HIGHER PARTIAL WAVES IN LOW ENERGY ππ SCATTERING

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Higher partial waves are taken into account in the equations for low-energy ππ scattering derived by the differential method. Their effect is shown to be small. A procedure for taking into account an infinite number of partial waves in the low energy equations is considered and found to be meaningless.

1. FORMULATION OF THE DIFFERENTIAL METHOD ON THE EXAMPLE OF SCATTERING OF NEUTRAL MESONS

IN the recent papers of Sarker[1] and Lovelace[2] questions were raised about the correspondence between the equations for partial waves for low-energy ππ scattering as derived by the differential [3-5] and the integral[6] methods (in what follows, the work of Chew and Mandelstam[6] will be referred to as CM). In view of some errors in [1,2] we discuss below in detail the question of inclusion of higher partial waves in the differential method, and also the relation of this method to the CM method.

The earlier[3] equations for ππ scattering were obtained by combining forward and backward dispersion relations, whereas in the work of Hsien, Ho, and Zöllner[4] there was also included information from first derivatives with respect to momentum transfer. Consequently, the real parts of only s and p waves were taken into account in [3], whereas in [4] d and f waves were also included.

We consider the question of including a larger and larger number of partial waves, up to the limiting case of an infinite number of such waves.

Let us write out first the formulas that express the low partial waves f_i in terms of the value and the derivatives of the function f(c) at the points $c = \pm 1$. Different formulas are obtained depending on how many harmonics are used to approximate the function f(c).

In the lowest approximation, keeping s and p waves only

$$f(c) \approx f_0 + 3f_1c,$$

we obtain

$$f_0 = \tfrac{1}{2}[f(1) + f(-1)], \qquad f_1 = \tfrac{1}{6}[f(1) - f(-1)]. \tag{1.1}$$

In the next approximation, which includes d and f waves,

$$f(c) = f_0 + 3cf_1 + \tfrac{5}{2}(3c^2 - 1)f_2 + \tfrac{7}{2}(5c^3 - 3c)f_3,$$

we obtain

$$\begin{aligned} f_0 &= \tfrac{1}{2}[f(1) + f(-1)] - \tfrac{1}{6}[f'(1) - f'(-1)],\\ f_1 &= \tfrac{1}{5}[f(1) - f(-1)] - \tfrac{1}{30}[f'(1) + f'(-1)],\\ f_2 &= \tfrac{1}{30}[f'(1) - f'(-1)],\\ f_3 &= -\tfrac{1}{70}[f(1) - f(-1)] + \tfrac{1}{70}[f'(1) + f'(-1)]. \end{aligned} \tag{1.2}$$

Finally, in the limiting case

$$f(c) = \sum_{n=0}^{\infty}(2n+1)f_nP_n(c)$$

we have the formulas

$$\begin{aligned} f_0 &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{f^{(n)}(-1) + (-)^n f^{(n)}(1)}{(n+1)!},\\ f_1 &= -\frac{1}{2}\sum_{n=0}^{\infty}\frac{f^{(n)}(-1) - (-)^n f^{(n)}(1)}{(n+2)!},\\ f_2 &= -\frac{1}{4}\sum_{n=0}^{\infty}\frac{n(n+5)}{(n+3)!}[f^{(n)}(-1) + (-)^n f^{(n)}(1)], \ldots, \end{aligned} \tag{1.3}$$

which express the partial waves of the function f(c) in terms of an infinite number of its derivatives at the points $c = \pm 1$.

Equations (1.3) can be obtained purely formally, by consecutive integration by parts of the integrals defining the partial amplitudes. The convergence of a series of the type (1.3) is determined by the singularities of the function f(c) in the complex c plane. In Sec. 3 below we consider this problem for the case of ππ scattering.

