COLLISION INTEGRAL FOR CHARGED PARTICLES IN A MAGNETIC FIELD

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The collision integral for Coulomb interactions is derived with shielding due to polarization of the medium taken into account. The polarization is introduced by means of a complex dielectric tensor in which both frequency and spatial dispersion are included.

1. In an investigation of the correlation function and the collision integral for charged particles in the absence of external fields one of the present authors [1] has pointed out that the Coulomb field should not be used in computing the collision probability; instead, in computing the field one must take account of the polarization of the medium by introducing the effect of shielding into the field produced by the charged particles. The explicit physical picture in this work makes it possible to write an expression for the collision integral in the presence of an external field without solving the equations for the correlation function. In the present work we obtain the collision integral for charged particles in the presence of a strong magnetic field.*

The collision integral for Coulomb particles in a strong magnetic field has been considered in turn by E. Lifshitz [3] and by Belyaev.[4] The collision integral obtained in the present work differs in two respects from those obtained by these authors: first, we take account of the polarization of the medium, thus making the collision integral convenient for describing remote collisions, second, we take account of quantum effects (in the Born approximation) thus making the collision integral suitable for analyzing close collisions (provided one is considering a plasma for which the Born approximation is adequate). For the particular case of classical collisions and spatially uniform distributions our results correspond to those obtained by Rostoker.[5] A complete correspondence holds only for electron-electron collisions.

2. Assuming that the Fourier components $[\sim \exp(-i\omega t + i\mathbf{k}\cdot\mathbf{r})]$ of the electric field and the electric induction are related by the expression [6]

$$D_i(\omega,\mathbf{k}) = \int d\mathbf{k}'\, \varepsilon_{ij}(\omega,\mathbf{k},\mathbf{k}')E_j(\omega,\mathbf{k}'), \qquad (1)$$

as in [1], we obtain an expression for the matrix element describing the scattering of particle α on particle β

$$\int d\mathbf{k}\, d\mathbf{k}'\, 4\pi e_\alpha e_\beta A^{-1}\left(\frac{E(\nu_\alpha') - E(\nu_\alpha)}{\hbar},\ \mathbf{k},\ \mathbf{k}'\right) \times \langle \nu_\alpha' | e^{i\mathbf{kr}} | \nu_\alpha\rangle \langle \nu_\beta' | e^{-i\mathbf{kr}} | \nu_\beta\rangle. \qquad (2)$$

Here, ν_α and ν_α' represent sets of quantum numbers describing the state of the particle before and after the collision, $E(\nu_\alpha)$ is the energy of the particle and $A^{-1}(\omega, \mathbf{k}, \mathbf{k}')$ represents a solution of the equation

$$\int d\mathbf{k}''\, k_i \varepsilon_{ij}(\omega,\mathbf{k},\mathbf{k}')\, k_j''\, A^{-1}(\omega,\mathbf{k}'',\mathbf{k}') = \delta(\mathbf{k}-\mathbf{k}'). \qquad (3)$$

In the particular case of a uniform medium

$$\varepsilon_{ij}(\omega,\mathbf{k},\mathbf{k}') = \varepsilon_{ij}(\omega,\mathbf{k})\,\delta(\mathbf{k}-\mathbf{k}'), \qquad (4)$$

where $\varepsilon_{ij}(\omega, \mathbf{k})$ is the complex dielectric tensor,[6] so that (2) assumes the form

$$\int d\mathbf{k}\, 4\pi e_\alpha e_\beta \frac{\langle \nu_\alpha' | e^{i\mathbf{kr}} | \nu_\alpha\rangle \langle \nu_\beta' | e^{-i\mathbf{kr}} | \nu_\beta\rangle}{k_i \varepsilon_{ij}([E(\nu_\alpha') - E(\nu_\alpha)]/\hbar,\ \mathbf{k})\, k_j}. \qquad (5)$$

Using the elements of the scattering matrix (2) we obtain the probability for transition of particles α and β from states ν_α, ν_β to the states ν_α', ν_β'

$$W(\nu_\alpha\nu_\beta;\ \nu_\alpha'\nu_\beta') = 2\pi\hbar^{-1}\delta[E(\nu_\alpha') + E(\nu_\beta') - E(\nu_\alpha) - E(\nu_\beta)] \times \left| 4\pi e_\alpha e_\beta \int d\mathbf{k}\, d\mathbf{k}'\, A^{-1}\left(\frac{E(\nu_\alpha') - E(\nu_\alpha)}{\hbar},\ \mathbf{k},\ \mathbf{k}'\right) \times \langle \nu_\alpha' | e^{i\mathbf{kr}} | \nu_\alpha\rangle \langle \nu_\beta' | e^{-i\mathbf{k}'\mathbf{r}} | \nu_\beta\rangle \right|^2. \qquad (6)$$

Using this expression we can write the collision integral in the following form:

$$I[f_\alpha(\nu_\alpha)] = \sum_{\beta\nu_\alpha'\nu_\beta'\nu_\beta} W(\nu_\alpha\nu_\beta,\ \nu_\alpha'\nu_\beta')\{f_\alpha(\nu_\alpha') f_\beta(\nu_\beta') - f_\alpha(\nu_\alpha) f_\beta(\nu_\beta)\}. \qquad (7)$$

We note here that the form of the collision integral

*Some of the results reported below have been published earlier in a brief communication.[2]

(7) is retained for distributions that are spatially nonuniform and time dependent, provided that the changes in the impact parameters and collision times characteristic of the distributions can be neglected.

