MOTION OF INDIVIDUAL PARTICLES IN HIGHLY DEFORMED NUCLEI

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The single-particle eigenfunctions and eigenvalues and the effective energies of individual particles have been investigated and calculated for the nuclei of the rare earths. The values of the electric quadrupole moments and decoupling parameters computed using these eigenfunctions are in good agreement with the experimental results.

IN recent years there have been numerous treatments of the problems of motion of individual particles in highly deformed nuclei.[1] In the present paper, we introduce the "effective energy" of a particle and give a more rational treatment of the problem of motion of a nucleon in a highly deformed nucleus.

The wave function of an axially symmetric nucleus has the form [2]

$$\psi_{IM} = \left(\frac{2I+1}{16\pi}\right)^{1/2} [D^I_{M\Omega}(\vartheta_i)\chi^\tau_\Omega(\mathbf{r}') + e^{iI\pi} D^I_{M-\Omega}(\vartheta_i) R_1 \chi^\tau_\Omega(\mathbf{r}')].$$

We shall find the wave functions for the internal motion of the nucleons, $\chi^T_\Omega(\mathbf{r}')$, and give specific results for some physical quantities for nuclei in the region $150 < A < 190$.

1. Let us assume that the Hamiltonian of a particle moving in the highly deformed field has the form

$$H = H_0 + H'; \tag{1}$$

$$H_0 = P^2/2M + \tfrac{1}{2} M [\omega_x^2(x^2+y^2) + \omega_z^2 z^2],$$
$$H' = C_N \mathbf{l}_t \mathbf{s} + D_N \mathbf{l}_t^2, \tag{2}$$

where C_N and D_N are two parameters, and M is the mass of the particle,

$$\omega_x = \omega_0(\varepsilon)(1+\varepsilon/3), \qquad \omega_z = \omega_0(\varepsilon)(1-2\varepsilon/3),$$
$$\omega_0(\varepsilon) = \omega_0^0 (1 - \varepsilon^2/3 - 2\varepsilon^3/27)^{-1/3};$$
$$(\mathbf{l}_t)_x = -i\hbar [\sqrt{\beta_\rho/\beta_z}\, y\,\partial/\partial z - \sqrt{\beta_z/\beta_\rho}\, z\,\partial/\partial y], \ldots,$$
$$\beta_\rho = M\omega_x/\hbar, \qquad \beta_z = M\omega_z/\hbar.$$

We know that the Hamiltonian H_0 has eigenfunctions of the form $(\rho^2 = x^2 + y^2)$

$$|nn_z m\sigma\rangle = A\rho^{|m|} \exp [im\varphi - \tfrac{1}{2}\beta_\rho \rho^2 - \tfrac{1}{2}\beta_z z^2] H_{n_z}(\sqrt{\beta_z} z) L_n^{|m|}(\beta_\rho \rho^2) |\sigma\rangle, \tag{3}$$

belonging to the eigenvalues

$$\lambda(N, n_z) = [(N + \tfrac{3}{2}) + \tfrac{1}{3}\varepsilon(N - 3n_z)]\hbar\omega_0. \tag{4}$$

Here A is a normalization constant, $L_n^{|m|}(\beta_\rho \rho^2)$ are Laguerre polynomials, $H_{n_z}(\sqrt{\beta_z}\, z)$ are Hermite polynomials,[3] $|\sigma\rangle$ is the spin wave function, and N is the principal quantum number:

$$N = 2n + |m| + n_z.$$

Let us expand the solution $|Nj\Omega\rangle$ for the Hamiltonian H in the functions $|nn_z m\sigma\rangle$, where j for $\epsilon = 0$ is the total angular momentum of the nucleon:

$$|Nj\Omega\rangle = \sum_{n_z \sigma} a_{n_z\sigma}(Nj\Omega)\,|nn_z m\sigma\rangle. \tag{5}$$

The summation here extends over all n_z and σ satisfying the conditions

$$N = 2n + |m| + n_z, \qquad \Omega = m + \sigma.$$

In this way we can find the eigenvalues $\lambda((Nj\Omega)$ of the Hamiltonian H.