We wish to note here that the passage from

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Eq. (1.1) to Eq. (1.2), and from Eq. (1.2) to Eq. (1.3), consists of not only the addition of terms with higher derivatives, but also requires the modification of the coefficients of terms already present. We shall apply formulas of type (1.1)—(1.3) to the scattering amplitude given in a spectral representation over the momentum or energy variable at fixed values of the cosine of the scattering angle c. For the scattering amplitude for neutral mesons this representation has the form

$$A(\nu, c) = \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im} A(\nu', c)}{\nu' - \nu} d\nu' + \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im} A(\nu', c_+)}{1 + \nu' + \nu(1+c)/2} d\nu' + \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im} A(\nu', c_-)}{1 + \nu' + \nu(1-c)/2} d\nu'. \quad (1.4)$$

Here the first integral refers to the physical cut of the first reaction, for which the square of the energy is $s = 4(\nu+1)$, the second integral refers to the crossed channel with energy squared equal to $u = -2\nu(1+c)$, and the third integral to the crossed channel with energy squared equal to $t = -2\nu(1-c)$, and where

$$c_+ = \frac{2 + 3\nu' - c(2+\nu')}{\nu'(1+c)}, \quad c_- = \frac{2 + 3\nu' + c(2+\nu')}{\nu'(1-c)}. \quad (1.5)$$

It is seen from Eq. (1.5) that

$$c_\pm(\nu', c = \pm 1) = \pm 1, \quad c_\mp(\nu', c = \pm 1) = \infty.$$

Therefore, in the limiting cases $c = \pm 1$ the numerator of one of the crossed integrands corresponds to forward (or backward) scattering, while the other contains the unphysical infinite cosine.

In the CM scheme, as well as in the Cini-Fubini [8] representation, the function $\operatorname{Im} A(\nu', c_\pm)$ is approximated by s waves in the entire interval $-1 \le c \le 1$, i.e., up to and including infinitely large values of the cosines $c_\pm$. Performing the indicated approximation, we obtain after integration over c the CM equation for the neutral model (compare [8,9])

$$A_0(\nu) = \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{\nu' - \nu} \operatorname{Im} A_0(\nu') - \frac{2}{\pi\nu}\int_0^\infty d\nu' \operatorname{Im} A_0(\nu') \ln\left(1 - \frac{\nu}{1 + \nu + \nu'}\right). \quad (1.6)$$

From here it is again clear that in the methods of CM and Cini-Fubini one makes use of an analytic continuation with the help of the first term in the Legendre polynomial expansion into a region in which the Legendre series does not exist.

Let us derive now the equation for the s wave by means of the differential approximation. Substituting Eq. (1.4) into the first of the formulas (1.1), and approximating $\operatorname{Im} A(\nu, \pm 1)$ by the s wave, we obtain

$$A_0(\nu) = \frac{1}{\pi}\int_0^\infty d\nu' \left(\frac{1}{\nu' - \nu} + \frac{1}{1 + \nu' + \nu}\right) \operatorname{Im} A_0(\nu') + \alpha, \quad (1.7)$$

where

$$\alpha = \frac{1}{2\pi}\int_0^\infty \frac{d\nu'}{1 + \nu'} [\operatorname{Im} A(\nu', \infty) + \operatorname{Im} A(\nu', -\infty)]. \quad (1.8)$$

Let us show that for solutions to Eq. (1.7) to exist it is necessary to set $\alpha = 0$. To this end we perform in Eq. (1.7) one subtraction, reducing it to the form of Eq. (2.5) of [7]. By repeating the considerations of Sec. 2 of that paper we arrive at the conclusion that $\operatorname{Im} A(\infty) = 0$, which, together with the unitarity condition

$$\operatorname{Im} A_0(\nu) = \sqrt{\frac{\nu}{1+\nu}} |A_0(\nu)|^2, \quad \nu > 0, \quad (1.9)$$

leads to

$$\operatorname{Re} A_0(\infty) = 0.$$

In other words, Eq. (1.7) has a solution only if

$$\alpha = 0. \quad (1.10)$$

We remark now that the quantity α represents the high-energy contributions. Indeed, for example, the third integral in Eq. (1.4) corresponds to the segment of the straight line $c = 1 - \epsilon = \text{const}$ from $-\infty$ to the point $a(\epsilon)$ with coordinates $t = 4$, $\nu = -2/\epsilon$, which approaches infinity as $\epsilon \to 0$. Consequently the first term on the right side of Eq. (1.8), which represents the limit of that integral as $c \to 1$, corresponds to the high energy contribution. The same applies to the second integral in Eq. (1.4) in the limit as $c \to -1$. Consequently α represents the contribution from a region which lies above the threshold of any state, with finite mass. It is therefore clear that we can neglect α entirely, since all intermediate states starting with the four-meson state have been ignored.