3. To write the collision integral in the presence of a magnetic field **B** we must obtain explicit expressions for the quantities given by (2), (6), and (7). Evidently it is sufficient to determine the form of the matrix element

$$\langle \nu'_\alpha | e^{-i\mathbf{kr}} | \nu_\alpha \rangle. \tag{8}$$

We use the Landau representation in what follows. Thus, if **B** is assumed to be along the z axis the energy eigenvalue $E(\nu_\alpha)$ and the eigenfunction $|\nu_\alpha\rangle$ are given by

$$E(\nu_\alpha) \equiv E(n_\alpha, p_z^\alpha) = \hbar\Omega_\alpha (n_\alpha + 1/2) + (p_z^\alpha)^2/2\mu_\alpha,$$

$$|\nu_\alpha\rangle \equiv |n_\alpha p_z^\alpha p_x^\alpha\rangle$$

$$= (2\pi\hbar)^{-1} \exp\{ip_x^\alpha x/\hbar + ip_z^\alpha z/\hbar\}\, \Phi_{n_\alpha}[(y - y_0^\alpha)/\lambda_\alpha], \tag{9}$$

where $\Omega_\alpha = |e_\alpha| B/c\mu_\alpha$, $\lambda_\alpha^2 = \hbar/\mu_\alpha\Omega_\alpha$, $y_0^\alpha = -cp_x^\alpha/e_\alpha B$ is the projection of the center of the Larmor orbit on the y axis and Φ_{n_α} is the normalized wave function for the one-dimensional oscillator.

Using (9) we obtain the following expression for the matrix element (8):

$$\langle \nu'_\alpha | e^{-i\mathbf{rk}} | \nu_\alpha \rangle = \delta\left(k_z - \frac{p_z^\alpha - p_z'^\alpha}{\hbar}\right) \delta\left(k_x - \frac{p_x^\alpha - p_x'^\alpha}{\hbar}\right)$$

$$\times \exp\left\{-\frac{i}{2} k_y (y_0^\alpha + y_0'^\alpha)\right\} \frac{\bar{n}_\alpha!}{\sqrt{n'_\alpha!\, n_\alpha!}} L_{\bar{n}_\alpha}^{|n'_\alpha - n_\alpha|}(|X_\alpha|^2)$$

$$\times X_\alpha^{|n'_\alpha - n_\alpha|} \exp(-|X_\alpha|^2/2), \tag{10}$$

where $\bar{n}_\alpha = \min(n_\alpha, n'_\alpha)$

$$X_\alpha = \sqrt{\frac{c\hbar}{2|e_\alpha|B}} \left[\frac{|e_\alpha| B}{c\hbar} (y_0^\alpha - y_0'^\alpha) \operatorname{sign}(n'_\alpha - n_\alpha) - ik_y\right],$$

and L_s^r is the Laguerre polynomial:

$$L_s^r(x) = \sum_{t=0}^{s} \binom{n+r}{n-t} \frac{(-x)^t}{t!}.$$

We can now write an explicit expression for the transition probability and the collision integral, which assumes the form

$$I[f_\alpha(n_\alpha, p_z^\alpha, y_0^\alpha)] = \sum_{\beta n'_\alpha n_\beta n'_\beta} (2\pi\hbar)^{-3} \int dp_z'^\alpha dp_z'^\beta dp_z^\beta \hbar\delta$$

$$\times (p_z^\alpha + p_z^\beta - p_z'^\alpha - p_z'^\beta)\ (2\pi\hbar)^{-3} \int dp_x'^\alpha dp_x'^\beta dp_x^\beta \hbar\delta$$

$$\times (p_x^\alpha + p_x^\beta - p_x'^\alpha - p_x'^\beta)\, \delta[E_\alpha(\nu'_\alpha) + E_\beta(\nu'_\beta) - E_\alpha(\nu_\alpha)$$

$$- E_\beta(\nu_\beta)] \frac{2\pi}{\hbar} \left| \int dk_y dk'_y 4\pi e_\alpha e_\beta A_0^{-1} \right.$$

