Z[\mathbf{p}(\varkappa); \mathbf{q}^0(\varkappa)] = \exp\left\{2\pi i\int p_j(\varkappa)\, q_j^0(\varkappa)\, d^3\varkappa\right\}$$

and the brackets $\langle\ \rangle_{\mathbf{q}^0}$ denote averaging over the random field q^0. Knowing $\Phi_0[\mathbf{p}(\varkappa)]$, we can find all the desired characteristics of the field $\mathbf{q}^0(\mathbf{k})$, for example

$$\langle q_i^0(\mathbf{k}_1)\, q_j^0(\mathbf{k}_2)\rangle_{\mathbf{q}^0} = \frac{1}{(2\pi i)^2}\frac{\delta^2\Phi_0[\mathbf{p}(\varkappa)]}{\delta p_i(\mathbf{k}_1)\,\delta p_j(\mathbf{k}_2)}\bigg|_{\mathbf{p}=0}.$$

It is assumed in (8) that the function $p_i(\mathbf{k})$ satisfies the condition $p_i^*(-\mathbf{k}) = p_i(\mathbf{k})$, from which it follows that the integral in the exponent is pure real and consequently $|\Phi_0| \leq 1$.

The solution $q_i(\mathbf{k}, t)$ of Eq. (5) for a random initial condition is also a random field, determined by the characteristic functional

$$\Phi_t[\mathbf{p}(\varkappa)] = \langle Z[\mathbf{p}(\varkappa); \mathbf{q}(\varkappa, t)]\rangle_{\mathbf{q}^0}, \tag{9}$$

and it is assumed here that $q_i(\varkappa, t)$ is expressed in terms of $q_i^0(\mathbf{k})$ and then averaged. If we introduce the functional

$$\Psi_t[\mathbf{p}(\varkappa)] = \langle Z[\mathbf{p}(\varkappa); \mathbf{g}(\varkappa, t)]\rangle_{\mathbf{q}^0}, \tag{10}$$

then, as is clear from (5), (9), and (10),

$$\Phi_t[\mathbf{p}(\varkappa)] = \Psi_t[\mathbf{p}(\varkappa)\exp(-\nu\varkappa^2 t)]. \tag{11}$$

(The transition from Φ_t to Ψ_t and from $\mathbf{q}$ to $\mathbf{g}$ is analogous to the transition from the Schrödinger representation to the interaction representation). We differentiate (10) with respect to time, insert (6), then interchange the order of averaging and integration with respect to $\mathbf{k}$:

$$\frac{\partial\Psi_t}{\partial t} = 2\pi i\int \exp(\nu k^2 t)\, p_i(\mathbf{k})\,\langle Z[\mathbf{p}(\varkappa); \mathbf{g}(\varkappa, t)]\, Q_i$$

$$\times[\exp(-\nu\varkappa^2 t)\,\mathbf{g}(\varkappa, t); \mathbf{k}]\rangle_{\mathbf{q}^0}\, d^3k. \tag{12}$$

The average value involved here can be expressed in terms of the variational derivatives of $\Psi_t[\mathbf{p}(\varkappa)]$.

*We shall adhere to the following notation: if a certain functional F depends on a functional argument $f(\mathbf{k}, t)$, it is written $F[f(\varkappa, \tau)]$ and actually is independent of $\varkappa$ or τ (Greek letters assume the role of 'dummy' indices in tensor algebra). If F is also dependent on certain $\mathbf{k}$ and t, then these arguments are denoted by Latin letters: for example $F[f(\varkappa, t)]$ or $F[f(\mathbf{k}, \tau), t]$.

Putting $D_i(\mathbf{k}) = \delta/\delta p_i(\mathbf{k})$, we have

$$D_l(\mathbf{k}_1) D_j(\mathbf{k}_2) \Psi_t[\mathbf{p}(\varkappa)] = (2\pi i)^2 \langle g_l(\mathbf{k}_1, t) g_j(\mathbf{k}_2, t) Z[\mathbf{p}(\varkappa); \mathbf{g}(\varkappa, t)]\rangle_{q^0}.$$

