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COLLECTIVE EXCITATIONS WITH NON-ZERO ANGULAR MOMENTUM PAIRING

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We have considered collective excitations in a system of fermions for which attraction in a state with non-vanishing angular momentum predominates. There are, in such a system, apart from sound excitations also other collective excitations without a gap in the energy spectrum. We express the velocity of these excitations in terms of the gap in the single-particle excitation spectrum. We show that the solution for the ground state function considered by Anderson and Morel [4] with $\Delta(n) \sim Y_{22}(n)$ is unstable.

IN the usual superconductivity theory [1-3] one considers the case where attraction in an S-state predominates in the interparticle interaction. Recently a number of authors [4-6] have also studied systems in which the predominant attraction is in a state with an angular momentum $l_0 \neq 0$. Pitaevskiĭ [7] has shown that this state occurs apparently in He^3 where attraction in a D-state predominates. We consider in the present paper, apart from the single-particle excitations, also the collective ones in such systems. The shape of the spectrum of these excitations is important for an elucidation of the properties of superfluidity. Moreover, a study of the collective excitations enables us to solve the problem of the stability of the state considered. In the case of non-zero angular momentum pairing such a study of the stability is not necessary, since the equation then obtained for the gap Δ in the energy spectrum has only two solutions, $\Delta = 0$ and $\Delta = \Delta_0 \neq 0$, and the state with the lower energy is, of course, the stable one. In the case of non-zero angular momentum pairing it is impossible to find an analytical solution of the equations for Δ in a general form. Each particular solution must thus be checked for stability.

1. It is convenient for a study of the collective excitations to use the "relativistic" technique developed earlier. [8] Following this earlier paper we introduce a four-component operator

$$\psi(x) = \begin{pmatrix} u(x) \\ \sigma_y u^+(x) \end{pmatrix} \tag{1}$$

and four-by-four matrices

$$\gamma_1 = 1, \quad \gamma_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\gamma_5 = -i\gamma_3\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} \sigma_y & 0 \\ 0 & -\sigma_y \end{pmatrix}. \tag{2}$$

In this notation the usual Lagrangian of a system of interacting particles

$$\mathcal{L} = iu^+ \frac{\partial u}{\partial t} + \frac{1}{2m} u^+ \frac{\partial^2}{\partial \mathbf{x}^2} u + \mu u^+ u$$
$$+ \frac{1}{2}\int d^4x\, d^4y\, u^+(x)\, u(x) D(x-y)\, u^+(y) u(y) \tag{3}$$

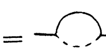
will become

$$\mathcal{L} = -\frac{i}{2}\bar{\psi}\hat{p}\psi$$
$$- \frac{1}{8}\int d^4x\, d^4y\, \bar{\psi}(x)\, \gamma_3\psi(x)\, D(x-y)\, \bar{\psi}(y)\, \gamma_3\psi(y). \tag{4}$$

Here $\bar{\psi} = \psi^+\gamma_0 = \psi C$, $\hat{p} = p_3\gamma_3 + p_4\gamma_4$, $p_4 = ip_0 = -\partial/\partial t$, $p_3 = -(2m)^{-1}\partial^2/\partial\mathbf{x}^2 - \mu$. The fermion Green's function is equal to

$$G(p) = i\int \langle T\psi(x)\bar{\psi}(y)\rangle e^{-ip(x-y)} d^4x = 1/(i\hat{p} + \hat{\Sigma}) \tag{5}$$

If we assume the interaction to be weak we can restrict ourselves for the calculation of the self-energy part Σ to the first-order diagram [8]

$\hat{\Sigma} =$  $= -i\int \gamma_3 G(p')\gamma_3 D(p-p')\, d^4p';$

$$d^4p = d\mathbf{p}dp_0/(2\pi)^{-4} \approx \rho d\Omega dp_3 dp_0/8\pi^2 \equiv \rho d\Omega d^2p/4\pi. \tag{6}$$

