GENERALIZATION OF SYMANZIK'S THEOREM ON THE MAJORIZATION OF FEYNMAN GRAPHS

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Submitted to JETP editor December 9, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **42**, 1285-1293 (May, 1962)

Any Feynman graph can be characterized algebraically by a certain rectangular matrix, the incidence matrix. An explicit expression is found for the quadratic form belonging to an arbitrary graph in the x and p representations in terms of the incidence matrix. The expressions obtained are used to prove and generalize the Symanzik theorem. With the help of these theorems a small number of graphs are found which majorize all strongly connected graphs of scattering processes involving π mesons and nucleons.

1. INTRODUCTION

IN a previous paper, [1] a method for the majorization of Feynman graphs has been developed. With the help of this method, the consideration of all strongly connected graphs* for a certain process (in the Euclidean region of the external momenta) is reduced to the consideration of a finite number of graphs. It should be noted that we regard the squares of the external momenta p_i^2 as independent variables, which in general do not satisfy the condition $p_i^2 = M_i^2$.

Let us denote by G(D) the maximal (connected) region of Euclidean external momenta p, including the point p = 0, in which the Feynman integral T_D corresponding to the (strongly connected) graph D has no singularity. Let R be the class of all strongly connected graphs for the given process. Then none of the integrals T_D has a singularity in the intersection of the regions G(D) (the intersection is denoted by G_R, $D \in R$).

For the scattering of mesons on mesons, nucleons on nucleons, and mesons on nucleons, a finite set R_0 of graphs has been found [2] such that

$$G_R = G_{R_0}.$$

For example, in the case of nucleon-nucleon scattering the set R_0 consists of the seven graphs shown in Fig. 1 (heavy lines: nucleons, thin lines: mesons; the circles symbolize external vertices; the numbering of the vertices in this figure will be used below in Sec. 5). In the case of meson-meson scattering, this set consists of three graphs,

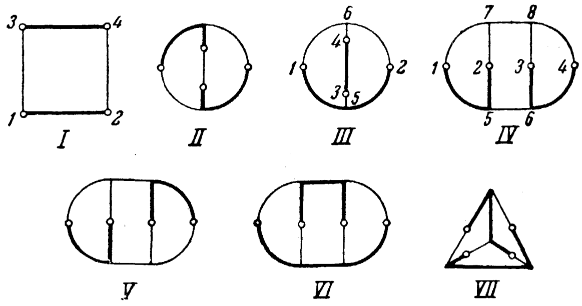

FIG. 1

and for meson-nucleon scattering we have fourteen graphs. A method for obtaining the class R_0 is given in Sec. 3.

In order to compare the graphs of the class R_0, we must investigate the quadratic forms belonging to a general Feynman graph in more detail. This will be the subject of the present paper. In Sec. 2 we shall obtain an explicit expression for the quadratic form of an arbitrary Feynman graph in terms of the so-called incidence matrix. We shall also give the expression for the conjugate (inverse) quadratic form. In Sec. 3 we shall determine the minimum of the conjugate quadratic form as a function of the Feynman parameters α (for Euclidean external momenta) and derive from this Symanzik's theorem [3] on the majorization of graphs. This theorem will also be somewhat generalized. The results of Sec. 3 will be used in Sec. 4 to narrow down the class of graphs R_0 for the above-mentioned scattering processes.

We note that the present paper consists basically of an enumeration of results. Many of the proofs have been omitted due to lack of space.*

*A graph is called strongly connected if it does not separate into two parts after an arbitrary single internal line has been cut.