Multiplying the last equation by $\exp[-\nu(k_1^2 + k_2^2)t] \times B_{l,ij}(\mathbf{k}_1 + \mathbf{k}_2)$, and then putting $\mathbf{k}_2 = \mathbf{k} - \mathbf{k}_1$ and integrating with respect to $\mathbf{k}_1$, we obtain an equation

$$Q_l[\exp(-\nu k^2 t)\mathbf{D}(\varkappa); \mathbf{k}]\Psi_t[\mathbf{p}(\varkappa)] = (2\pi i)^2 \times \langle Z[\mathbf{p}(\varkappa); \mathbf{g}(\varkappa, t)] Q_l[\exp(-\nu k^2 t)\mathbf{g}(\varkappa, t); \mathbf{k}]\rangle_{q^0},$$

which when substituted in (12) results in a variational differential equation equivalent to the Hopf equation: [1]

$$2\pi i \frac{\partial \Psi_t}{\partial t} = \int d^3k \exp(\nu k^2 t) p_l(\mathbf{k}) Q_l \times [\exp(-\nu \varkappa^2 t)\mathbf{D}(\varkappa); \mathbf{k}]\Psi_t[\mathbf{p}(\varkappa)]. \quad (13)$$

If we introduce the time-dependent operators and functions*

$$D_l(\mathbf{k}, t) = \exp(-\nu k^2 t) D_l(\mathbf{k}), \quad p_l(\mathbf{k}, t) = \exp(\nu k^2 t) p_l(\mathbf{k}), \quad (14)$$

then (13) becomes

$$2\pi i \frac{\partial \Psi_t}{\partial t} = \int d^3k p_l(\mathbf{k}, t) Q_l[\mathbf{D}(\varkappa, t); \mathbf{k}]\Psi_t[\mathbf{p}(\varkappa)]. \quad (15)$$

It is necessary to add to (15) the initial condition

$$\Psi_0[\mathbf{p}(\varkappa)] = \Phi_0[\mathbf{p}(\varkappa)], \quad (16)$$

where Φ_0 satisfies the condition $k_i D_i(\mathbf{k})\Phi_0[\mathbf{p}(\boldsymbol{\kappa})] = 0$, which is a consequence of the condition $k_i q_i^0(\mathbf{k}) = 0$.

3. Equation (15) is analogous to the equation of nonlinear quantum theory for a certain vector Bose field, with $p_i(\mathbf{k})$ and $D_i(\mathbf{k})$ regarded as operators for the creation and annihilation of bosons with momentum $\mathbf{k}$. It is easy to check that these operators satisfy the same commutation relation as the creation and annihilation operators†

$$[D_i(\mathbf{k}), p_j(\mathbf{k}')] = \delta_{ij}\delta(\mathbf{k} - \mathbf{k}'). \quad (17)$$

From the definition of (14) we then obtain

$$[D_l(\mathbf{k}, t), p_j(\mathbf{k}', t')] = \exp[-\nu k^2(t - t')]\delta_{lj}\delta(\mathbf{k} - \mathbf{k}'). \quad (18)$$

The state vectors in the Schrödinger and in the interaction representations are analogous to the functionals Φ_t and Ψ_t. The operator

$$H(t) = \int d^3k p_l(\mathbf{k}, t) Q_l[\mathbf{D}(\varkappa, t); \mathbf{k}] = \int d^3k_1 \times \int d^3k_2 B_{l,ij}(\mathbf{k}_1 + \mathbf{k}_2) p_l(\mathbf{k}_1 + \mathbf{k}_2, t) D_l(\mathbf{k}_1, t) D_j(\mathbf{k}_2, t) \quad (19)$$

is interpreted as the interaction Hamiltonian in the interaction representation. Equation (15) in the form

$$2\pi i \partial\Psi_t/\partial t = H(t)\Psi_t[\mathbf{p}(\varkappa)] \quad (20)$$

is analogous to the Schrödinger equation in the representation interaction. The essential difference from quantum field theory is that the Hamiltonian (19) has no crossing symmetry and describes only the single process of fusion of two bosons.

4. The indicated singularity enables us to construct for (20) a certain class of exact solutions, corresponding to special initial conditions. For this purpose we use the well known series expansion

$$\Psi_t[\mathbf{p}(\varkappa)] = S(t)\Phi_0[\mathbf{p}(\varkappa)] = \left\{1 + \ldots + \frac{1}{(2\pi i)^m}\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \times \int_0^{\tau_{m-1}} d\tau_m H(\tau_1)\ldots H(\tau_m) + \ldots\right\}\Phi_0[\mathbf{p}(\varkappa)]. \quad (21)$$

Let Φ_0 be a functional of n-th degree

$$\Phi_0 = \theta_n[\mathbf{p}(\varkappa)] = \int\ldots\int f_{i_1\ldots i_n}(\mathbf{k}_1, \ldots, \mathbf{k}_n) p_{i_1} \times (\mathbf{k}_1)\ldots p_{i_n}(\mathbf{k}_n) d^3k_1 \ldots d^3k_n$$