We drop in the self-energy part $\hat{\Sigma}$ the off-diagonal elements which lead to a renormalization of the chemical potential and of the effective mass, and retain only the diagonal terms which determine the magnitude of the gap in the energy spectrum. If the attraction in a state with an even angular momentum predominates in the interaction $D(\mathbf{nn}')$, the diagonal part of $\hat{\Sigma}$ does not contain the spin operators and is of the form

$$\hat{\Sigma} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta^* \end{pmatrix} = \Delta_1\gamma_1 + i\Delta_2\gamma_5 \equiv \Delta_1 + i\Delta_2\gamma_5,$$
$$\Delta_1 = \mathrm{Re}\,\Delta, \qquad \Delta_2 = \mathrm{Im}\,\Delta. \tag{7}$$

In the case considered Δ depends, generally speaking, on the angles and this dependence may be different for Δ_1 and Δ_2; it is thus impossible, in contradistinction to the scalar case, to choose the phase of Δ to be real and Σ to be proportional to the unit matrix.

Substituting (7) into Eq. (5) for the Green function, we get

$$G = \frac{1}{i\hat{p} + \Delta_1 + i\Delta_2\gamma_5} = \frac{-i\hat{p} + \Delta_1 - i\Delta_2\gamma_5}{p^2 |\Delta|^2}. \qquad (8)$$

2. To determine Δ_1 and Δ_2 we substitute Eq. (8) for G into Eq. (6). With logarithmic accuracy the region near the Fermi surface is the important one in that equation. In that region $D(p - p')$ depends only on the angle between p and p′: $D(p - p') = D(\mathbf{nn}')$, $\mathbf{n} = \mathbf{p}/p$, $\mathbf{n}' = \mathbf{p}'/p'$; $\Delta(p)$ depends also on the angles of the vector p: $\Delta(p) = \Delta(\mathbf{n})$. We get from Eq. (6) equations for Δ_1 and Δ_2:

$$\Delta_{1,2} = \rho \int D(\mathbf{nn}') \frac{\Delta_{1,2}(\mathbf{n}')}{p^2 + |\Delta(\mathbf{n}')|^2} \frac{d\mathbf{n}'}{4\pi} d^2p. \qquad (9)$$

Performing the integration over the two-dimensional vector p and changing again to the complex variable $\Delta = \Delta_1 + i\Delta_2$, we arrive at the integral equation [4,5]

$$\Delta(\mathbf{n}) = \frac{1}{2}\rho \int D(\mathbf{nn}') \ln \frac{\Lambda^2}{|\Delta(\mathbf{n}')|^2} \Delta(\mathbf{n}') \frac{d\mathbf{n}'}{4\pi}, \qquad (10)$$

where Λ is the energy width of the region in which the interaction takes place.

We expand $\Delta(\mathbf{n})$ and $D(\mathbf{nn}')$ in series in terms of spherical functions

$$\Delta(\mathbf{n}) = \sum_{lm} \Delta^{lm} Y_{lm}(\mathbf{n}), \qquad (11)$$

$$\rho \frac{D(\mathbf{nn}')}{4\pi} = \sum_l g_l \frac{2l+1}{4\pi} P_l(\mathbf{nn}') = \sum_l g_l Y_{lm}(\mathbf{n}) Y^*_{lm}(\mathbf{n}'). \qquad (12)$$

Equation (10) is then reduced to a system of algebraic equations

$$\Delta^{lm} = g_l \sum_{l'm'} L^{lm}_{l'm'} \Delta^{l'm'}; \qquad (13)$$

$$L^{lm}_{l'm'} = \int d\mathbf{n} Y^*_{lm}(\mathbf{n}) \ln \frac{\Lambda}{|\Delta(\mathbf{n})|} Y_{l'm'}(\mathbf{n}). \qquad (14)$$