We have also found the quantities $a_{n_z\sigma}(Nj\Omega)$, $\lambda(Nj\Omega)$ and $\bar{n}_z(Nj\Omega)$,[4] where

$$\bar{n}_z(Nj\Omega) = \sum_{n_z\sigma} n_z a^2_{n_z\sigma}(Nj\Omega),$$

for the proton system with N = 4, 5 and the neutron system with N = 5, 6. In the numerical computations we used the following values of the parameters C_N and D_N (in units of $\hbar\omega_0^0$):

	Proton system		Neutron system	
N:	4	5	5	6
C_N:	—0.18	—0.18	—0.16	—0.16
D_N:	—0.050	—0.046	—0.033	—0.035

2. We know that the nuclear field is mainly produced by the two-particle interactions between nucleons. Thus when we describe the nuclear field by means of some averaged field, the total energy of the individual particles is

$$E_t = \sum_i \left[\langle |H_i|\rangle - \tfrac{1}{2}\langle |T_i|\rangle\right] = \sum_i E_i. \tag{6}$$

Let us introduce the "effective energy" of an individual particle

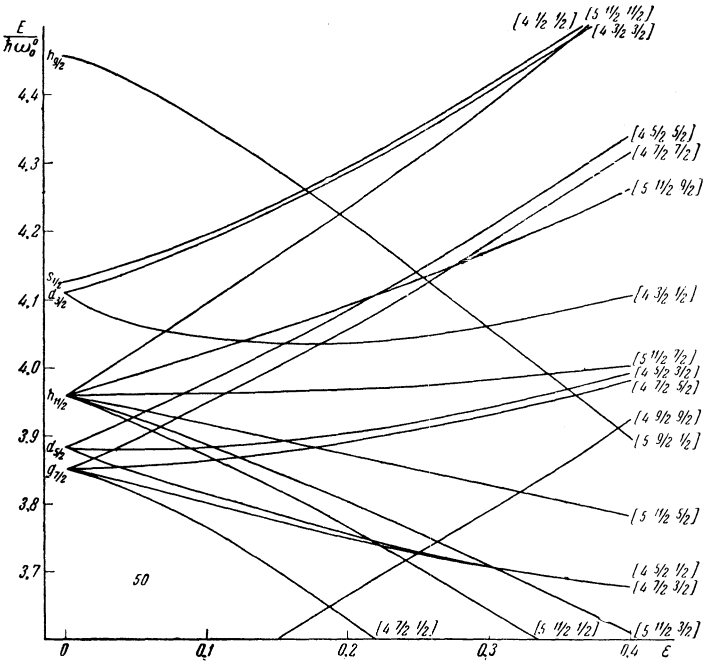

FIG. 1. Scheme of effective individual levels of particles for the proton system $50 < Z < 82$. The levels are labelled by the quantum numbers $[Nj\Omega]$.

$$E_i = \langle | H_i | \rangle - \tfrac{1}{2} \langle | T_i | \rangle = \tfrac{1}{2} \langle | H_i + T_i | \rangle. \quad (7)$$

We easily find that

$$\langle | T_i | \rangle = \tfrac{1}{2}\left(N + \tfrac{3}{2}\right) \hbar\omega_0 + \tfrac{1}{6}\, \varepsilon\, (N - 3\bar{n}_z)\, \hbar\omega_0. \quad (8)$$

Substituting (8) in (7), we have

$$E_i = \tfrac{1}{2} \lambda\, (Nj\Omega) + \tfrac{1}{4}\left(N + \tfrac{3}{2}\right) \hbar\omega_0 + \tfrac{1}{4}\, \varepsilon \left(\tfrac{1}{3} N - \bar{n}_z\right) \hbar\omega_0. \quad (9)$$

The level schemes of effective energies E_i are shown in Figs. 1 and 2.

The sum of the effective energies of the nucleons filling the levels with a given N is

$$E_N(\varepsilon) = \tfrac{3}{4}(N+1)\left(N + \tfrac{3}{2}\right)(N+2)\, \hbar\omega_0 + \sum_l l\,(l+1)\,(2l+1)\, D_N, \quad (10)$$

where l for $\epsilon = 0$ is the orbital angular momentum quantum number.