*A detailed account of all proofs can be found in a preprint of the authors.[2]

2. EXPRESSION FOR THE QUADRATIC FORM OF A GRAPH IN TERMS OF ITS INCIDENCE MATRIX

Let us consider an arbitrary graph with n vertices and l internal lines. All lines in the graph are oriented, the external lines going into the corresponding vertices whereas the directions of the internal lines are fixed arbitrarily. The structure of such a graph can be completely characterized by a certain matrix E with n rows and l columns. For this purpose we number separately the vertices and internal lines of the graph. The elements $\epsilon_{i\nu}$ of the matrix E are determined in the following way: $\epsilon_{i\nu} = 1$ if the line ν comes out of the vertex i, $\epsilon_{i\nu} = -1$ if the line ν goes into the vertex i, and $\epsilon_{i\nu} = 0$ if the vertex i does not belong to the line ν. The matrix E is well known in topology and is called the incidence matrix.*

The incidence matrix E allows for a simple description of the momentum conservation law in the vertices of the graph. Let k_ν be the momenta of the internal lines and p_i the external momenta of the graph. If no external line goes into the vertex i, the momentum p_i is identically zero. The law of conservation of momentum in the vertices of the graph takes the form †

$$\sum_{\nu=1}^{l} \varepsilon_{i\nu} k_\nu = p_i, \qquad i = 1, \ldots, n. \tag{2.1}$$

The self-consistency of the system (2.1) requires the conservation of the external momenta:

$$\sum_{i=1}^{n} p_i = 0. \tag{2.2}$$

With the help of the incidence matrix it is comparatively easy to obtain an explicit expression for the quadratic form $Q(\alpha, p)$ belonging to an arbitrary graph. The form $Q(\alpha, p)$ is defined as the extremum of the function

$$K(\alpha, k) = \sum_{\nu=1}^{l} \alpha_\nu (k_\nu^2 - m_\nu^2) \tag{2.3}$$

as a function of the internal momenta (see [11]) and, for a connected graph with the incidence matrix E, is equal to

$$Q(\alpha, p) = A(\alpha, p) - M^2(\alpha); \tag{2.4}$$

$$A(\alpha, p) = -\begin{vmatrix} 0 & p_j \\ p_i & d_{ij} \end{vmatrix} d^{-1}(\alpha), \qquad i, j = 1, \ldots, n-1,$$

$$M^2(\alpha) = \sum_{\nu=1}^{l} \alpha_\nu m_\nu^2. \tag{2.5}$$

Here

$$d_{ij} = \sum_{\nu=1}^{l} \varepsilon_{i\nu}\varepsilon_{j\nu}/\alpha_\nu, \tag{2.6}$$

$d(\alpha)$ is a determinant of order $n-1$:

$$d(\alpha) = \begin{vmatrix} d_{11} & \ldots & d_{1n-1} \\ \ldots & \ldots & \ldots \\ d_{n-11} & \ldots & d_{n-1\,n-1} \end{vmatrix}$$

(we assume that there is an external line going into the vertex n with a momentum p_n which is not identically zero). The expression for the quadratic form $\overline{A}(\alpha, x)$, the conjugate (i.e., the inverse) of the form $A(\alpha, p)$, is particularly simple:

$$\overline{A}(\alpha, x) = \sum_{\nu=1}^{l} \frac{1}{\alpha_\nu}\Big(\sum_{i=1}^{n-1} \varepsilon_{i\nu} x_i\Big)^2 = \sum_{i,j=1}^{n} d_{ij} x_i x_j. \tag{2.7}$$

The fact that the forms (2.5) and (2.7) are mutually conjugate implies that

$$A(\alpha, \tfrac{1}{2}\partial\overline{A}/\partial x) = \overline{A}(\alpha, x). \tag{2.8}$$

Here $\partial\overline{A}/\partial x$ is a symbolic notation for the set of $n-1$ four-vectors $\partial\overline{A}/\partial x_{i\mu}$ ($\mu = 0, 1, 2, 3$; $i = 1, 2, \ldots, n-1$).

Formulas (2.7) and (2.5) can be obtained in different ways. They are apparently most easily derived from the expression for the contribution from the Feynman graph in the x representation by a Fourier transformation (cf. [3]). In this case the quadratic form (2.7) appears in the exponent and Gaussian quadrature leads to the form (2.5) (cf. [8], where this derivation is given). The same formulas can also be derived by purely algebraic transformations, starting with the definition of the form $Q(\alpha, p)$ as the extremum of the form $K(\alpha, k)$ (cf. [2]).