Let us pass now to the second approximation described by the formulas (1.2). Approximating $\operatorname{Im} A(\nu, \pm 1)$ by s waves and ignoring high energy contributions of the form $\alpha' \pm \beta\nu$ we obtain from the first of the Eqs. (1.2)

$$A_0(\nu) = \frac{1}{\pi}\int_0^\infty d\nu' \operatorname{Im} A_0(\nu') \times \left[\frac{1}{\nu' - \nu} + \frac{1}{\nu' + \nu + 1}\left(1 - \frac{\nu}{6(\nu' + \nu + 1)}\right)\right]. \quad (1.11)$$

It is of interest to investigate the asymptotic behavior of the solution of Eq. (1.11). To this end let us represent the second integral in Eq. (1.11) in the form

$$\left(1-\frac{\nu}{6}\frac{\partial}{\partial\nu}\right)\frac{1}{\pi}\int\frac{\operatorname{Im}A_0(\nu')}{1+\nu'+\nu}d\nu'.$$

It is clear from here that the asymptotic behavior of Eq. (1.11) coincides with the asymptotic behavior of Eq. (1.7) for $\alpha = 0$:

$$\operatorname{Re}A_0(\nu)\to\pi b/\ln\nu,\quad b={}^1/_2, \tag{1.12}$$

since $\nu\partial\ln^{-1}\nu/\partial\nu = -\ln^{-2}\nu$. It therefore follows that the inclusion of d waves in the real part of the scattering amplitude does not appreciably change the behavior of the logarithmic branch of the solution of the neutral model, Eq. (1.7).

We turn now to the limiting case, Eq. (1.3). After ignoring the power series in ν with coefficients due to high energy contributions we obtain for the s waves

$$A_0(\nu)=\frac{1}{\pi}\int_0^\infty\frac{d\nu'}{\nu'-\nu}\operatorname{Im}A_0(\nu')$$
$$+\frac{1}{\pi}\int_0^\infty d\nu'\operatorname{Im}A_0(\nu')\frac{2}{\nu}\sum_{n=1}^\infty\frac{1}{n}\left(\frac{\nu}{2(1+\nu+\nu')}\right)^n.$$

The sum in the crossed integral may be written in closed form with the result

$$A_0(\nu)=\frac{1}{\pi}\int_0^\infty\frac{d\nu'}{\nu'-\nu}\operatorname{Im}A_0(\nu')$$
$$-\frac{2}{\pi\nu}\int_0^\infty d\nu'\operatorname{Im}A_0(\nu')\ln\left(1-\frac{\nu}{2(\nu'+\nu+1)}\right). \tag{1.13}$$

It is not difficult to verify that Eq. (1.13) admits logarithmic asymptotic behavior, Eq. (1.12).

It is important to note that Eq. (1.13) differs substantially from the CM Equation (1.6) which possesses the logarithmic asymptotic behavior of the form (1.12) for b = ⅓.

What has been said above proves that Sarker's [1] assertion is false. His conclusion that equations of the type (1.7), (1.11) may be obtained from (1.6) by expanding the logarithm is based on an insufficiently precise study of the numerical coefficients of the corresponding series.