From the condition for minimum total energy of the individual particles,

$$\partial E_t(\varepsilon)/\partial\varepsilon = 0$$

one can determine the equilibrium deformation and the corresponding configuration for the ground state of the nucleus. It should be emphasized that the order of filling of the levels in the schemes considered is identical with the order determined from the condition of minimum total energy. It should be noted that the order of the levels E_i does not agree with the order of the levels $\lambda\,(Nj\Omega)$, i.e., even though $\lambda_\alpha > \lambda_\beta$ it may happen that $E_\alpha < E_\beta$, where λ_α, λ_β are the single-particle eigenvalues for states α and β, while E_α, E_β are the effective energies of an individual particle in these states. For precisely this reason the actual filling of nucleons for the ground state should go in accordance with the level scheme for the effective energies of individual particles, and not in accordance with the level scheme of single-particle eigenvalues.

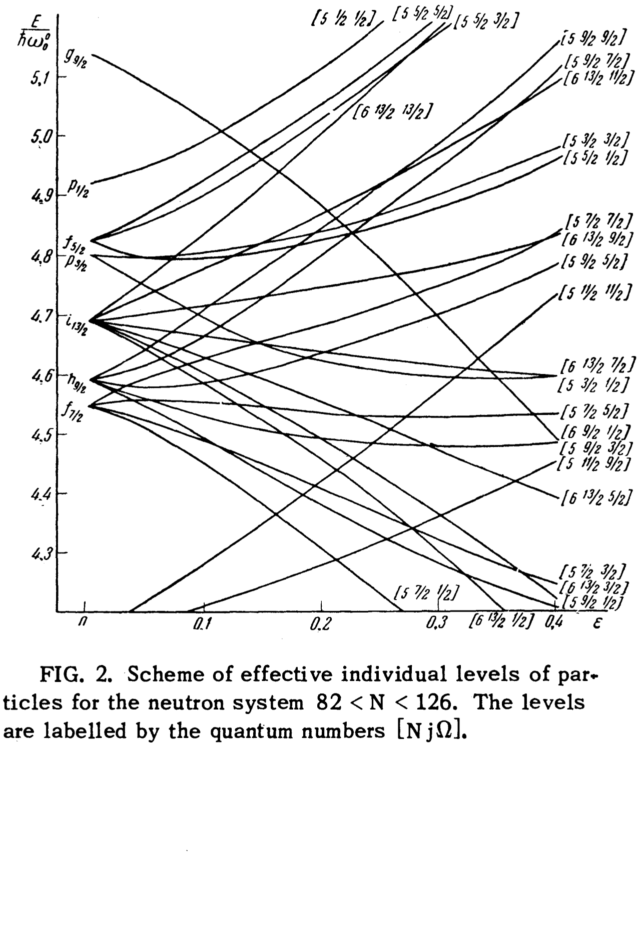

FIG. 2. Scheme of effective individual levels of particles for the neutron system $82 < N < 126$. The levels are labelled by the quantum numbers $[Nj\Omega]$.

Table I. Equilibrium deformations and electric quadrupole moments of highly deformed nuclei

Nucleus	A	ε_0	Configuration of odd nucleus	Q_0, b (exp.)	$A^{-1/3} Q_0$, b	
					exp.	theory
$_{62}$Sm	152	0,298		5,86	1,10	1,36
$_{63}$Eu	153	0,294	$4\,{}^7/_2\,{}^5/_2$	6,94	1,29	1,35
$_{64}$Gd	154	0,290		5,88	1,10	1,35
	155	0,290	$5\,{}^9/_2\,{}^3/_2$	6,50	1,21	1,35
	156	0,304		6,79	1,26	1,40
	157	0,304	$5\,{}^9/_2\,{}^3/_2$	6,60	1,22	1,40
	158	0,305		7,41	1,37	1,40
$_{65}$Tb	159	0,300	$4\,{}^5/_2\,{}^3/_2$	7,41	1,37	1,34
$_{66}$Dy	160	0,297		6,72	1,24	1,39
	161	0,290	$6\,{}^{13}/_2\,{}^5/_2$	7,3	1,34	1,36
	162	0,297		7,19	1,32	1,39
	163	0,271	$5\,{}^7/_2\,{}^5/_2$	7,3	1,34	1,29
	164	0,271		7,55	1,38	1,29
$_{67}$Ho	165	0,270	$5\,{}^{11}/_2\,{}^7/_2$	7,56	1,38	1,31
$_{68}$Er	164	0,286		7,14	1,30	1,39
	166	0,269		7,56	1,37	1,33
	167	0,271	$6\,{}^{13}/_2\,{}^7/_2$	7,8	1,42	1,33
	168	0,272		7,60	1,38	1,34
	170	0,290		7,42	1,34	1,40
$_{69}$Tm	169	0,270	$4\,{}^3/_2\,{}^1/_2$	7,52	1,36	1,35
$_{70}$Yb	172	0,272		7,48	1,34	1,37
	173	0,266	$5\,{}^9/_2\,{}^5/_2$	7,8	1,40	1,35