For example, if all internal lines represent the scalar propagator $(k_\nu^2 - m_\nu^2 + i0)^{-1}$ and the graph D contains no divergences, the contribution from this graph has the following form:

$$T_D(p) = C\int_0^1 \cdots \int_0^1 \frac{\delta(1 - \sum_{\nu=1}^{l} \alpha_\nu)}{d^2(\alpha)\,[Q_D(\alpha, p) + i0]^{2n-l-2}} \prod_{\nu=1}^{l} \frac{d\alpha_\nu}{\alpha_\nu^2}, \tag{2.9}$$

where C is a constant.

We note that (2.5) and (2.7) have direct meaning only for positive $\alpha_1, \ldots, \alpha_l$. As was noted by Landau,[9] if some α_ν is zero, then the form $Q(\alpha, p)$, as a function of the remaining α, coincides with the form corresponding to the graph obtained from

*The incidence matrix was introduced by Poincaré[4] in 1901. For the topological properties of the graphs and the role of the incidence matrix see [5,6].

†The momentum conservation law in the form (2.1) has been given in the work of Bogolyubov and Parasyuk.[7] Formula (2.1) is a particular case of the general expression for the Δ boundary of a one-dimensional chain (cf. [5], Chapter 7).

the original graph by contracting the line ν into a point. In this case the external momentum at this point is equal to the sum of the external momenta at the ends of the line ν. In the following we shall assume that all $\alpha_\nu > 0$ and use formulas (2.5) and (2.7).

3. METHOD OF DETERMINATION OF THE CLASS R_0

As is known (cf., e.g., [3]), the region of regularity G(D) defined in the Introduction consists of those (Euclidean) momenta p for which

$$Q_D(\alpha, p) < 0 \text{ for all } \alpha_\nu \geqslant 0 \left(\sum_{\nu=1}^{l} \alpha_\nu > 0\right). \quad (3.1)$$

For example, the expression (2.9), as a function of p, has obviously no singularities in the region (3.1). In the general case of a graph with arbitrary internal lines and containing, in general, divergences, an analogous assertion can be proved with the help of the regularized expression for the function $T_D(p)$ obtained by Bogolyubov and Parasyuk.[7]

The majorization of graphs for Euclidean external momenta* is based on the following simple lemma:[1]

Lemma 1. If the momenta along the internal lines k_ν take the values of the set P of linear combinations of the external momenta (with real coefficients) and satisfy the conservation law (2.1) in the vertices of the graph, then the form $Q(\alpha, p)$ is equal to the minimum (with respect to the k_ν) of the function $K(\alpha, k)$.

With the help of this lemma one can prove two theorems[1] which play an important role in the majorization of graphs. In order to formulate the first theorem, we introduce the notion of a subgraph: if the removal of some internal lines and internal vertices† from a graph $D \in R$ leads to a graph $D' \in R$, then the graph D′ is called a subgraph of the graph D.

Theorem 1. Every graph is majorized by any one of its subgraphs.

*The set P consisting of linear combinations of the external momenta p is called Euclidean if the matrix composed of the scalar products p_ip_j is positive definite, i.e., if for arbitrary real a_i

$$(\Sigma a_ip_i)^2 \geq 0,$$

where only the null vector has a length equal to zero. These conditions are necessary and sufficient for the existence of a basis in the space P in which the scalar product of a vector with itself is equal to the sum of the squares of its components. As is known, the physical momenta are pseudo-Euclidean, not Euclidean.

†An internal vertex is a vertex in which the external momentum is identically equal to zero.

Theorem 2. The graph D contain a closed polygon with n+1 sides, n of which are associated with the mass M and one with the mass $m \leq M$. Interchange the masses corresponding to these sides in the following way: $M \rightleftarrows m$. As a result one obtains a new graph D′ which majorizes the original graph D.