In first order in the interaction we can retain on the right-hand side of Eq. (13) only the terms with the largest $g_l = g_{l_0}$. The terms with $l \neq l_0$ introduce small corrections of order $g_l^2 \Delta_{l_0} \times (g_{l_0} - g_l)^{-1}$. In first approximation there remains only the summation over m in the set (13):

$$\Delta^{l_0 m} = g_{l_0} \sum_{m'} L^{l_0 m}_{l_0 m'} \Delta^{l_0 m'}. \qquad (15)$$

We assume that $\Delta^{l_0 m}$ is non-vanishing only for a well-defined value of $m = m_0$. It is then clear from Eq. (14) that $L^{l_0 m}_{l_0 m'}$ is diagonal in m and m′ and thus such an assumption does not contradict Eq. (15). In that case (15) is of the form

$$1 = g_{l_0} L^{l_0 m_0}_{l_0 m_0} = g_{l_0} \int d\mathbf{n} \, |Y_{l_0 m_0}(\mathbf{n})|^2 \ln (\Lambda / \Delta^{l_0 m_0} |Y_{l_0 m_0}(\mathbf{n})|). \qquad (16)$$

In the case $l_0 = 2$ we get from (16)

$$\Delta^{20} = \left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)^{2/3\sqrt{3}} \frac{e^{16/15}}{\sqrt{5}} \Lambda \exp\left(-\frac{1}{g_2}\right) = 0.78\Lambda \exp\left(-\frac{1}{g_2}\right),$$

$$\Delta^{21} = \Delta^{22} = 30^{-1/2} e^{47/30} \Lambda \exp\left(-\frac{1}{g_2}\right) = 0.87\Lambda \exp\left(-\frac{1}{g_2}\right).$$

The difference between the energy of the ground state of non-interacting particles and that of a state with pairing is given by the formula

$$\delta E = \frac{1}{4}\rho \int |\Delta(\mathbf{n})|^2 d\mathbf{n} = \frac{1}{4}\rho \sum_{lm} |\Delta_{lm}|^2. \qquad (17)$$

In the case considered, where $l = 2$, the states with m = 2 and m = 1 are energetically more advantageous than the one with m = 0. [4,5] The general statement that the most advantageous state is the one with the largest $m = l$ [4,5] is, however, erroneous. One can easily solve Eq. (16) in the limiting case $l_0 \gg 1$. Substituting instead of $|Y_{lm}|^2$ its quasi-classical expression

$$|Y_{lm}|^2 = (4\pi)^{-1} (\sin^2\theta - m^2/l^2)^{-1/2},$$

we get

$$\Delta^{l_0 m} = \Delta^{l_0 l_0} (1 - m^2/l_0^2)^{1/2}. \qquad (18)$$

For $l_0 \gg 1$ the state with m = 0 is thus at any rate energetically more favorable than the one with $m = l_0$. Moreover, even in the case $l_0 = 2$ the state with $m = 2$ is not the most favorable. We shall show in the following that all states in which $\Delta(\mathbf{n})$ contains only one harmonic Y_{2m} turn out to be unstable. They have thus a higher energy than some states in which $\Delta(\mathbf{n})$ is a superposition of several harmonics. In the present investigation we did not find the coefficients Δ^{2m} for this state, since this entails the evaluation of the integrals in (14) and the solution of the set (15) of five non-linear equations, which can apparently only be done numerically.

3. We turn now to a consideration of collective excitations. It turns out that many properties of these excitations can be established without a knowledge of the explicit form of the function $\Delta(\mathbf{n})$.