$$\times \left(\frac{E_\alpha(\nu'_\alpha) - E_\alpha(\nu_\alpha)}{\hbar}, \frac{p_x'^\alpha - p_x^\alpha}{\hbar}, \frac{p_z'^\alpha - p_z^\alpha}{\hbar}; k_y, k'_y\right)$$

$$\times \exp\left\{+\frac{i}{2} k_y (y_0^\alpha + y_0^{\alpha\prime}) - \frac{i}{2} k'_y (y_0^\beta + y_0'^\beta)\right.$$

$$\left. - \frac{1}{2}|X_\alpha|^2 - \frac{1}{2}|X'_\beta|^2\right\}$$

$$\times \frac{\bar{n}_\alpha!}{\sqrt{n'_\alpha!\, n_\alpha!}} L_{\bar{n}_\alpha}^{|n'_\alpha - n_\alpha|}(|X_\alpha|^2) \frac{\bar{n}_\beta!}{\sqrt{n'_\beta!\, n_\beta!}} L_{\bar{n}_\beta}^{|n'_\beta - n_\beta|}(|X'_\beta|^2)$$

$$\left. \times X_\alpha^{|n'_\alpha - n_\alpha|} X_\beta'^{|n'_\beta - n_\beta|} \right|^2 \{f_\alpha(n'_\alpha, p_z'^\alpha, y_0'^\alpha) f_\beta(n'_\beta, p_z'^\beta, y_0'^\beta)$$

$$- f_\alpha(n_\alpha, p_z^\alpha, y_0^\alpha) f_\beta(n_\beta, p_z^\beta, y_0^\beta)\}. \tag{11}$$

Here

$$X'_\beta = \sqrt{\frac{c\hbar}{2|e'_\beta|B}} \left[\frac{|e_\beta| B}{c\hbar} (y_0^\beta - y_0'^\beta) \operatorname{sign}(n'_\beta - n_\beta) - ik'_y\right].$$

In writing (11) we take account of the fact that if the density matrix is diagonal in the Landau representation it is possible to have a spatially nonuniform distribution which depends on the y coordinate only. In this case

$$\varepsilon_{ij}(\omega, \mathbf{k}, \mathbf{k}') = \delta(k_x - k'_x)\, \delta(k_z - k'_z)\, \varepsilon_{ij}(\omega, k_x, k_z; k_y, k'_y), \tag{12}$$

so that

$$A^{-1}(\omega, \mathbf{k}, \mathbf{k}') = \delta(k_x - k'_x)\, \delta(k_z - k'_z)\, A_0^{-1}(\omega, k_x, k_z; k_y, k'_y). \tag{13}$$

A further simplification is possible when the particle distribution is spatially uniform. This case occurs when the density matrix is diagonal in the Landau representation and is independent of y_0. Then

$$I[f_\alpha(n_\alpha, p_z^\alpha)]$$

$$= \sum_{\beta n'_\alpha n'_\beta n_\beta} (2\pi\hbar)^{-3} \int dp_z'^\alpha dp_z'^\beta dp_z^\beta \hbar\delta\, (p_z^\alpha + p_z^\beta - p_z'^\alpha - p_z'^\beta)$$

$$\times \delta[E_\alpha(n_\alpha, p_z^\alpha) + E_\beta(n_\beta, p_z^\beta)$$

$$- E_\alpha(n'_\alpha, p_z'^\alpha) - E_\beta(n'_\beta, p_z'^\beta)]$$

$$\times |e_\beta| Bc^{-1} (2\pi)^{-2} \int dk_x dk_y \frac{2\pi}{\hbar} \left| \varepsilon \right.$$

$$\left. \times \left(\frac{E_\alpha(n'_\alpha, p_z'^\alpha) - E_\alpha(n_\alpha, p_z^\alpha)}{\hbar}, k_\perp, \frac{p_z'^\alpha - p_z^\alpha}{\hbar}\right)\right|^{-2}$$

$$\times \left[\frac{4\pi e_\alpha e_\beta}{k_\perp^2 + [(p_z^\alpha - p_z'^\alpha)/\hbar]^2}\right]^2$$

$$\times \left[\frac{c\hbar k_\perp^2}{2|e_\alpha|B}\right]^{|n'_\alpha - n_\alpha|} \left[\frac{c\hbar k_\perp^2}{2|e_\beta|B}\right]^{|n'_\beta - n_\beta|}$$

$$\times \exp\left\{-\frac{c\hbar k_\perp^2}{2B}\left[\frac{1}{|e_\alpha|} + \frac{1}{|e_\beta|}\right]\right\} \frac{(\bar{n}_\alpha!\, \bar{n}_\beta!)^2}{n'_\alpha!\, n_\alpha!\, n'_\beta!\, n_\beta!}$$

$$\times \left[L_{\bar{n}_\alpha}^{|n'_\alpha - n_\alpha|}\left(\frac{c\hbar k_\perp^2}{2|e_\alpha|B}\right)\right]^2 \left[L_{\bar{n}_\beta}^{|n'_\beta - n_\beta|}\left(\frac{c\hbar k_\perp^2}{2|e_\beta|B}\right)\right]^2$$

$$\times \{f_\alpha(n'_\alpha, p_z'^\alpha) f_\beta(n'_\beta, p_z'^\beta) - f_\alpha(n_\alpha, p_z^\alpha) f_\beta(n_\beta, p_z^\beta)\}. \tag{14}$$