Let us prove the following lemma:

Lemma 2. Every graph is majorized by some graph which has three lines in every vertex.

Assume, for example, that $n > 3$ meson lines go into some vertex O of the graph D. Replace the vertex O by a polygon with n vertices such that at each vertex of this polygon one and only one of the lines converging in the point O enters. The vertex O is thus replaced by n vertices each having only three lines. The resulting graph D′ majorizes the graph D by virtue of the fact that D′ goes over into D if the Feynman parameters α on all sides of the n vertex polygon are set equal to zero. A vertex containing other lines besides meson lines can be treated in an analogous way by making use of the conservation of baryon charge and strangeness.

Let us consider the class R of strongly connected graphs of a certain process whose vertices contain either two nucleon lines and one π meson line or four meson lines.* This class is characteristic of the usual theory of the interaction of pseudoscalar mesons with nucleons. According to Lemma 2 every graph of this class is majorized by some graph of the class R′ in which each vertex has three lines: two nucleon lines and one meson line or three meson lines. The class R′ corresponds to the model of the interaction of scalar mesons with nucleons. From the previous discussion $G_R \supset G_{R'}$ (cf. the definition of the intersection G_R in the Introduction).

In every graph of the class R′ the nucleon lines form several nonintersecting polygons and broken lines ("chains"). The number of nucleon chains is equal to half the number of nucleons participating in the reaction. If the nucleon polygons of an arbitrary graph are replaced by meson polygons, the value of the form $Q(\alpha, p)$ increases. Therefore $G_{R'} = G_{R^*}$, where R^* is the subset of R′ consisting of graphs without nucleon polygons.

The following lemmas hold for the graphs of the class R:

Lemma 3. Let Ω be a set of lines and vertices

*The class R is a subclass of a wider class of graphs whose internal lines can correspond to strange particles.[1] The class R is obtained from this wider class by replacing all baryon lines in a given graph by nucleon lines and all meson lines by π meson lines.

of a graph D containing, together with each line, the vertices at the end of that line. Assume that the vertex a does not belong to Ω and is connected with the vertex b of Ω through some chain $(a, b)_0$ which has no points belonging to Ω except b. Then there exist two chains without common lines, $(a, b)_1$ [which, in general, is different from $(a, b)_0$] and $(a, b')_2$ $(b' \in \Omega)$, connecting the vertex a with the set Ω.

Lemma 4. Under the assumptions of Lemma 3, where the first chain $(a, b)_0$ consists of nucleon lines and the remaining nucleons of the graph are all contained in Ω, the graph D is majorized by the graph D′ containing the meson chain $(a, b')_2$ $(b' \in \Omega)$ and differing from D only in that the nucleon lines not belonging to Ω form the chain $(a, b)_1$ [instead of the chain $(a, b)_0$].

The proof of Lemma 4 makes use of Theorem 2. With the help of the assertions above it can be shown that any strongly connected graph is majorized by a graph of the class R_0 [cf. formula (1.1)].

4. THE SYMANZIK THEOREM AND ITS GENERALIZATION

It follows from (3.1) that the region of regularity G of a graph with the quadratic form (2.4) (with Euclidean momenta) can be described in the following fashion:

$$p \in G, \quad \text{if} \quad \overline{L}^2(p) \equiv \sup_{\alpha_\nu \geqslant 0} \frac{A(\alpha, p)}{M^2(\alpha)} < 1. \tag{4.1}$$

The function $\overline{L}(p)$ is homogeneous of the first degree [under multiplication of the $4(n-1)$-component vector argument p by a positive factor]. If the momenta are Euclidean, $\overline{L}(p)$ is negative and satisfies the triangular condition

$$\overline{L}(p+q) \leqslant \overline{L}(p) + \overline{L}(q),$$

it vanishes only if $p = 0$. In other words, if the external momenta are Euclidean, the function $\overline{L}(p)$ is a norm.*