The Bethe-Salpeter equations for the spectrum of the two-particle excitations are of the form

$$\Gamma_i(\mathbf{n}, k) = \frac{i}{4}\rho \int \frac{d\mathbf{n}'}{4\pi} D(\mathbf{nn}') \operatorname{Sp} \gamma_k G\left(p' - \frac{q}{2}\right) \gamma_3 \gamma_i \gamma_3 G\left(p' + \frac{q}{2}\right) \times d^2p' \Gamma_k(\mathbf{n}', k) - \delta_{i3} \frac{i}{2} \rho D(k) \int \operatorname{Sp} \gamma_k G\left(p' - \frac{q}{2}\right) \times \gamma_3 G\left(p' + \frac{q}{2}\right) d^2p' \int \frac{d\mathbf{n}'}{4\pi} \Gamma_3(\mathbf{n}', k). \qquad (19)$$

The two-dimensional vector p′ is defined here by

(9) and the vector q has the components $q_0 = \omega$, $q_3 = v\mathbf{k}\cdot\mathbf{n}'$. The system (19) of linear, homogeneous integral equations has a solution only for certain well-defined relations between energy ω and momentum k, and these also determine the excitation spectrum. We shall in the following be interested in the form of the spectrum for small ω and k. The integrals over p′ tend in that case to zero when one of the indices i or k is equal to 3 or 4. Moreover, we can neglect the annihilation term—the second term on the right hand side of (19) (bearing in mind the application to He^3 we assume that there is no Coulomb interaction between the particles). The set (19) for the quantities $\Gamma_+ = \Gamma_1 + \Gamma_5$ and $\Gamma_- = \Gamma_1 - \Gamma_5$ becomes then of the form

$$\Gamma_\alpha(\mathbf{n}, k) = \rho\int D(\mathbf{nn}')\,\Pi_{\alpha\beta}(\mathbf{n}'k)\,\Gamma_\beta(\mathbf{n}', k)\frac{dn'}{4\pi},$$

$$\Pi_{\alpha\beta} = \frac{i}{4}\int d^2p\ \mathrm{Sp}\ \gamma_\beta\, G\left(p' - \frac{q}{2}\right)\gamma_3\gamma_\alpha\gamma_3 G\left(p' + \frac{q}{2}\right). \quad (20)$$

The indices α and β take on the values + or −; $\gamma_\pm = \gamma_1 \pm \gamma_5$.

We shall show that Eqs. (20) have solutions with $\omega = 0$ for $k = 0$. To do this we note that the Green's function G of (5) does not change into itself under several transformations which leave the initial Lagrangian invariant. Such transformations are the gauge transformation $u \to ue^{i\alpha}$ for which G changes into $G' = \exp(i\alpha\gamma_5)G$, and the transformation consisting of a rotation around an arbitrary axis. The invariance of the initial Lagrangian means that the new G must satisfy the same equations as the original one. We shall show that the equations obtained for the change in the self-energy part $\hat{\Sigma}$ are the same as Eqs. (20) as $k = \omega = 0$. We write the equation for $\hat{\Sigma}$ in the form

$$\hat{\Sigma} = -\frac{i\rho}{2}\int D(\mathbf{nn}')\,\gamma_3\frac{1}{i\hat{p} + \hat{\Sigma}}\gamma_3 d^2p\frac{dn'}{4\pi}. \quad (21)$$

Performing an infinitesimal transformation and subtracting Eq. (21) from the equation thus obtained for $\hat{\Sigma} + \hat{\Sigma}'$ we get

$$\hat{\Sigma}' = \frac{i\rho}{4}\int D(\mathbf{nn}')\,\gamma_3\frac{1}{i\hat{p} + \hat{\Sigma}}\hat{\Sigma}'\frac{1}{i\hat{p} + \hat{\Sigma}'}\gamma_3 d^2p\frac{dn'}{4\pi}. \quad (22)$$

This equation is the same as the set (20) as $k = 0$ if we put

$$\Gamma_\pm = \tfrac{1}{4}\mathrm{Sp}\,(1 \pm \gamma_5)\,\hat{\Sigma}'(\mathbf{n}). \quad (23)$$

The equations (20) for the collective excitations possess thus non-vanishing solutions.