Here, $k_\perp^2 = k_x^2 + k_y^2$ and ε is defined by the relation

$$k^2 \varepsilon(\omega, k_\perp, k_z) = k_i k_j \varepsilon_{ij}(\omega, \mathbf{k}). \tag{15}$$

The fact that ε depends on $k_\perp$ follows because E_α depends only on n and p_z, corresponding to the isotropic distribution in the plane perpendicular to the fixed magnetic field.

The collision integral we seek is given by (11) and (14). The particular importance of this collision integral is the fact that it does not diverge at small values of the transferred momentum. This result is obtained because we have taken account of the polarization of the medium and this leads to a shielding of the Coulomb interaction.

4. In kinetic plasma theory wide use is made of the kinetic equations written in the form given by Landau and the corresponding Fokker-Planck equation.[7] Equations of this type can be easily obtained from those written above by taking the limit $\hbar = 0$. In this case the asymptotic expressions for the Laguerre polynomials[8] are used:

$$e^{-x/2} x^{m/2} L_n^m(x) = \frac{(n+m)!\, J_m\,[\sqrt{(4n+2m+2)x}]}{n!\,[(2n+m+1)/2]^{m/2}} + O(n^{n/2-3/4}), \tag{16}$$

where J_m is the Bessel function. Using (11) we obtain a collision integral for distribution functions depending on the longitudinal and transverse momenta and the y component of the radii of the Larmor orbits:

$$\begin{aligned} I\,[f_\alpha(p_z^\alpha, p_\perp^\alpha, y_0^\alpha)] &= \sum_{\beta,\, m_\alpha,\, m_\beta} (2\pi\hbar)^{-3} \int dp_\perp^\beta p_\perp^\beta \int dp_z^\beta \int (2\pi)^{-3} dk_x dk_z dy_0^\beta \\ &\times \left(k_z \frac{\partial}{\partial p_z^\alpha} - \frac{ck_x}{e_\alpha B}\frac{\partial}{\partial y_0^\alpha} + \frac{m_\alpha \Omega_\alpha}{v_\alpha^\perp}\frac{\partial}{\partial p_\alpha^\perp}\right) \\ &\times \pi\delta(k_z v_z^\alpha + m_\alpha\Omega_\alpha - k_z v_z^\beta - m_\beta\Omega_\beta) \left| \int dk_y dk_y' \, 4\pi e_\alpha e_\beta A_0^{-1} \right. \\ &\times (m_\alpha\Omega_\alpha + k_z v_z^\alpha,\ + k_x,\ + k_z;\ k_y,\ k_y') \\ &\times \exp\{-ik_y' y_0^\beta + ik_y y_0^\alpha\}\, J_{|m_\alpha|}\left(\frac{v_\alpha^\perp k_\perp}{\Omega_\alpha}\right) J_{|m_\beta|}\left(\frac{v_\beta^\perp k_\perp'}{\Omega_\beta}\right) \\ &\times \left[\frac{k_x}{k_\perp}\operatorname{sign}(e_\alpha m_\alpha) + \frac{ik_y}{k_\perp}\right]^{|m_\alpha|} \\ &\times \left. \left[\frac{k_x}{k_\perp'}\operatorname{sign}(e_\beta m_\beta) - \frac{ik_y'}{k_\perp'}\right]^{|m_\beta|} \right|^2 \\ &\times \left\{k_z \frac{\partial}{\partial p_z^\alpha} - \frac{ck_x}{e_\alpha B}\frac{\partial}{\partial y_0^\alpha} + \frac{m_\alpha\Omega_\alpha}{v_\alpha^\perp}\frac{\partial}{\partial p_\perp^\alpha} - k_z\frac{\partial}{\partial p_z^\beta} \right. \\ &\left. + \frac{ck_x}{e_\beta B}\frac{\partial}{\partial y_0^\beta} - \frac{m_\beta\Omega_\beta}{v_\beta^\perp}\frac{\partial}{\partial p_\perp^\beta}\right\} f_\alpha(p_z^\alpha, p_\perp^\alpha, y_0^\alpha)\, f_\beta(p_z^\beta, p_\perp^\beta, y_0^\beta). \end{aligned} \tag{17}$$

Here, $k_\perp^2 = k_x^2 + k_y^2$ and $k_\perp'^2 = k_x^2 + k_y'^2$.