Using the fact that the quadratic forms (2.5) and (2.7) are mutually conjugate, it can be shown that the norm conjugate to $\overline{L}(p)$ is equal to

$$L(x) = \inf_{\alpha_\nu \geqslant 0} \sqrt{M^2(\alpha)\overline{A}(\alpha, x)}. \tag{4.2}$$

We recall that the norms $L(x)$ and $\overline{L}(p)$ are called mutually conjugate if

$$L(x) = \max_{\overline{L}(p) \leqslant 1} \left| \sum_{i=1}^{n-1} x_i p_i \right|. \tag{4.3}$$

[An equivalent definition of the conjugate norm can be given by starting from a condition of the type (2.8)].

It turns out that the minimum of the right-hand side of (4.2), i.e., the norm $L(x)$, can be found explicitly. Let us assume initially that, for given x, the minimum occurs at an internal point of the region of values α (i.e., all $\alpha_\nu > 0$, $\nu = 1, \ldots, l$). This point is found by setting the derivatives with respect to α_ν of the expression $M^2(\alpha)\overline{A}(\alpha, x)$ equal to zero:

$$\alpha_\nu = \frac{1}{m_\nu} \left| \sum_{i=1}^{n-1} \varepsilon_{i\nu} x_i \right| \left\{ \left[\sum_{s=1}^{l} \frac{1}{\alpha_s} \left(\sum_{i=1}^{n-1} \varepsilon_{is} x_i \right)^2 \right]^{-1} \sum_{s=1}^{l} \alpha_s m_s^2 \right\}^{1/2}. \tag{4.4}$$

The variables α are determined by (4.4) only up to a positive factor, which is unessential in view of the homogeneity of the functions $\overline{L}(p)$ and $L(x)$. If, for example, we assume that

$$\sum_{\nu=1}^{l} m_\nu^2 \alpha_\nu = 1,$$

then we obtain from (4.4) that

$$\alpha_\nu = \left\{ \sum_{s=1}^{l} m_s \left| \sum_{i=1}^{n-1} \varepsilon_{is} x_i \right| \right\}^{-1} \frac{1}{m_\nu} \left| \sum_{i=1}^{n-1} \varepsilon_{i\nu} x_i \right|.$$

Substituting (4.4) in (4.2), we find

$$L(x) = \sum_{\nu=1}^{l} m_\nu \left| \sum_{i=1}^{n-1} \varepsilon_{i\nu} x_i \right|. \tag{4.5}$$

Let now some $\alpha_{\nu_0} = 0$. According to the remarks at the end of the preceding section, the forms A and M^2 then go over into the forms corresponding to a graph in which the line ν_0 is contracted into a point, so that the variables x_i at the ends of this line coincide. Hence,

$$\sum_{i=1}^{n-1} \varepsilon_{i\nu_0} x_i = 0,$$

and formula (4.5) is again the correct expression for the minimum (4.3). Thus formula (4.5) is valid in all cases.

If not all vertices of a graph are external, some $p_i \equiv 0$ and the norm $\overline{L}(p)$ is actually a norm in a space with a lower number of dimensions [than $4(n-1)$]. It can be shown that the corresponding conjugate norm is equal to the minimum of the norm $L(x)$ with respect to the variables x_i corresponding to the internal vertices:

$$\text{If } p_{int} \equiv 0, \text{ then } L(x) = \min_{x_{int}} \sum_{\nu=1}^{l} m_\nu \left| \sum_{i=1}^{n-1} \varepsilon_{i\nu} x_i \right|. \tag{4.6}$$

We are now in a position to formulate the Symanzik theorem:

Theorem 3. (Symanzik theorem). Consider two graphs D_1 and D_2 for the same process and let

*The general properties of a norm, which will be used in the following, can be found in any treatise on functional analysis (cf. e.g., [10], Chapter 11).

$\bar{L}_1(p)$ and $\bar{L}_2(p)$ be the norms corresponding to these graphs according to (4.1). Then the graph D_1 majorizes the graph D_2, i.e.,

$$\bar{L}_1(p) \geqslant \bar{L}_2(p)$$

for all Euclidean p, if and only if

$$L_1(x) \leqslant L_2(x),$$

for all Euclidean x, where $L_i(x)$ is given by (4.6).