Let us investigate also the influence of higher partial waves in the imaginary part of the scattering amplitude on the s wave. To this end we repeat our considerations including, with the help of Eq. (1.2), s and d waves in the real as well as in the imaginary part of the scattering amplitude. Carrying out the calculations with the high-energy terms ignored, with the help of the formulas

$$\left.\frac{\partial c_+}{\partial c}\right|_{c=+1}=\left.\frac{\partial c_-}{\partial c}\right|_{c=-1}=-\frac{1+\nu'}{\nu'}, \tag{1.14}$$

we obtain for the s and p waves the system of equations

$$A_0(\nu)=\frac{1}{\pi}\int_0^\infty\frac{\operatorname{Im}A_0(\nu')}{\nu'-\nu}d\nu'$$
$$+\frac{1}{\pi}\int_0^\infty\frac{d\nu'}{1+\nu'+\nu}\left(1+\frac{\nu}{6(\nu'+\nu+1)}\right)(\operatorname{Im}A_0(\nu')$$
$$+5\operatorname{Im}A_2(\nu'))+\frac{10}{3\pi}\int_0^\infty\frac{d\nu'}{1+\nu'+\nu}\frac{1+\nu'}{\nu'}\operatorname{Im}A_2(\nu'), \tag{1.15}$$

$$A_2(\nu)=\frac{1}{\pi}\int_0^\infty\frac{\operatorname{Im}A_2(\nu')}{\nu'-\nu}d\nu'-\frac{1}{\pi}\int_0^\infty\frac{d\nu'}{\nu'+\nu+1}\frac{\nu'+1}{\nu'}\operatorname{Im}A_2(\nu')$$
$$-\frac{\nu}{30\pi}\int_0^\infty\frac{d\nu'}{(1+\nu'+\nu)^2}(\operatorname{Im}A_0(\nu')+5\operatorname{Im}A_2(\nu')). \tag{1.16}$$

It follows from Eq. (1.16) that the logarithmic asymptotic behavior of the function A_2 is determined by the crossed integral containing A_0, and is of the form

$$\operatorname{Re}A_2(\nu')\sim-\ln^{-2}\nu,\qquad \operatorname{Im}A_2(\nu)\sim\ln^{-4}\nu. \tag{1.17}$$

Consequently, the term containing Im A_2 in the crossed integral in Eq. (1.15) goes like $\ln^{-3}\nu$ for large ν and does not change the asymptotic behavior, Eq. (1.12), of the s wave. This leads to two important conclusions:

1) When higher partial waves are taken into account the approximations in the real and imaginary parts of the scattering amplitude must be correlated. Thus, in the approximation following Eq. (1.11) one must take into account Im A_2 in addition to Re A_4. Consequently Eq. (1.13) does not represent an improvement in accuracy in comparison with Eq. (1.11).

2) The logarithmic asymptotic behavior, Eq. (1.12), is not changed when higher partial waves are taken into account in the imaginary as well as in the real part of the scattering amplitude.

As will become clear in what follows, the conclusion 1) is a special property of the neutral model and is due to the absence of the p wave. It will be shown below that for the scattering of charged mesons the coefficient in the logarithmic asymptotic behavior does change, but the change is insignificant.

2. SCATTERING OF CHARGED PIONS

We now consider the realistic case of the scattering of charged pions. The formulas (1.1), (1.2) will be applied to the functions

$$A^0=3A+B+C,\qquad A^1=B-C,\qquad A^2=B+C,$$

specified by the representations

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix}(\nu, c) = \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{\nu' - \nu}\,\mathrm{Im}\begin{bmatrix} A \\ B \\ C \end{bmatrix} + \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{1+\nu'+\nu(1+c)/2}\,\mathrm{Im}\begin{bmatrix} C \\ B \\ A \end{bmatrix}(\nu' c_+) + \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{1+\nu'+\nu(1-c)/2}\,\mathrm{Im}\begin{bmatrix} B \\ A \\ C \end{bmatrix}(\nu', c_-). \quad (2.1)$$

The cosines of the crossed reactions c_+ and c_- are defined by Eq. (1.5).