(in field theory it describes the n-particle state). It is clear that the operator $H(t)$, acting on θ_n, reduces its degree by unity; the $(m+1)$-th member of the series (21) reduces the degree of this functional by m; all members of the series starting with the $(n+1)$-st, acting on θ_n, yield zero. Thus, if we specify the n-particle state as the initial state, then the first n members of the series (21) yield the exact solution of the problem. As is well known, the series (21) is an expansion in the interaction constant, which in our case is the Reynolds number (this is clear from the fact that the small parameter ν precedes not the nonlinear terms of (1), but the linear term Δv_i). Consequently, the finite partial sequences of the series (21), which yield the exact solution

*Expressions (14) can also be obtained in the form $\exp(iH_0t)D_i(\mathbf{k})\exp(-iH_0t)$ and $\exp(iH_0t)p_i(\mathbf{k})\exp(-iH_0t)$, where H_0 is the 'free-field Hamiltonian,' which we do not write out here. Our method of obtaining (14) and (15), however, is shorter.

†In field theory it is preferable to use as the creation operator the Hermitian adjoint operators $A_i^+(\mathbf{k}) = \frac{1}{2}p_i(\mathbf{k}) - D_i(\mathbf{k})$ and $A_i^-(\mathbf{k}) = \frac{1}{2}p_i(\mathbf{k}) + D_i(\mathbf{k})$ [4], which satisfy the same commutation relations $[A_i^-(\mathbf{k}), A_j^+(\mathbf{k}')] = \delta_{ij}\delta(\mathbf{k} - \mathbf{k}')$. Since we do not use the adjoint condition for the creation and annihilation operators, we can employ the foregoing analogy.

considered above, are polynomials of degree n in the Reynolds number.

However, the initial state θ_n, which has physical meaning in field theory, has no meaning in turbulence theory. Indeed, from the definition (8) it follows that $|\Phi_0| \leq 1$. But the functional θ_n which we are considering does not have this property, and neither does any linear combination of such functionals with different n. We can expand Φ_0 only in an infinite power series corresponding to an infinite number of particles in the initial state

$$\Phi_0[\mathbf{p}(\varkappa)] = \sum_{n=0}^{\infty} \frac{(2\pi i)^n}{n!} \int \cdots$$
$$\times \int \langle q^0_{i_1}(\mathbf{k}_1) \ldots q^0_{i_n}(\mathbf{k}_n)\rangle_{q^0} p_{i_1}(\mathbf{k}_1) \ldots p_{i_n}(\mathbf{k}_n) d^3\mathbf{k}_1 \ldots d^3\mathbf{k}_n.$$

In virtue of the linearity of (20), we can seek a solution in the form of a sum of terms corresponding to the individual terms in the expansion of Φ_0. For each of these terms we can write out an exact solution which is a polynomial of finite degree in the Reynolds number. The series constructed in this fashion, which we do not write out here, is a series in powers of the Reynolds number. Even if we assume that this series converges, the number of terms that must be taken in it is of the same order as the Reynolds number of the initial state, which is unrealistic in practice. However, such a construction indicates the nature of the difficulties that arise when attempts are made to obtain a solution by neglecting moments of relatively low order.

5. Our next problem is to represent the solution of (20) in the form of a continuous integral. For this purpose we transform the expression

$$S(t) = T(K), \qquad K = \exp\left[\frac{1}{2\pi i}\int_0^t H(\tau)\, d\tau\right] \tag{22}$$

for the S-matrix with the aid of the Hori operator [5] to the normal form

$$S(t) = N\{(\exp \Delta)\cdot K\}, \tag{23}$$

where Δ is the operator

$$\Delta = \int d^3k_1 \int d^3k_2 \int_0^t d\tau_1 \int_0^t d\tau_2 \dot{p}_i(\mathbf{k}_1, \tau_1) \dot{D}_j(\mathbf{k}_2, \tau_2)$$
$$\times \frac{\delta}{\delta p_i(\mathbf{k}_1, \tau_1)} \frac{\delta}{\delta D_j(\mathbf{k}_2, \tau_2)}. \tag{24}$$

Convolution of $\dot{\mathbf{p}}(\mathbf{k}_1, \tau_1)\dot{\mathrm{D}}_j(\mathbf{k}_2, \tau_2)$, yields

$$\dot{p}_i(\mathbf{k}_1, \tau_1) \dot{D}_j(\mathbf{k}_2, \tau_2)$$
$$= \theta(\tau_2 - \tau_1) \exp[-\nu k_1^2(\tau_2 - \tau_1)]\, \delta_{ij}\delta(\mathbf{k}_1 - \mathbf{k}_2),$$

where $\theta(x) = 1$ when $x > 0$ and $\theta(x) = 0$ when $x < 0$, and Δ assumes the form

$$\Delta = \int d^3k \int_0^t d\tau_2 \int_0^{\tau_2} d\tau_1$$
$$\times \exp[-\nu k^2(\tau_2 - \tau_1)] \frac{\delta}{\delta p_i(\mathbf{k}, \tau_1)} \frac{\delta}{\delta D_i(\mathbf{k}, \tau_2)}. \tag{25}$$