An important feature of the collision integral (17) is the fact that it is an integral operator with respect to the dependence of f_β on y_0^β. In the particular case of a spatially uniform particle distribution, using (17) or (14) [in the latter case we must take account of (16)] we have

$$\begin{aligned} I\,[f_\alpha(p_z^\alpha, p_\perp^\alpha)] &= \sum_{\beta,\, m_\beta,\, m_\alpha} (2\pi\hbar)^{-3} \int 2\pi dp_\perp^\beta p_\perp^\beta \int dp_z^\beta \int \frac{dk}{(2\pi)^3} \\ &\times \left(k_z\frac{\partial}{\partial p_z^\alpha} + \frac{m_\alpha\Omega_\alpha}{v_\alpha^\perp}\frac{\partial}{\partial p_\alpha^\perp}\right) \pi\delta(k_z v_z^\alpha + m_\alpha\Omega_\alpha - k_z v_z^\beta - m_\beta\Omega_\beta) \\ &\times \frac{(4\pi e_\alpha e_\beta)^2}{k^4\,|\varepsilon(m_\alpha\Omega_\alpha + k_z v_z^\alpha,\ k_\perp,\ k_z)|^2} J_{m_\alpha}^2\left(\frac{v_\alpha^\perp k_\perp}{\Omega_\alpha}\right) J_{m_\beta}^2\left(\frac{v_\beta^\perp k_\perp}{\Omega_\beta}\right) \\ &\times \left\{k_z\frac{\partial}{\partial p_z^\alpha} + \frac{m_\alpha\Omega_\alpha}{v_\alpha^\perp}\frac{\partial}{\partial p_\alpha^\perp} - k_z\frac{\partial}{\partial p_z^\beta} - \frac{m_\beta\Omega_\beta}{v_\beta^\perp}\frac{\partial}{\partial p_\perp^\beta}\right\} f_\alpha f_\beta. \end{aligned} \tag{18}$$

In the case of electron-electron collisions (18) is the same expression as that obtained by Rostoker[5] except that it is written in somewhat different form.

5. In order to indicate the value of the results given above we consider applications to a case which could not be studied before either by means of a theory of two-body collisions, neglecting the effect of waves in the plasma, or by means of the diagram temperature technique, which is suitable only for weak deviations from total thermodynamic equilibrium. As an example we compute the coefficient in front of the second derivative in y_0^α in (17); this quantity represents the diffusion coefficient for the guiding centers of the Larmor orbits.

We have

$$f_\alpha = [n_\alpha + \delta n_\alpha(y_0^\alpha)]\frac{\delta(p_\perp^\alpha)}{2\pi p_\perp^\alpha}\varphi_\alpha(p_z^\alpha - \mu_\alpha v_0), \quad n_\alpha = \text{const},$$

$$f_\beta = n_\beta \frac{\delta(p_\perp^\beta)}{2\pi p_\perp^\beta} F_\beta(p_z^\beta), \quad n_\beta = \text{const}.$$

We assume that the density δn_α is small and that its effect on the dielectric properties of the plasma can be neglected. Furthermore, we assume F_β to be constant for $|p_z^\beta| < \mu_\beta v_\beta < \mu_\beta v_0$ and zero for large values of the argument while φ_α to be nonvanishing only when $|p_z^\alpha - \mu_\alpha v_0| < \mu_\alpha \Delta v$, in which case

$$\varphi_\alpha = \frac{\mu_\alpha(v_0 + \Delta v) - p_z^\alpha}{\mu_\alpha^2(\Delta v)^2}, \qquad \mu_\alpha v_0 < p_z^\alpha < \mu_\alpha(v_0 + \Delta v),$$

$$\varphi_\alpha = \frac{p_z^\alpha - \mu_\alpha(v_0 - \Delta v)}{\mu_\alpha^2(\Delta v)^2}, \qquad \mu_\alpha v_0 > p_z^\alpha > \mu_\alpha(v_0 - \Delta v).$$

After some elementary calculations we obtain the coefficient in question in the form

$$D = \pi \sum_\beta \frac{e_\beta^2 c^2}{B^2}\frac{n_\beta}{v_\beta}\ln\frac{v_z^\alpha + v_\beta}{v_z^\alpha - v_\beta}\int_0^{k_{max}} \frac{dk_\perp}{k_\perp}|\varepsilon(0, k_\perp)|^{-2},$$

where, in the usual way k_{max} is determined by the limits of applicability of perturbation theory or the classical expansion, while

$$\varepsilon(0, k_\perp) = 1 - \frac{1}{k_\perp^2}\sum_\beta \frac{\omega_{L\beta}^2}{(v_z^\alpha)^2 - v_\beta^2} + \left(\frac{\omega_{L\alpha}}{k_\perp \Delta v}\right)^2 \ln\left[1 - \left(\frac{\Delta v}{v_z^\alpha - v_0}\right)^2\right].$$