We consider as an example the case of scalar pairing when $D(\mathbf{nn}')$ is independent of the angles. Under a gauge transformation the phase is changed:

$$\Delta \to \Delta e^{i\alpha}, \quad \hat{\Sigma} \to \hat{\Sigma}e^{i\alpha\gamma_5} \approx \hat{\Sigma} + \Sigma_{i\alpha\gamma_5}.$$

If Δ were chosen to be real, $\Sigma' = \Delta i\alpha\gamma_5$: Equations (20) must then be satisfied if $\Gamma_+ = -\Gamma_-$. Indeed, the equation $1 = g_0\Pi_{55}$ has a corresponding solution as $k = 0$ which describes the gap-less sound vibrations.[8]

In our case Δ depends on the angles between the vector p and an arbitrarily chosen direction; apart from sound vibrations there must also exist excitations which correspond to a periodically changing rotation of the chosen direction. An infinitely large wavelength corresponds to the general rotation of the chosen direction which does not change the energy of the system. Thus the spectrum of these excitations has likewise no gap. The form of the functions $\Gamma_1(\mathbf{n})$ and $\Gamma_5(\mathbf{n})$ for $k = 0$ can be determined using Eq. (23) if the function $\Delta(\mathbf{n})$ is known. Under a rotation over a small angle the unit vector $\mathbf{n}$ transforms into $\mathbf{n} + \boldsymbol{\epsilon}\times\mathbf{n}$, where $\boldsymbol{\epsilon}$ is an infinitesimal vector along the axis of rotation. In that case $\hat{\Sigma}(\mathbf{n})$ changes into $\hat{\Sigma}(\mathbf{n}) + (\boldsymbol{\epsilon}\times\mathbf{n}\nabla\hat{\Sigma})$, i.e.,

$$\Gamma_+ = \Gamma_-^* = (\varepsilon\,[\mathbf{n}\,\nabla\Delta\,(\mathbf{n})]). \quad (24)$$

Under a gauge transformation $\hat{\Sigma} \to \hat{\Sigma} + i\alpha\gamma_5\hat{\Sigma}$, i.e.,

$$\Gamma_+ = \Gamma_-^* = i\alpha\Delta. \quad (25)$$

4. From the form of the functions (24) and (25) which are the solutions of Eqs. (20) as $k = 0$ we can find the excitation spectrum for small k. We write $\Pi_{\alpha\beta}$ in the form $\Pi_{\alpha\beta}^{(0)} + \Pi_{\alpha\beta}^{(1)}$ where $\Pi_{\alpha\beta}^{(1)}$ is a small correction tending to zero as $k = 0$. We also write the function $\Gamma_\alpha(\mathbf{n})$ in the form of a sum

$$\Gamma_\alpha(\mathbf{n}) = \sum_m c_m\Gamma_\alpha^{(0)m}(\mathbf{n}) + \Gamma_\alpha^{(1)}(\mathbf{n}),$$

where m is the number of the solution for $k = \omega = 0$. The $\Gamma_\alpha^{(0)m}$, i.e., the functions (24) and (25), satisfy the equations

$$\Gamma_\alpha^{(0)m}(\mathbf{n}) = \frac{i\rho}{4}\int D(\mathbf{nn}')\,\Pi_{\alpha\beta}^{(0)}(\mathbf{n}')\,\Gamma_\beta^{(0)m}(\mathbf{n}')\frac{dn'}{4\pi}. \quad (26)$$

Retaining in Eqs. (20) the first-order terms, we get

$$\Gamma_\alpha^{(1)}(\mathbf{n}) = -\frac{i\rho}{4}\int D(\mathbf{nn}')\,\Pi_{\alpha\beta}^{(0)}(\mathbf{n}')\,\Gamma_\beta^{(1)}(\mathbf{n}')\frac{dn'}{4\pi} + \frac{i\rho}{4}\int D(\mathbf{nn}')\,\Pi_{\alpha\beta}^{(1)}(\mathbf{n}')\,\Gamma_\beta^{(0)m}c_m\frac{dn'}{4\pi}. \quad (27)$$