Applying the operator (23) to Φ_0 written in the form (8), and including the operator S under the averaging sign, we obtain

$$\Psi_t[\mathbf{p}(\varkappa)] = \langle\{N[(\exp \Delta)\cdot K]\}\cdot Z[\mathbf{p}(\varkappa); \mathbf{q}^0(\varkappa)]\rangle_{q^0}. \tag{26}$$

We now note that all the differentiation operators $D_i(\mathbf{k}, t)$ in (23) are on the right side of the functions $p_i(\mathbf{k}, t)$, and consequently act only on Z. But when applied to this exponential, the operators D_i can be replaced by the eigenvalues in accordance with the rule

$$D_i(\mathbf{k}, t) \to 2\pi i \tilde{q}_i(\mathbf{k}, t), \qquad \tilde{q}_i(\mathbf{k}, t) = \exp[-\nu k^2 t].q_i^0(\mathbf{k}). \tag{27}$$

This substitution can obviously be made also in expression (23), provided the operator $\delta/\delta D_i(\mathbf{k}, t)$ in Δ is replaced in accord with the rule

$$\frac{\delta}{\delta D_i(\mathbf{k}, \tau)} \to \frac{1}{2\pi i} \frac{\delta}{\delta \tilde{q}_i(\mathbf{k}, \tau)}. \tag{28}$$

Then $H \to \tilde{H}$ and $\Delta \to \tilde{\Delta}$, where

$$\tilde{H}[\mathbf{p}(\varkappa, t); \tilde{\mathbf{q}}(\varkappa, t)]$$
$$= -4\pi^2 \int d^3k p_i(\mathbf{k}, t)\, Q_i[\tilde{\mathbf{q}}(\varkappa, t); \mathbf{k}], \tag{29}$$

$$\tilde{\Delta} = \frac{1}{2\pi i} \int d^3k \int_0^t d\tau_2 \int_0^{\tau_2} d\tau_1 \exp[-\nu k^2(\tau_2 - \tau_1)]$$
$$\times \frac{\delta}{\delta p_i(\mathbf{k}, \tau_1)} \frac{\delta}{\delta \tilde{q}_i(\mathbf{k}, \tau_2)}. \tag{30}$$

If we introduce the notation

$$\tilde{K}_t[\mathbf{p}(\varkappa, \tau); \tilde{\mathbf{q}}(\varkappa, \tau)] = \exp\left\{\frac{1}{2\pi i}\int_0^t \tilde{H}[\mathbf{p}(\varkappa, t'); \tilde{\mathbf{q}}(\varkappa, t')]\, dt'\right\},$$

then (26) assumes the form (the sign of N can now be omitted)

$$\Psi_t[\mathbf{p}(\varkappa)] = \langle\{(\exp \tilde{\Delta})$$
$$\times \tilde{K}_t[\mathbf{p}(\varkappa, \tau); \tilde{\mathbf{q}}(\varkappa, \tau)]\}\cdot Z[\mathbf{p}(\varkappa); \mathbf{q}^0(\varkappa)]\rangle_{q^0}. \tag{31}$$

We note that by virtue of the conditions $p_i^*(-\mathbf{k}, t) = p_i(\mathbf{k}, t)$, $\tilde{q}_i^*(-\mathbf{k}, t) = \tilde{q}_i(\mathbf{k}, t)$, and $B^*_{l,ij}(-\mathbf{k}) = B_{l,ij}(\mathbf{k})$ the quantity $\tilde{H}$ is pure real. The quantity

$$\tilde{S}_t[\mathbf{p}(\varkappa, \tau); \tilde{\mathbf{q}}(\varkappa, \tau)] = (\exp \tilde{\Delta})\cdot \tilde{K}_t[\mathbf{p}(\varkappa, \tau); \tilde{\mathbf{q}}(\varkappa, \tau)] \tag{32}$$

is now no longer an operator, but a functional of the c-functions $\mathbf{p}(\kappa, \tau)$ and $\tilde{q}(\kappa, \tau)$. With the aid of (32) we rewrite (31) in the form

$$\Psi_t [\mathbf{p}(\varkappa)] = \langle \widetilde{S}_t [\mathbf{p}(\varkappa, \tau); \widetilde{\mathbf{q}}(\varkappa, \tau)] \cdot Z [\mathbf{p}(\varkappa); \mathbf{q}^0(\varkappa)] \rangle_{q^0}. \tag{33}$$