We average D over the distribution φ_α. Since $\Delta v \ll v_0$, we can write

$$\langle D\rangle = \pi \sum_\beta \frac{e_\beta^2 c^2}{B^2} \frac{n_\beta}{v_\beta} \ln \frac{v_0 + v_\beta}{v_0 - v_\beta} \int_0^{k_{max}} \frac{dk_\perp}{k_\perp} \langle |\varepsilon(0, k_\perp)|^{-2}\rangle,$$

where

$$\langle |\varepsilon(0, k_\perp)|^{-2}\rangle = 2k_\perp^4 \int_0^1 \frac{dx\,(1-x)}{|k_\perp^2 - a^2 + b^2(i\pi + \ln[x^{-2} - 1])|^2},$$

$$a^2 = \sum_\beta \frac{\omega_{L\beta}^2}{v_0^2 - v_\beta^2},$$

$$b^2 = \left(\frac{\omega_{L\alpha}}{\Delta v}\right)^2.$$

In the limit of low beam density, in which case $a^2 \gg b^2$, we have

$$\langle D\rangle = \pi \sum_\beta \frac{e_\beta^2 c^2}{B^2} \frac{n_\beta}{v_\beta} \ln \frac{v_0 + v_\beta}{v_0 - v_\beta} \left\{\ln \frac{k_{max}}{a} + \left(\frac{a}{b}\right)^2\right\}.$$

Whence it follows that for the particular case $v_0 \gg v_\beta$,

$$\langle D\rangle = 2\pi \sum_\beta \frac{e_\beta^2 c^2}{B^2 v_0} \left\{\ln \frac{v_0 k_{max}}{\left(\sum_\beta \omega_{L\beta}^2\right)^{1/2}} + \sum_\beta \left(\frac{\omega_{L\beta}}{\omega_{L\alpha}} \frac{\Delta v}{v_0}\right)^2\right\}.$$

This expression differs in two ways from that which is obtained when plasma waves are neglected: first, by the more exact logarithmic term (when the logarithmic term is important the difference can be appreciable at large values of v_0); second, and this is more important, by the appearance of the additional term (the nonlogarithmic term), which becomes important at low beam densities. At low densities there are oblique plasma waves, characterized by the spectrum

$$\omega^2 = (k_z/k)^2 \sum_\beta \omega_{L\beta}^2 \gg k_z^2 v_\beta^2$$

which have a rather small damping factor $\gamma = \pi\omega_{L\alpha}^2 \omega / 2(k\Delta v)^2$ (or growth rate γ). Under these conditions the magnitude of the diffusion coefficient is determined by the second term in the curly brackets in the last expression for $\langle D\rangle$ and may be considerably greater than the value obtained from the theory of two-body collisions. In this case our results hold as long as the amplitudes of the oblique waves do not become appreciably greater than the amplitude of the stationary noise wave with positive γ given by

$$(E^2)_{k_z v_0 + 0, k} \approx \frac{\omega_{L\alpha}^2}{|k_z|\Delta v} \frac{4\pi k^2}{[k^2 - \sum_\beta \omega_{L\beta}^2 / v_0^2]^2 + \pi^2 [\omega_{L\alpha}/\Delta v]^2}.$$

APPENDIX A

In this appendix we obtain the formulas for $k_i \varepsilon_{ij}(\omega, \mathbf{k}, \mathbf{k}') k_j'$. We consider the charge density of the plasma particles due to a scalar potential φ which is a periodic function of time and coordinates $\exp(-i\omega t + i\mathbf{k}\cdot\mathbf{r})$. In the linear approximation in φ we have*

$$\sum_{\alpha\nu_\alpha} \langle \nu_\alpha | e_\alpha \delta(\mathbf{r} - \mathbf{r}_\alpha) | \nu_\alpha\rangle f_\alpha(\nu_\alpha) = -\sum_\alpha e_\alpha^2 \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\mathbf{r} - i\omega t}\varphi \times \sum_{\nu_\alpha \nu_\alpha'} \langle \nu_\alpha | e^{-i\mathbf{k}\mathbf{r}_\alpha} | \nu_\alpha'\rangle \langle \nu_\alpha' | e^{i\mathbf{k}'\mathbf{r}_\alpha} | \nu_\alpha\rangle \times \frac{f_\alpha(\nu_\alpha') - f_\alpha(\nu_\alpha)}{E(\nu_\alpha') - E(\nu_\alpha) - \hbar\omega - i\hbar\Delta}. \quad \text{(A.1)}$$