The simplest equations for s and p waves (see [3,5]) may be obtained from Eq. (2.1) with the help of the formulas (1.1). Restricting oneself in the amplitudes A^J to s and p waves only

$$A^0(\nu, c) \cong A_0^0(\nu) \equiv A_0(\nu), \quad A'(\nu, c) \cong 3cA_1^1(\nu) \equiv 3cA_1(\nu),$$
$$A^2(\nu, c) \cong A_0^2(\nu) \equiv A_2(\nu) \quad (2.2)$$

and taking into account the inverse relations

$$A(\nu, \pm 1) = \tfrac{1}{3}(A_0 - A_2), \qquad B(\nu, \pm 1) = \tfrac{1}{2}(A_2 \pm 3A_1),$$
$$C(\nu, \mp 1) = \tfrac{1}{2}(A_2 \pm 3A_1),$$

we obtain consecutively from Eq. (2.2), ignoring high energy constants of the type of Eq. (1.8),

$$A(\nu, 1) = A(\nu, -1) = \frac{1}{3\pi}\int_0^\infty \frac{d\nu'}{\nu'-\nu}[\mathrm{Im}\,A_0(\nu') - \mathrm{Im}\,A_2(\nu')] + \frac{1}{2\pi}\int_0^\infty \frac{d\nu'}{1+\nu+\nu'}[\mathrm{Im}\,A_2(\nu') - 3\,\mathrm{Im}\,A_1(\nu')],$$

$$B(\nu, 1) = C(\nu, -1) = \frac{1}{2\pi}\int_0^\infty \frac{d\nu'}{\nu'-\nu}[\mathrm{Im}\,A_2(\nu') + 3\,\mathrm{Im}\,A_1(\nu')] + \frac{1}{2\pi}\int_0^\infty \frac{d\nu'}{\nu'+\nu+1}[\mathrm{Im}\,A_2(\nu') + 3\,\mathrm{Im}\,A_1(\nu')],$$

$$B(\nu, -1) = C(\nu, 1) = \frac{1}{2\pi}\int_0^\infty \frac{d\nu'}{\nu'-\nu}[\mathrm{Im}\,A_2(\nu') - 3\,\mathrm{Im}\,A_1(\nu')] + \frac{1}{3\pi}\int_0^\infty \frac{d\nu'}{\nu'+\nu+1}[\mathrm{Im}\,A_0(\nu') - \mathrm{Im}\,A_2(\nu')]. \quad (2.3)$$

On going over to partial waves we obtain the equation

$$A_l(\nu) = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\,A_l(\nu')}{\nu'-\nu} + \frac{1}{\pi}\int_0^\infty \frac{f_l(\nu)}{1+\nu+\nu'}d\nu', \quad (2.4)$$

where

$$f_l(\nu) = \mathrm{Im}\,A_l(\nu) + l_l\varphi(\nu),$$
$$\varphi(\nu) = 2\,\mathrm{Im}\,A_0(\nu) + 9\,\mathrm{Im}\,A_1(\nu) - 5\,\mathrm{Im}\,A_2(\nu),$$
$$l_0 = -{}^1/_3, \qquad l_1 = -{}^1/_{18}, \qquad l_2 = {}^1/_6 \quad (2.5)$$

with the additional threshold condition on the p wave

$$A_1(0) = 0, \quad (2.6)$$

which follows from the crossing symmetry condition

$$B(s, u, t) = C(s, t, u).$$

Equations (2.4) have been studied in detail previously.[5] In particular the existence of a logarithmic branch of the solution was established, with asymptotic behavior given by (see also [2]):

$$A_l(\nu) \sim d_l/\ln\nu, \qquad d_0 = 2.13,$$
$$d_1 = -0.118, \qquad d_2 = 0.640. \quad (2.7)$$