6. To find S_t we use the Fourier transformation. We first determine the continuous integral of the function $F[f(x)]$, $a \le x \le b$, with the aid of the relation [6]

$$\int F[f(\xi)] \mathfrak{D}f(\xi) = \lim_{n\to\infty} \ldots \int \ldots \int F(f_1, \ldots, f_n) \times (\sqrt{\Delta x}\, df_1) \ldots (\sqrt{\Delta x}\, df_n), \tag{34}$$

where $\Delta x = (b-a)/n$ and $F(f_1, \ldots, f_n)$ is the value of $F[f(\xi)]$ on a step function that assumes values $f_1, \ldots, f_n$. Symbolically we can write (34) as $\mathfrak{D}f(\xi) = \prod_\xi \sqrt{d\xi}\, df(\xi)$. If

$$\delta_\infty [\varphi(\xi)] = \int \exp\left[2\pi i \int f(\xi)\varphi(\xi) d\xi\right] \mathfrak{D}f(\xi), \tag{35}$$

then, as shown by Novikov, [6] the following formula holds true

$$\int F[\varphi(\xi)]\, \delta_\infty [\varphi(\xi) - f(\xi)]\, \mathfrak{D}\varphi(\xi) = F[f(\xi)]. \tag{36}$$

By taking the limits $a \to -\infty$ and $b \to -\infty$, we extend (34)—(36) to include the case of an infinite interval. Substituting (35) in (36) we obtain the formula for the continuous Fourier integral

$$F[f(\xi)] = \int \mathfrak{D}\varphi(\xi) \int \mathfrak{D}\psi(\xi)\, F[\varphi(\xi)] \times \exp\left\{2\pi i \int \psi(\xi)[\varphi(\xi) - f(\xi)] d\xi\right\}. \tag{37}$$

Formula (37) holds true for real functions f, φ, ψ, but by introducing new variables, which are the Fourier transforms of these functions, formula (37) can be extended also to complex functions which are Fourier transforms of real functions (for which the integral in the exponential is real). In the case of functions of many variables, all the formulas remain in force if we put

$$\mathfrak{D}f(\varkappa, \tau) = \prod_{\varkappa,\tau} \sqrt{d^3\varkappa\, d\tau}\, df(\varkappa, \tau).$$

7. We represent the functional $\widetilde{K}_t$ in the form of a continuous integral in the variable $\widetilde{q}_i(\kappa, \tau)$. Using an analog of (37), we obtain

$$\widetilde{K}_t [\mathbf{p}(\varkappa, \tau); \widetilde{\mathbf{q}}(\varkappa, \tau)] = \int \mathfrak{D}^3 a(\varkappa, \tau) \int \mathfrak{D}^3 A(\varkappa, \tau)\, \widetilde{K}_t [\mathbf{p}(\varkappa, \tau); \mathbf{a}(\varkappa, \tau)] \times \exp\left\{2\pi i \int d^3k \int_0^t d\tau A_i(\mathbf{k}, \tau) [\widetilde{q}_i(\mathbf{k}, \tau) - a_i(\mathbf{k}, \tau)]\right\}, \tag{38}$$

where the integration extends to the functions **a** and **A**, which satisfy the conditions $a^*(-\kappa, \tau) = a(\kappa, \tau)$, $\mathbf{A}^*(-\kappa, \tau) = \mathbf{A}(\kappa, \tau)$. In expression (38), the functions $p_i(\kappa, \tau)$, $\widetilde{q}_i(\kappa, \tau)$ enter in the exponential linearly. We can therefore readily calculate the action of the operator $\widetilde{\Delta}$ on $\widetilde{K}_t$: it multiplies the integrand in (38) by a factor

$$\widetilde{\Delta} \to \frac{1}{2\pi i} \int_0^t \widetilde{H} [\mathbf{L} [\mathbf{A}(\varkappa, \tau); t']; \mathbf{a}(\varkappa, t')]\, dt',$$

where

$$L_i [\mathbf{A}(\mathbf{k}, \tau); t'] = \int_{t'}^t \exp[-\nu k^2 (\tau - t')]\, A_i(\mathbf{k}, \tau)\, d\tau.$$