This theorem follows immediately from the fact that $L(x)$ and $\bar{L}(p)$ are mutually conjugate norms.

In an analogous fashion one proves the following generalization of the Symanzik theorem:

Theorem 4. Consider k+1 graphs D_σ ($\sigma = 0, 1, \ldots, k$) for the same process and let $\bar{L}_\sigma(p)$ be the norms corresponding to these graphs. The graphs $D_1, \ldots, D_k$ majorize the graph D_0, i.e.,

$$\bar{L}_0(p) \leqslant \max_{\sigma=1,\ldots,k} (\bar{L}_\sigma(p)) \equiv \bar{L}_{1\ldots k}(p), \tag{4.7}$$

if and only if the conjugate norms satisfy the inverse inequality

$$L_{1\ldots k'}(x) \leqslant L_0(x).$$

Here $L_{1\ldots k}(x)$ is the maximal norm satisfying the inequality

$$L_{1\ldots k}(x) \leqslant \min_{\sigma=1,\ldots,k} L_\sigma(x). \tag{4.8}$$

Theorem 4 and (4.8) imply a simple sufficient condition for the majorization of one graph by a set of others which we shall use in the following.

Corollary to Theorem 4. A sufficient condition for the validity of (4.7), i.e., that the graph D_0 is majorized by the set of graphs $D_1, \ldots, D_k$, is

$$\min_{\sigma=1,\ldots,k} L_\sigma(x) \leqslant L_0(x). \tag{4.9}$$

5. APPLICATION TO NUCLEON-NUCLEON SCATTERING

We can use Theorem 3 to show that graph I of Fig. 1 majorizes graphs IV, V, and VI of the same figure. For example, for the graph IV with the norm $L_{IV}(x)$, the assertion follows from the following chain of inequalities:

$$\begin{aligned} L_{IV}(x) &= \min_{x_5,\ldots,x_8} \{M[|x_1 - x_5| + |x_5 - x_2| \\ &+ |x_3 - x_6| + |x_6|] + m[|x_1 - x_7| + |x_7 - x_2| \\ &+ |x_7 - x_8| + |x_5 - x_6| + |x_3 - x_8| + |x_8|]\} \\ &\geqslant (M - \tfrac{1}{2}m) \min_{x_5, x_6} [(|x_1 - x_5| + |x_5 - x_2|) + (|x_3 - x_6| \\ &+ |x_6|)] + \tfrac{1}{2} m \min_{x_5 \ldots, x_8} [(|x_1 - x_7| + |x_7 - x_2|) \\ &+ (|x_3 - x_8| + |x_8|) \\ &+ (|x_2 - x_5| + |x_5 - x_6| + |x_6|) + (|x_1 - x_5| \\ &+ |x_5 - x_6| + |x_6 - x_3|) + (|x_2 - x_7| + |x_7 - x_8| \\ &+ |x_8|) + (|x_1 - x_7| + |x_7 - x_8| + |x_8 - x_3|)] \\ &\geqslant L_I(x_1, x_2, x_3). \end{aligned} \tag{5.1}$$

Here $L_I(x)$ is the norm of graph I:

$$\begin{aligned} L_I(x_1, x_2, x_3) &= M(|x_1 - x_2| + |x_3|) \\ &+ m(|x_1 - x_3| + |x_2|) \end{aligned} \tag{5.2}$$

(everywhere we set $x_4 = 0$). It is shown in an analogous manner that the graphs V and VI also are majorized by graph I.