It follows immediately that

$$k_i \varepsilon_{ij}(\omega, \mathbf{k}, \mathbf{k}') k_j' = k^2 \delta(\mathbf{k} - \mathbf{k}') - \sum_{\alpha\nu_\alpha\nu_\alpha'} 4\pi e_\alpha^2 \langle \nu_\alpha | e^{-i\mathbf{k}\mathbf{r}_\alpha} | \nu_\alpha'\rangle \langle \nu_\alpha' | e^{i\mathbf{k}'\mathbf{r}_\alpha} | \nu_\alpha\rangle \times \frac{f_\alpha(\nu_\alpha') - f_\alpha(\nu_\alpha)}{E(\nu_\alpha') - E(\nu_\alpha) - \hbar\omega - i\hbar\Delta}. \quad \text{(A.2)}$$

It is convenient to use the Landau representation for the case of charged particles in a fixed magnetic field when the distribution function is inhomogeneous in the y direction. Then (A.2) assumes the form

$$k_i \varepsilon_{ij}(\omega, \mathbf{k}, \mathbf{k}') k_j' - k^2 \delta(\mathbf{k} - \mathbf{k}') = -\delta(k_x - k_x')\,\delta(k_z - k_z') \times \sum_{\alpha n_\alpha n_\alpha'} 4\pi e_\alpha^2 \int \frac{dp_z^\alpha dp_x^\alpha}{(2\pi\hbar)^2} \exp\left\{\frac{ic}{e_\alpha B}[k_y - k_y']\left(p_x^\alpha + \frac{1}{2}\hbar k_x\right)\right\}$$
$$\times \frac{(\bar{n}_\alpha !)^2}{n_\alpha! \, n_\alpha'!} \exp\left\{-\frac{1}{2}|X_\alpha|^2 - \frac{1}{2}|X_\alpha'|^2\right\} L_{\bar{n}_\alpha}^{|n_\alpha' - n_\alpha|} \times (|X_\alpha|^2) L_{\bar{n}_\alpha}^{|n_\alpha' - n_\alpha|}(|X_\alpha'|^2)\,(X_\alpha X_\alpha')^{|n_\alpha' - n_\alpha|}$$
$$\times \frac{f_\alpha(n_\alpha', p_z^\alpha + \hbar k_z, p_x^\alpha + \hbar k_x) - f_\alpha(n_\alpha, p_z^\alpha, p_x^\alpha)}{(n_\alpha' - n_\alpha)\hbar\Omega_\alpha + (\hbar k_z^2 / 2\mu_\alpha) + \hbar k_z v_z^\alpha - \hbar\omega - i\hbar\Delta},$$

where

$$X_\alpha = \sqrt{\frac{c\hbar}{2|e_\alpha|B}}\,[k_x \operatorname{sign}(e_\alpha\{n_\alpha - n_\alpha'\}) + ik_y], \quad \text{(A.3)}$$

$$X_\alpha' = \sqrt{\frac{c\hbar}{2|e_\alpha|B}}\,[k_x \operatorname{sign}(e_\alpha\{n_\alpha - n_\alpha'\}) - ik_y']. \quad \text{(A.4)}$$

For a spatially uniform distribution, in which case the density matrix $f_\alpha(\nu_\alpha)$ is independent of y_0^α, the right side of (A.3) becomes

*Here, Δ is an infinitesimally small positive correction corresponding to the adiabatic switching on of the interaction in the infinitely remote past.

$$\delta(\mathbf{k}-\mathbf{k}')\sum_{\alpha n_\alpha n'_\alpha} 4\pi e_\alpha^2 \frac{|e_\alpha|B}{c}\int \frac{dp_z^\alpha}{(2\pi\hbar)^2}\frac{(\bar{n}_\alpha!)^2}{n_\alpha!\,n'_\alpha!}\left[\frac{c\hbar k_\perp^2}{2|e_\alpha|B}\right]^{|n'_\alpha-n_\alpha|}$$

$$\times \exp\left\{-\frac{c\hbar k_\perp^2}{2|e_\alpha|B}\right\}\left[L_{\bar{n}_\alpha}^{|n'_\alpha-n_\alpha|}\left(\frac{c\hbar k_\perp^2}{2|e_\alpha|B}\right)\right]^2$$

$$\times \frac{f_\alpha(n'_\alpha, p_z^\alpha+\hbar k_z)-f_\alpha(n_\alpha, p_z^\alpha)}{(n'_\alpha-n_\alpha)\hbar\Omega_\alpha+(\hbar^2k_z^2/2\mu_\alpha)+\hbar k_z v_z^\alpha-\hbar\omega-i\hbar\Delta}. \qquad \text{(A.4)}$$