Table II. Values of decoupling parameter

Nucleus	Configuration	Energy, keV	ε_0	a		a	
				exp	theory	$\varepsilon=0.2$	$\varepsilon=0.3$
$_{69}$Tm^{169}	$4\,{}^3/_2\,{}^1/_2$	0	0,30	−0.77	−1.00	1,0960	−0.9956
$_{69}$Tm^{171}	$4\,{}^3/_2\,{}^1/_2$	0	0,30	−0.86	−1.00	−1,0960	−0.9956
$_{68}$Er^{165}	$5\,{}^3/_2\,{}^1/_2$	243	0,30	1	0,93	0,8071	0.9288
$_{70}$Yb^{171}	$5\,{}^3/_2\,{}^1/_2$	0	0,27	0,85	0,89	0,8071	0.9288
$_{74}$W^{181}	$5\,{}^5/_2\,{}^1/_2$	515	0,20	0,17	0,11	0,1111	−0.0933
$_{74}$W^{183}	$5\,{}^5/_2\,{}^1/_2$	0	0,20	0,19	0,11	0,1111	−0,0933

Table I gives theoretical values of equilibrium deformations of nuclei and the configurations of nuclear ground states. A is the mass number, ϵ_0 is the theoretical value of the nuclear deformation, and Q_0 is the electric quadrupole moment.

3. Using these functions, one can compute some of the physical quantities for highly deformed nuclei.

A. Electric quadrupole moments.

The intrinsic electric quadrupole moment is equal to

$$Q_0(\varepsilon) = \frac{\hbar}{M\omega_0^0} \sum_{Nj\Omega} \frac{3(1-\varepsilon^2/3-2\varepsilon^3/27)^{1/3}}{(1+\varepsilon/3)(1-2\varepsilon/3)} \left[\frac{2}{9}\varepsilon\left(N+\frac{3}{2}\right) + \left(\bar{n}_z - \frac{N}{3}\right)\right]. \quad (11)$$

The summation goes over all protons, and $\hbar\omega_0^0 = 38A^{-1/3}$ MeV (corresponding to $r_0 = 1.25$ F). The sum of the quadrupole moments of the protons filling all the levels for a given N is

$$Q_N(\varepsilon) = \tfrac{2}{3}\hbar\varepsilon(N+1)\left(N+\tfrac{3}{2}\right) \times (N+2)/M\omega_0(\varepsilon)\left(1+\tfrac{1}{3}\varepsilon\right)\left(1-\tfrac{2}{3}\varepsilon\right). \quad (12)$$

Table I gives experimental [5] and theoretical values of nuclear quadrupole moments. From the table we see that the theoretical results essentially coincide with the experimental data.

B. Decoupling parameter.

If the coordinate system fixed in the nucleus is rotated through 180° around the x axis, the wave function is changed as follows:

$$R_1 \mid Nj\Omega\rangle = -i(-1)^N \mid Nj-\Omega\rangle. \quad (13)$$

In this representation, the decoupling parameter is equal to

$$a = (-)^N \delta_{\Omega,1/2}\Big[\sum_{n_z} a^2_{n_z,1/2}(Nj\Omega) - \left(\sqrt{\beta_\rho/\beta_z} + \sqrt{\beta_z/\beta_\rho}\right) \times \Big\{\sum_{n_z} \sqrt{(n_z+1)(N-n_z+1)}\, a_{n_z+1,1/2} \times (Nj^1/_2)\, a_{n_z,-1/2}(Nj^1/_2) + \sum_{n_z} \sqrt{(N-n_z)(n_z+1)}\, a_{n_z+1,-1/2}(Nj^1/_2)\, a_{n_z,1/2}(Nj^1/_2)\Big\}\Big]. \quad (14)$$

In Table II we give theoretical and experimental values of the decoupling parameter (in the last two columns we give computed values of the decoupling parameter for different deformations). We note that these results are also in good agreement with the experimental data. In the case of W^{181} and W^{183}, the decoupling parameter has the correct sign, in contrast to the computations of Nilsson.

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