With the help of the corollary to Theorem 4 we can show that a pair of graphs I with different arrangements of the external momenta majorizes the graphs III and VII of Fig. 1. As an example, we give the proof for graph III:

$$\begin{aligned} L_{III}(x) &= \min_{x_5, x_6} \{M(|x_1 - x_5| + |x_5 - x_2| + |x_3|) + m(|x_5 - x_3| \\ &+ |x_1 - x_6| + |x_6| + |x_6 - x_2|)\} \geqslant M|x_3| \\ &+ (M - m) \min_{x_5} (|x_1 - x_5| + |x_5 - x_2|) \\ &+ \tfrac{1}{2} m \min_{x_5, x_6} [(|x_1 - x_5| + |x_5 - x_2|) + (|x_1 - x_6| \\ &+ |x_6 - x_2|) + (|x_1 - x_5| + |x_5 - x_3|) + (|x_2 - x_5| \\ &+ |x_5 - x_3|) + (|x_1 - x_6| + |x_6|) \\ &+ (|x_2 - x_6| + |x_6|)] \geqslant M(|x_1 - x_2| + |x_3|) \\ &+ \tfrac{1}{2} m(|x_1 - x_3| + |x_2 - x_3| + |x_1| + |x_2|) \\ &\geqslant \min\{L_I(x_1, x_2, x_3), L_I(x_2, x_1, x_3)\}, \end{aligned} \tag{5.3}$$

where $L_I(x)$ is given by (5.2). The proof for graph VII of Fig. 1 is analogous. We note that we would not be able to eliminate graphs III and VII with the help of the Symanzik theorem (Theorem 3) alone, for the graph I with a fixed arrangement of the external momenta does not majorize graphs III and VII.

It has thus been shown that the graphs I and II of Fig. 1 majorize all strongly connected graphs for the nucleon-nucleon scattering process.

Setting M = m, we conclude from the above-given results that all strongly connected graphs for the scattering of a meson by a meson are majorized by a quadrangular graph consisting only of meson lines.

It can also be shown that all strongly connected graphs for meson-nucleon scattering are majorized by the set of four graphs shown in Fig. 2.

We note that, according to the discussion in

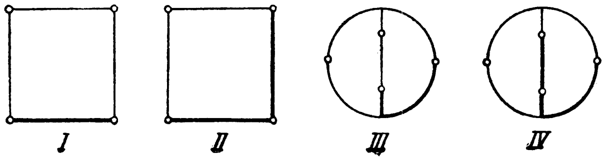

FIG. 2

Sec. 3, these results are true not only for scalar but also for pseudoscalar mesons. The graphs I and II of Fig. 1 still have physical meaning if the scalar mesons are replaced by pseudoscalar ones. They can therefore be used for a precise determination of the region G_R. The graphs of Fig. 2 and the quadrangular graph for meson-meson scattering cease to have physical meaning after such a replacement, i.e., they no longer belong to the class R. In the pseudoscalar case it can, therefore, only be asserted that these graphs allow us to determine some region containing the region G_R;

6. DISCUSSION OF RESULTS

In the present paper we have carried out the majorization of graphs in the Euclidean region without any restriction on the squares of the external momenta. The results obtained in this work can therefore be used in the investigation of the analytic properties of the amplitudes in the entire real region of values of the independent scalar invariants [3,11] and even in a certain complex region of values of these invariants. [11] This makes it possible to obtain single dispersion relations in energy and momentum transfer for elastic scattering.

The expressions for the quadratic forms A and $\bar{A}$ obtained in Sec. 2 and the functions L(x) and $\bar{L}$(p) determined in Sec. 4 can be used not only for the majorization of graphs in the Euclidean region but also for the determination of the singularities of graphs with arbitrary complex values of the momenta. [8]

In conclusion the authors express their deep gratitude to Academician N. N. Bogolyubov for his interest in this work and valuable comments and to A. N. Tavkhelidze, who took part in obtaining some of the results of this paper.

Note added in proof (April 17, 1962). We have been able to show with the help of the corollary of Theorem 4 that graph II of Fig. 1 is also majorized by two graphs I with different arrangements of the external momenta. Thus the graph I of Fig. 1 majorizes all strongly connected graphs for nucleon-nucleon scattering.

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Translated by R. Lipperheide
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