Multiplying (27) by $\Gamma_\gamma^{*(0)n}(\mathbf{n})\,\Pi_{\gamma\alpha}^{(0)}(\mathbf{n})$, summing over α, integrating over $\mathbf{n}$, and using (26) and the fact that $\Pi_{\alpha\beta}$ is Hermitian, we get

$$\int dn\,\Gamma_\gamma^{*(0)n}(\mathbf{n})\,\Pi_{\gamma\beta}^{(1)}(\mathbf{n})\,\Gamma_\beta^{(0)m}(\mathbf{n})\,c_m = 0. \quad (28)$$

We get the quantities $\Pi^{(1)}_{\alpha\beta}$ by expanding (20) in a series in q; we take into account that for the excitations considered $\Gamma_- = \Gamma_+^*$ (in the following we omit the superscripts 0 and 1). As a result the set (28) becomes of the form

$$\sum_m \int dn \frac{\omega^2 - (v\mathbf{kn})^2}{|\Delta|^2}\Big(2\Gamma_+^{*n}\Gamma_+^m + 2\Gamma_+^n\Gamma_+^{*m} - \frac{\Delta^{*2}}{|\Delta|^2}\Gamma_+^n\Gamma_+^m - \frac{\Delta^2}{|\Delta|^2}\Gamma_+^{*n}\Gamma_+^{*m}\Big)c_m = 0. \tag{29}$$

The condition that this set have a solution determines the spectrum of the collective excitations. It is clear from (29) that ω depends linearly on k. There is, apparently, always a branch representing sound waves with the hydrodynamic velocity $v/\sqrt{3}$; one can verify this from a number of examples, one of which is considered below. The velocity of the remaining excitations depends, generally speaking, on the direction of the vector **k** and may tend to zero for certain directions.

Equations (20) may have solutions which are not described by Eqs. (24) and (25). The frequency of the corresponding excitations does not tend to zero as $k \to 0$. If $\omega^2 > 0$, these are the normal excitations with a gap. If, however, for some excitation it turns out that $\omega^2 < 0$, this means that the state considered is unstable with respect to these excitations.

5. As an example we consider a state for which the angular dependence of Δ is given by the formula $\Delta(\mathbf{n}) = \Delta_{22}Y_{22}(\mathbf{n})$. From (24) and (25) we find the form of the functions Γ^n. A rotation around the z axis is in this case equivalent to a change in the phase of Δ. The functions Γ obtained from (24) and (25) are thus the same and equal to $i\Delta$ if ϵ is along the z axis. For ϵ along the x or the y axis the functions Γ obtained from (24) are proportional to $Y_{21}(\mathbf{n})$. Substituting such a form of Γ into (29) we find the excitation spectrum. Taking it into account that the off-diagonal elements with $m \neq n$ vanish, we get for the excitation spectrum with $\Gamma = i\Delta$ the equation

$$\int dn \frac{\omega^2 - (v\mathbf{kn})^2}{|\Delta|^2} 6|\Delta|^2 = 0,$$

and hence $\omega^2 = v^2k^2/3$, i.e., the excitation is a sound wave. We find the excitation spectrum with $\Gamma \sim Y_{21}(\mathbf{n})$ from the equation

$$\int dn\,[\omega^2 - (v\,\mathbf{kn})^2]\frac{|Y_{21}(\mathbf{n})|^2}{|Y_{22}(\mathbf{n})|^2} = 0,$$

whence $\omega^2 = v^2k_z^2$: the velocity of these excitations is thus equal to vk_z/k and vanishes for directions of k perpendicular to the chosen axis. We note that if $k_z^2/k^2 < 1/3$, then ω has an imaginary part indicating the presence of damping due to the possible decay of a collective excitation into two single-particle ones. For small k, however, the imaginary part tends to zero faster than the real one and it is thus meaningful to speak of collective excitations if $vk \ll \Delta$.