APPENDIX B

The quantum-mechanical and classical collision integrals can both be simplified considerably when the spatial inhomogeneities have a weak effect on the nonlocal polarization of the medium because the elements of the scattering matrix for α and β are given by (5) rather than (2). Using (5), (6), (7), and (10) we have

$$I[f_\alpha(n_\alpha, p_z^\alpha, y_0^\alpha)] = \sum_{\beta n'_\alpha n_\beta n'_\beta}(2\pi\hbar)^{-6}\int dp_z'^\alpha dp_z'^\beta dp_z^\beta dp_x'^\alpha dp_x'^\beta dp_x^\beta$$

$$\times \hbar\delta[p_z^\alpha+p_z^\beta-p_z'^\alpha-p_z'^\beta]\,\hbar\delta(p_x^\alpha+p_x^\beta-p_x'^\alpha-p_x'^\beta)$$

$$\times \delta[E_\alpha(n'_\alpha, p_z'^\alpha)+E_\beta(n'_\beta, p_z'^\beta)$$

$$-E_\alpha(n_\alpha, p_z^\alpha)-E_\beta(n_\beta, p_z^\beta)]$$

$$\times \frac{2\pi}{\hbar}\left|\int dk_y \frac{4\pi e_\alpha e_\beta\hbar^2}{(p_x'^\alpha-p_x^\alpha)^2+\hbar^2k_y^2+(p_z'^\alpha-p_z^\alpha)^2}\right.$$

$$\times \exp\{\tfrac{1}{2}ik_y(y_0^\alpha+y_0'^\alpha)-\tfrac{1}{2}ik_y(y_0^\beta+y_0'^\beta)$$

$$-\tfrac{1}{2}|X_\alpha|^2-\tfrac{1}{2}|X_\beta|^2\}$$

$$\times \varepsilon^{-1}[(E'_\alpha-E_\alpha)/\hbar,\ (p_x'^\alpha-p_x^\alpha)/\hbar,\ k_y,$$

$$\times (p_z'^\alpha-p_z^\alpha)/\hbar]\ \frac{\bar{n}_\alpha!\,\bar{n}_\beta!}{[n'_\alpha!\,n_\alpha!\,n'_\beta!\,n_\beta!]^{1/2}}$$

$$\times X_\alpha^{|n'_\alpha-n_\alpha|}X_\beta^{|n'_\beta-n_\beta|}L_{\bar{n}_\alpha}^{|n'_\alpha-n_\alpha|}(|X_\alpha|^2)\,L_{\bar{n}_\beta}^{|n'_\beta-n_\beta|}$$

$$\times (|X_\beta|^2)\Big|^2\ \{f_\alpha(n'_\alpha, p_z'^\alpha, y_0'^\alpha)f_\beta(n'_\beta, p_z'^\beta, y_0'^\beta)$$

$$-f_\alpha(n_\alpha, p_z^\alpha, y_0^\alpha)f_\beta(n_\beta, p_z^\beta, y_0^\beta)\}. \qquad \text{(B.1)}$$

When the distribution function is independent of y_0 we again obtain (14). The quantity $\varepsilon(\omega, \mathbf{k})$ in (B.1) is given by

$$\varepsilon(\omega, \mathbf{k}) = 1-\frac{4\pi}{k^2}\sum_{\beta n_\beta n'_\beta} e_\beta^2\int \frac{dp_x^\beta dp_z^\beta}{(2\pi\hbar)^2}\frac{(n_\beta!)^2}{n'_\beta!\,n_\beta!}$$

$$\times \left[\frac{c\hbar k_\perp^2}{2|e_\beta|B}\right]^{|n'_\beta-n_\beta|}\left[L_{\bar{n}_\beta}^{|n'_\beta-n_\beta|}\left(\frac{c\hbar k_\perp^2}{2|e_\beta|B}\right)\right]^2\exp\left\{-\frac{c\hbar k_\perp^2}{2|e_\beta|B}\right\}$$

$$\times \frac{f_\beta(n'_\beta, p_z^\beta+\hbar k_z, p_x^\beta+\hbar k_x)-f_\beta(n_\beta, p_z^\beta, p_x^\beta)}{(n'_\beta-n_\beta)\hbar\Omega_\beta+\hbar^2k_z^2/2\mu_\beta+\hbar k_z v_z^\beta-\hbar\omega-i\hbar\Delta}. \qquad \text{(B.2)}$$

If the distribution is independent of p_x^β (B.2) becomes (A.4). In the classical limit the collision integral (B.1) becomes (17) if we write

$$A_0^{-1}(m\Omega_\alpha+k_z v_z^\alpha,\ k_x,\ k_z;\ k_y,\ k'_y) = \delta(k_y-k'_y)$$

$$\times \{\varepsilon_{\rm cl}(m_\alpha\Omega_\alpha+k_z v_z^\alpha,\ k_x,\ k_y,\ k_z)[k_x^2+k_y^2+k_z^2]\}^{-1},$$

in the latter, where $\varepsilon_{\rm cl}(\omega, \mathbf{k})$ is the classical limit of (B.2).

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