Since this factor is independent of the functions $p_i(\kappa, \tau)$, $\widetilde{q}_i(\kappa, \tau)$, the repeated application of the operator $\widetilde{\Delta}$ is given by the same formula and then

$$\exp \widetilde{\Delta} \to \exp\left\{\frac{1}{2\pi i} \int_0^t \widetilde{H} [\mathbf{L} [\mathbf{A}(\varkappa, \tau); t']; \mathbf{a}(\varkappa, t')]\, dt'\right\} = \widetilde{K}_t [\mathbf{L} [\mathbf{A}(\varkappa, \tau'), \tau]; \mathbf{a}(\varkappa, \tau)].$$

Substituting the last expression into the formula for $\widetilde{S}_t$, we obtain

$$\widetilde{S}_t [\mathbf{p}(\varkappa, \tau); \widetilde{\mathbf{q}}(\varkappa, \tau)] = \int \mathfrak{D}^3 a(\varkappa, \tau) \int \mathfrak{D}^3 A(\varkappa, \tau)\, \widetilde{K}_t [\mathbf{p}(\varkappa, \tau) + \mathbf{L} [\mathbf{A}(\varkappa, \tau'); \tau]; \mathbf{a}(\varkappa, \tau)] \times \exp\left\{2\pi i \int d^3k \int_0^t d\tau A_i(\mathbf{k}, \tau) [\widetilde{q}_i(\mathbf{k}, \tau) - a_i(\mathbf{k}, \tau)]\right\}. \tag{39}$$

8. Formula (39) can be greatly simplified by integrating explicitly over the functions $\mathbf{a}(\kappa, \tau)$, which are contained in the exponential of (39) quadratically. This integration can be carried out by using the method indicated by Feynman. [7]

We represent (39) in the form

$$\widetilde{S}_t [\mathbf{p}, \widetilde{\mathbf{q}}] = \int \exp\left\{2\pi i \int d^3k \int_0^t d\tau A_i(\mathbf{k}, \tau)\, \widetilde{q}_i(\mathbf{k}, \tau)\right\} \times G_t [\mathbf{p}(\varkappa, \tau); \mathbf{A}(\varkappa, \tau)]\, \mathfrak{D}^3 A(\varkappa, \tau), \tag{40}$$

where $G_t[\mathbf{p}(\kappa, \tau); \mathbf{A}(\kappa, \tau)]$ denotes the functional

$$G_t [\mathbf{p}(\varkappa, \tau); \mathbf{A}(\varkappa, \tau)] = \int \mathfrak{D}^3 a(\varkappa, \tau)\, \widetilde{K}_t [\mathbf{p}(\varkappa, \tau) + \mathbf{L} [\mathbf{A}(\varkappa, \tau'); \tau]; \mathbf{a}(\varkappa, \tau)] \times \exp\left\{-2\pi i \int d^3k \int_0^t d\tau A_i(\mathbf{k}, \tau)\, a_i(\mathbf{k}, \tau)\right\}. \tag{41}$$

The integrand in (41) is an exponential function that contains terms both quadratic and linear in $a_i(k, \tau)$. To calculate (41) we introduce new integration variables

$$a_i(\varkappa, \tau) = a_i^0(\varkappa, \tau) + u_i(\varkappa, \tau), \qquad \mathfrak{D}^3 a(\varkappa, \tau) = \mathfrak{D}^3 u(\varkappa, \tau),$$

and choose the fixed functions $a_i^0(\kappa, \tau)$ such that the exponential has no terms linear in u. This leads to a linear integral equation of the convolution type, which can be solved with the aid of the Fourier transformation. Its solution has the following form

$$a_i^0(\mathbf{k}, t') = \int I_{ij}[\mathbf{p}(\varkappa, t') + \mathbf{L}[\mathbf{A}(\varkappa, \tau); t']; \mathbf{k} + \mathbf{k}'] A_j(\mathbf{k}', t')\, d^3k', \quad (42)$$

where the following notation is used:*

$$I_{il}[\mathbf{f}(\varkappa, t'); \mathbf{k}] = \frac{1}{16\pi^3}\int (\exp i\mathbf{k}\mathbf{x}) \frac{\partial \ln T[\mathbf{f}(\varkappa, t'); \mathbf{x}]}{\partial T_{ij}[\mathbf{f}(\varkappa, t'); \mathbf{x}]} d^3x,$$

$$T = \operatorname{Det}\| T_{il}\|,$$

$$T_{ij}[\mathbf{f}(\varkappa, t'); \mathbf{x}] = \int (\exp i\mathbf{k}\mathbf{x})\, \Gamma_{ij}[\mathbf{f}(\mathbf{k}, t')]\, d^3k,$$

$$\Gamma_{ij}[\mathbf{f}(\mathbf{k}, t')] = \tfrac{1}{2}[B_{l,jl}(\mathbf{k}) + B_{j,ll}(\mathbf{k})] f_l(\mathbf{k}, t').$$