Solving (20) we find the spectrum of the remaining excitations. For the case considered, $\Delta \sim Y_{22}(\mathbf{n})$, the functions Γ have the form $\Gamma_+ \sim Y_{2m}(\mathbf{n})$, $\Gamma_- = \pm\Gamma_+^*$. In the case $m = -1$, the spectrum is the same as for the case considered above where $m = 1$. For $\Gamma \sim Y_{2-2}$. Eqs. (20) have no solution for ω^2 near the real axis; this means that there are no long-lived excitations of such symmetry.

6. We show now that the solution considered with $\Delta \sim Y_{22}$ is unstable. To do this we consider the excitation spectrum with $\Gamma \sim Y_{20}$. It is determined from the equation

$$1 = g_2\int Y_{20}^2\Pi_{++}(\mathbf{n})\,dn. \tag{30}$$

We verify that for $k = 0$ this equation has a solution with negative ω^2. Indeed, when $-\omega^2 \gg \Delta^2$, the right hand side of (30) is equal to $g_2 \ln(\Lambda/|\omega|)$ and less than unity, while if $\omega^2 = 0$ we get, taking (16) into account that the right hand side is equal to $1 + 1/3\,g_2^2 > 1$. There is thus a solution with some $\omega^2 < 0$. As we noted earlier this means that the state considered, for which $\Delta \sim Y_{22}$ is unstable, i.e., does not have a minimum energy. Indeed, let us determine the energy of the system as the average of the Hamiltonian over a Bardeen-type wave function:[1,4]

$$\psi = \prod_{\mathbf{p}}(u_{\mathbf{p}} + v_{\mathbf{p}}a_{\mathbf{p}}^+a_{-\mathbf{p}}^+). \tag{31}$$

We can then check that the solution with $\Delta \sim Y_{22}(\mathbf{n})$ corresponds not to a minimum of the energy, but to a saddle point: an extra term of the form

$$\delta(uv^*) = \alpha\frac{|\xi|Y_{20}(\mathbf{n})}{\xi^2 + |\Delta|^2}$$

leads in second order in α to a decrease in the energy.

As indicated, of all solutions for which Δ is proportional to only one of the harmonics Y_{2m}, the solution with m =.2 corresponds to the lowest energy. Its instability means thus that all these states are unstable and that the energy minimum corresponds to a solution which is a superposition of harmonics with different m. Thouless (private communication) was the first to draw attention to this fact.

In a paper by Gor'kov and one of the authors[6] a solution was proposed with a Δ independent of the angles. Such a solution does not satisfy Eq.

(10), i.e., the wave function is not of the form (31). We were unable to find a consistent scheme leading to the results of that paper.

Note added in proof (April 12, 1962). We have recently seen a paper by Anderson and Morel[9] in which a number of solutions of Eq. (15) were obtained. One can use the method developed in our paper to check the stability of these solutions and to determine the spectrum of the collective excitations.

[1] Bardeen, Cooper, and Schrieffer, Phys. Rev. **108**, 1175 (1957).

[2] Bogolyubov, Tolmachev, and Shirkov, Novyĭ metod v teorii sverkhprovodimosti (A New Method in the Theory of Superconductivity) AN SSSR, 1958, Fortschr. Physik **6**, 605 (1958).

[3] L. P. Gor'kov, JETP **34**, 735 (1958), Soviet Phys. JETP **7**, 505 (1958).

[4] P. W. Anderson and P. Morel, Phys. Rev. Letters **5**, 136 (1960).

[5] V. M. Galitskiĭ, JETP **34**, 1011 (1958), Soviet Phys. JETP **7**, 698 (1958).

[6] L. P. Gor'kov and V. M. Galitskiĭ, JETP **40**, 1124 (1961), Soviet Phys. JETP **13**, 792 (1961).

[7] L. P. Pitaevskiĭ, JETP **37**, 1794 (1959), Soviet Phys. JETP **10**, 1267 (1960).

[8] Vaks, Galitskiĭ, and Larkin, JETP **41**, 1655 (1961), Soviet Phys. JETP **14**, 1177 (1962).

[9] P. W. Anderson and P. Morel, Phys. Rev. **123**, 1911 (1961).

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