We pass now to the next approximation, in which d and f waves are taken into account in the real parts of the amplitudes. Evaluating the derivatives with the help of the relations (1.16) and

$$\frac{\partial\,\mathrm{Im}\,A(\nu, c)}{\partial c} = 0, \qquad \frac{\partial\,\mathrm{Im}\,B(\nu, c)}{\partial c} = \frac{\partial\,\mathrm{Im}\,C(\nu, c)}{\partial c} = \frac{2}{3}\,\mathrm{Im}\,A_1(\nu),$$

we find

$$A'(\nu, 1) = A'(\nu, -1) = I_1(\nu) - \frac{\nu}{2\pi}\int_0^\infty \frac{d\nu'}{2(1+\nu'+\nu)^2}[\mathrm{Im}\,A_2(\nu') - 3\,\mathrm{Im}A_1(\nu')],$$

$$B'(\nu, 1) = -C'(\nu, -1) = -I_1(\nu) - \frac{\nu}{2\pi}\int_0^\infty \frac{d\nu'}{2(1+\nu'+\nu)^2}[\mathrm{Im}\,A_2(\nu') + 3\,\mathrm{Im}\,A_1(\nu')],$$

$$C'(\nu, 1) = -B'(\nu, -1) = \frac{\nu}{2\pi}\int_0^\infty \frac{d\nu'}{3(1+\nu'+\nu)^2}[\mathrm{Im}\,A_0(\nu') - \mathrm{Im}\,A_2(\nu')], \quad (2.8)$$

where we have used the notation

$$I_1(\nu) = \frac{3}{2\pi}\int_0^\infty \frac{d\nu'}{1+\nu+\nu'}\,\frac{\nu'+1}{\nu'}\,\mathrm{Im}\,A_1(\nu'). \quad (2.9)$$

Substituting Eqs. (2.3) and (2.8) into Eq. (2.1) we obtain

$$A_0(\nu) = \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{\nu'-\nu}\mathrm{Im}\,A_0(\nu') + \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{1+\nu'+\nu}\left(1 + \frac{\nu}{6(\nu'+\nu+1)}\right)f_0(\nu') - \frac{2}{3}I_1(\nu),$$

$$A_1(\nu) = \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{\nu'-\nu}\mathrm{Im}\,A_1(\nu') + \frac{6}{5\pi}\int_0^\infty \frac{d\nu'}{1+\nu+\nu'}\left(1 + \frac{\nu}{12(1+\nu+\nu')}\right)f_1(\nu') + \frac{1}{15}I_1(\nu),$$

$$A_2(\nu) = \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{\nu'-\nu}\mathrm{Im}\,A_2(\nu') + \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{1+\nu+\nu'} \times \left(1 + \frac{\nu}{6(\nu'+\nu+1)}\right)f_2(\nu') + \frac{1}{3}I_1(\nu). \quad (2.10)$$

The f_i appearing above are defined in Eq. (2.5).

Equations (2.10) are analogous to the equations (21)—(23) of Hsien, Ho, and Zöllner.[4] There is, however, one important difference. The point is that the Eqs. (21)—(23) contain terms of the form

$$\int_0^\infty \frac{d\nu'}{\nu'} \operatorname{Im} A_1(\nu'), \qquad (2.11)$$

which are independent of ν and do not vanish in the limit of large ν. Therefore the Eqs. (21)—(23) are not satisfied by a logarithmic asymptotic behavior and, consequently, have no solutions. This remark does not apply to the subtracted equations (25)—(27) of the above-mentioned paper (see [4]) which, consequently, are not equivalent to the unsubtracted equations (21)—(23).

The presence of the terms (2.11) in the equations of Hsien, Ho, and Zöllner[4] is due to the fact that these authors started out not from dispersion relations at fixed cosine c of the type of Eq. (1.4), but rather from dispersion relations at fixed t, which for $t \neq 0$ contain unphysical low energy contributions from regions for which the cosine of the scattering angle varies between the limits $1 \leq |c| \leq \infty$. Let us study the logarithmic asymptotic behavior of the system (2.10). Setting

$$A_l(\nu) \approx d_l / \ln \nu, \qquad (2.12)$$

we obtain from Eq. (2.10) the system of equations for the coefficients d_i:

$$\pi d_l = d_i^2 + \sum_k \sigma_{ik} d_k^2 ; \qquad (2.13)$$

$$\sigma_{ik} = \begin{pmatrix} \frac{1}{3} & -4 & \frac{5}{3} \\ -\frac{2}{15} & \frac{7}{10} & \frac{1}{3} \\ \frac{1}{3} & 2 & \frac{1}{6} \end{pmatrix}. \qquad (2.14)$$