With the aid of this change of variables, the integral (41) reduces to

$$G_t[\mathbf{p}(\varkappa, \tau); \mathbf{A}(\varkappa, \tau)] = P[\Gamma_{ij}, \mathbf{A}]\cdot Q[\Gamma_{ij}], \quad (43)$$

where

$$\Gamma_{ij} = \Gamma_{ij}[\mathbf{p}(\mathbf{k}, t) + \mathbf{L}[\mathbf{A}(\mathbf{k}, \tau); t]];$$

$$P[\Gamma_{ij}/\mathbf{A}] = \exp\left\{-i\pi\int d^3k' \int d^3k'' \int_0^t dt' I_{ij}[\Gamma_{ln}; \mathbf{k}' + \mathbf{k}''] A_i(\mathbf{k}', t') A_j(\mathbf{k}'', t')\right\}, \quad (44)$$

$$Q[\Gamma_{ij}] = \int \exp\left\{2\pi i\int d^3k' \times \int d^3k'' \int_0^t dt'\, \Gamma_{ij} u_i(\mathbf{k}', t')\, u_j(\mathbf{k}'', t')\right\} \mathfrak{D}^3u. \quad (45)$$

Since the functional Q depends on the function A, in which integration is subsequently carried out, we must evaluate also the integral (45).† This can be done following the same procedure of Feynman. Differentiating (41), we can write the equation

$$-2\pi i \frac{\delta G_t}{\delta \Gamma_{ij}(\mathbf{k}, t')} = \int \frac{\delta^2 G_t}{\delta A_i(\mathbf{k}', t')\, \delta A_j(\mathbf{k} - \mathbf{k}', t')} d^3k'. \quad (46)$$

Substituting (43), (44), and (45) into this equation, we can obtain after rather cumbersome manipulations the following equation for the functional Q:

$$\delta \ln Q/\delta\Gamma_{ij}(\mathbf{k}, t') = -8\pi^3\delta_4(0)\, I_{ij}[\Gamma_{ln}(\mathbf{k}, t')], \quad (47)$$

where $\delta_4(0)$ is an infinite constant, equal to $\delta(\mathbf{k})\,\delta(t)\big|_{\mathbf{k}=0,\, t=0}$. In carrying out the repeated integration with respect to **A**, this constant should vanish by virtue of the normalization of Φ_t. When the continuous integral is written in the finite-dimensional form (34), we must replace $\delta_4(0)$ by $(\Delta^3\mathbf{k}\cdot\Delta t)^{-1}$. It is easy to check that the solution of (46) has the form

$$\ln Q = \ln Q_0 - \frac{\delta_4(0)}{2}\int d^3x \int_0^t dt' \ln T[\mathbf{f}(\varkappa, t'); \mathbf{x}],$$

where Q_0 is a numerical constant, and $\mathbf{f}(\mathbf{k}, t')$ should be replaced by $\mathbf{p}(\mathbf{k}, t') + \mathbf{L}[\mathbf{A}(\mathbf{k}, \tau); t']$. G_t is finally written in the form

$$G_t[\mathbf{p}(\varkappa, \tau); \mathbf{A}(\varkappa, \tau)] = M[\mathbf{p}(\varkappa, \tau) + \mathbf{L}[\mathbf{A}(\varkappa, \tau'); \tau]; \mathbf{A}(\varkappa, \tau)],$$

where

$$M[\mathbf{f}(\varkappa, \tau); \mathbf{A}(\varkappa, \tau)] = Q_0 \exp\left\{-\frac{\delta_4(0)}{2}\int d^3x \int_0^t dt' \ln T[\mathbf{f}(\varkappa, t'); \mathbf{x}] - i\pi\int d^3k_1 \int d^3k_2 \int_0^t dt' I_{ij}[\mathbf{f}(\varkappa, t'), \times \mathbf{k}_1 + \mathbf{k}_2] A_i(\mathbf{k}_1, t') A_j(\mathbf{k}_2, t')\right\}. \quad (48)$$