This system possesses the unique notrivial solution

$$d_0 = 2.13, \quad d_1 = -0.137, \quad d_2 = 0.653, \qquad (2.15)$$

that has been recently found by Lovelace.[2] A remarkable property of this solution is its closeness to Eq. (2.7). From a comparison of the numbers it is seen that the logarithmic asymptotic behavior is very stable with respect to the inclusion of d and f waves.

It is of interest to investigate the impact on the asymptotic behavior of the real parts of the higher waves. We discuss right away the limiting case of all waves, making use of formula (1.3). To this end it is sufficient to substitute Eqs. (2.3) and (2.8) into Eq. (1.3). In the equations for the partial waves given below we have kept only the terms that contribute to the logarithmic asymptotic behavior:

$$A_0(\nu) = \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{\nu'-\nu} \operatorname{Im} A_0(\nu') + \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{1+\nu+\nu'} f_0(\nu')$$
$$- 4(2\ln 2 - 1)\, I_1(\nu),$$
$$A_1(\nu) = \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{\nu'-\nu} \operatorname{Im} A_1(\nu') + \frac{3}{2\pi}\int_0^\infty \frac{d\nu'}{1+\nu+\nu'} f_1(\nu')$$
$$+ (3 - 4\ln 2)\, I_1(\nu),$$
$$A_2(\nu) = \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{\nu'-\nu} \operatorname{Im} A_2(\nu') + \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{1+\nu+\nu'} f_2(\nu')$$
$$+ 2(2\ln 2 - 1)\, I_1(\nu). \qquad (2.16)$$

To these equations corresponds the matrix.

$$\sigma_{ik} = \begin{pmatrix} \frac{1}{3} & -3(4\ln 2 - 1) & \frac{5}{3} \\ -\frac{1}{6} & \frac{21}{4}\left(1 - \frac{24}{21}\ln 2\right) & \frac{5}{12} \\ \frac{1}{3} & \frac{3}{4}(4\ln 2 - 1) & \frac{1}{6} \end{pmatrix} \qquad (2.17)$$

and, correspondingly, the asymptotic coefficients

$$d_0 = 2.15, \quad d_1 = -0.167, \quad d_2 = 0.667. \qquad (2.18)$$

Comparing the numbers (2.18) with (2.15) and (2.7) we conclude that the logarithmic asymptotic behavior of the equations for the partial waves obtained by the differential method converges rapidly to its limit (2.18). We see also that the equations of the differential method do not at all go over into the corresponding equations of CM when the real part of the scattering amplitude is approximated more and more precisely.

3. THE PROBLEM OF INCLUSION OF HIGHER PARTIAL WAVES

We discuss now the meaning of our results. In the construction of low-energy schemes one has to deal with two approximations: the elastic approximation in the unitarity condition and the restriction to a few lower partial waves in the scattering. The second approximation affects both the real and the imaginary parts of the scattering amplitude and for this reason it may be realized in a variety of ways. As was shown in Sec. 1 on the example of scattering of neutral mesons, the approximations in Re A and Im A should be correlated. It makes no sense to include Re A_4 if Im A_2 is ignored.

The equations of CM provide an example of an unbalanced scheme. In these equations no approximation is made in Re A, while Im A is approximated by s and p waves. As a result the equations of CM do not allow the introduction into Im A of even d and f waves and, apparently, possess no solutions at all.[2,10]

It is appropriate to pose the question whether equations of the type (2.4), (2.10) can be improved by taking into account higher partial waves in the scattering amplitude with the help of formulas of the type (1.3). There may arise here the temptation to take into account an infinite number of partial waves or, which is equivalent, an infinite number of terms in the summations (1.3) for the real and imaginary parts of the scattering amplitude, and to obtain in this manner "exact" equations, not containing the neglect of higher partial waves.