9. Bearing in mind formula (48) for the functional $G_t[\mathbf{p}, \mathbf{A}]$, we rewrite (33) for $\Psi_t[\mathbf{p}(\kappa)]$ in the form

$$\Psi_t[\mathbf{p}(\varkappa)] = \left\langle G_t[\mathbf{p}(\varkappa, \tau); \mathbf{A}(\varkappa, \tau)]\cdot \exp\left\{2\pi i \int d^3k[p_i(\mathbf{k})\, q_i^0(\mathbf{k}) + \int_0^t d\tau A_i(\mathbf{k}, \tau) \exp(-\nu k^2\tau)\, q_i^0(\mathbf{k})]\right\} \mathfrak{D}^3A(\mathbf{k}, \tau)\right\rangle_{q^0}, \quad (49)$$

where we use the explicit expression for the functional $Z[\mathbf{p}(\kappa), \mathbf{q}^0(\kappa)]$ and make the substitution $\mathbf{q}(\mathbf{k}, \tau) = \exp(-\nu k^2\tau)\mathbf{q}^0(\mathbf{k})$. Replacing $\mathbf{p}(\mathbf{k})$ by $\exp(-\nu k^2 t)\mathbf{p}(\mathbf{k})$ in accord with (11) we change over from Ψ_t to Φ_t; in addition, we put $\mathbf{p}(\mathbf{k}, \tau) = \exp(\nu k^2\tau)\mathbf{p}(\mathbf{k})$. As a result we obtain

$$\Phi_t[\mathbf{p}(\varkappa)] = \left\langle \int G_t[\mathbf{p}(\varkappa)\exp[-\nu\varkappa^2(t-\tau); \mathbf{A}(\varkappa, \tau)] \times \exp\left\{2\pi i\int d^3k q_i^0(\mathbf{k})[p_i(\mathbf{k})\exp(-\nu k^2 t) + \int_0^t \exp(-\nu k^2\tau) A_i(\mathbf{k}, \tau)\, d\tau]\right\} \mathfrak{D}^3A(\mathbf{k}, \tau)\right\rangle_{q^0}. \quad (50)$$

Using the definition (8), we average the $q_i^0(\mathbf{k})$-dependent factors in (50), obtaining $\Phi_0[\mathbf{p}(\kappa)\times \exp(-\nu\kappa^2 t) + \mathbf{L}[\mathbf{A}(\kappa, \tau), 0]]$. As a result we obtain a final expression for the characteristic

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*We note that $\operatorname{Det}\| T_{ij}\| \neq 0$, although the incompressibility condition yields $\operatorname{Det}\| \Gamma_{ij}\| = 0$. The proof is usually obtained by writing $\operatorname{Det}\| T_{ij}\|$ in the form of a triple integral of Γ_{ij}.

†We note that Rosen[2] did not take this into account, erroneously, an analogous circumstance and the numerical constant appearing in this paper is actually a functional of a function over which integration is subsequently carried out.

functional during the process of degeneration of the turbulence:

$$\Phi_t[\mathbf{p}(\varkappa)] = \int G_t[\mathbf{p}(\varkappa)\exp[-\nu\varkappa^2(t-\tau)]; \mathbf{A}(\varkappa,\tau)] \times \Phi_0[\mathbf{p}(\varkappa)\exp(+\nu\varkappa^2 t) + \mathbf{L}[\mathbf{A}(\varkappa,\tau);0]]\,\mathfrak{D}^3 A(\varkappa,\tau). \quad (51)$$

10. On the basis of (51) we can draw certain conclusions concerning the character of the solution in the case when the Reynolds number of the initial state tends to infinity. In this case the characteristic functional of the initial state $\Phi_0[\mathbf{p}]$ is concentrated in an ever narrowing region near the point $\mathbf{p} = 0$. This is directly clear if $\Phi_0[\mathbf{p}(\kappa)]$ is a Gaussian functional of the form

$$\Phi_0[\mathbf{p}(\varkappa)] = \exp\left\{-\frac{1}{2}\int W_{ij}(\varkappa)\,p_i(\varkappa)\,p_j(-\varkappa)\,d^3\varkappa\right\},$$

where $W_{ij}(\kappa)$ is the spectral density of the initial-state energy, proportional to $v_0^2 L_0^3$, L_0 is the characteristic dimension and v_0 the characteristic velocity when $t = 0$. It is clear that as $v_0 L_0 \to \infty$ the functional Φ_0 is concentrated near the point $\mathbf{p} = 0$. The situation is similar not only for a Gaussian function, but in the general case: as the characteristic velocity and the characteristic scale tend to infinity, the characteristic functional is concentrated near the point $\mathbf{p} = 0$. This is a consequence of the fact that the probability density and the characteristic functions are Fourier transforms of each other, and therefore when the scale of one of the functions tends to infinity the scale of the transform turns to zero. But it follows from (51) that in this case $\Phi_t[\mathbf{p}]$ is independent of the specific form of the initial functional Φ_0, for as $L_0 v_0 \to \infty$ and for arbitrary Φ_0 this functional is concentrated near the point $\mathbf{p} = 0$. Thus, at infinitely large Reynolds numbers in the initial state, the law governing the degeneration of the turbulence does not depend on the form of the probability distribution of the initial state.

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