Such "exactness" makes no sense, however, for two distinct reasons.

The first reason has to do with the fact that in the region of not too small energies, in which higher partial waves may be important, a substantial role is also played by the contributions from inelastic processes which were ignored.

Let us suppose, however, that for some reason the contributions due to inelastic processes are small in absolute magnitude or, equivalently, let us consider a model in which inelastic processes are forbidden. Even in this case the "exactness" claimed above cannot be achieved because of the existence of spectral functions. We clarify this circumstance on the example of s waves. We note to this end that the first of the formulas (1.3) may be viewed as the result of integration over c of two Taylor series for the function f(c) at the points c = +1 and c = −1. At that the Taylor series at the point c = −1 is integrated in the interval [−1, 0], and the series at the point c = +1 in the interval [0, +1]. Consequently, in order that the sum (1.3) exist it is necessary for the region of convergence of these two Taylor series to cover entirely the physical interval [1, −1].

This requirement is satisfied for arbitrary $\nu > 0$ for the functions given by spectral representations of the type (1.4) under the condition that the numerators of the integrals have no singularities in c (i.e., in the absence of spectral functions) and under the condition that the polynomials in ν with high-energy coefficients of the type (1.8) are discarded. These two conditions together give rise, for example, to the result that the second integral in Eq. (1.4) is expanded in a Taylor series only about the point c = +1, with the singularities in c given by its numerator.

This picture changes substantially when the spectral functions are taken into account. Then the region of analyticity is defined by the Lehmann ellipse and, as is easily seen, the series (1.3) for the real part of the amplitude ceases to exist in the region $\nu > 2$.

However, even under these conditions the finite sum of terms from Eq. (1.3) may yield a good approximation. Integrating by parts N times we obtain for the s wave the expression

$$f_0 = \frac{1}{2}\int_{-1}^{1} dc\, f(c) = \frac{1}{2}\sum_{n=0}^{N-1} \frac{f^{(n)}(-1) + (-)^n f^{(n)}(1)}{(n+1)\,!} + \frac{(-)^N}{2N!}\int_{-1}^{1} f^{(N)}(c)\, c^N dc. \quad (3.1)$$

Assuming that in the energy region of interest higher partial waves, starting with f_m, are small we may discard in Eq. (3.1) the last term for N = m and obtain expressions of the form (1.1), (1.2).

From this we conclude that a scheme that takes into account a small number of low partial waves may yield a good approximation in the low energy region. Taking into account an infinite number of terms in the summation (1.3) does not, on the one hand, produce a real improvement in the precision because of the existence of inelastic processes and, on the other hand, is mathematically impossible because of the existence of the spectral functions. In that case the series (1.3) must be viewed as an asymptotic series.

To illustrate this thesis we discuss one more scheme for consecutive inclusion of higher partial waves. We expand the second integral in Eq. (1.4) about the point c = +1, the third integral about the point c = −1, and use these expansions in the entire interval [+1, −1] in place of discarding the high energy constants. At that the imaginary parts are approximated by s and p waves (we have in mind scattering of charged mesons). If we now go over to an infinite number of terms then the unsubtracted equations will not exist, because the imaginary parts of the crossed integrals will, owing to the presence of p waves, have poles in c at the point c = +1 or c = −1. One subtraction gives rise to precisely the equations of CM, with all the mathematical complications connected with them.[10]

The difficulty with the spectral functions may be partially overcome by taking into account the elastic two-particle parts of the spectral functions. This leads to the "strip approximation" program of Chew and Frautschi.[11] However, in contrast to these authors, we expect that the inclusion of the spectral functions in the elastic strips will change insignificantly the low energy approximation. On the contrary, the behavior of the scattering amplitude in the region of high energies and low momentum transfers may turn out to be completely determined by the properties of low energy scattering. This perspective seems to be particularly likely in view of the recent results (see [12]).

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