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COVARIANT EXPANSION OF THE ELECTROMAGNETIC FIELD

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We present a covariant form of the expansion of the photon field in multipoles and a covariant expansion of the current in multipole moments. We show the connection between the expansions and the representations of the so-called little group. We indicate a generalization of the Lorentz representations which leads to a covariant definition of the Stokes parameters.

1. INTRODUCTION

In nonrelativistic quantum mechanics, extensive use is made of expansions of amplitudes for processes in spherical functions—irreducible representations of the three-dimensional rotational group. But the generalization of this expansion to the relativistic case, i.e., a truly covariant expansion, is not usually considered. To find such a form, one must use the well-known irreducible representations of the Lorentz group. In various papers, for example, those of Yu. Shirokov,[1] Chou Kuang-chao and M. I. Shirokov,[2] and Dolginov,[3] the apparatus of the irreducible representations of the Lorentz group is applied to the covariant description of relativistic systems. The methods developed in these papers are used successfully by their authors to obtain general formulas. But these formulas are either not a direct generalization of the nonrelativistic formulas, or use a description which is not fully covariant.

It seems useful to us to give another type of covariant expansion, which directly generalizes the nonrelativistic formulas. In the present paper, we restrict ourselves to the covariant expansion of the free electromagnetic field. We consider two problems: a) expansion of the radiation in electric and magnetic multipoles; b) expansion of the field sources (currents) in multipole moments. In addition we give a brief treatment of the properties of the covariant polarization of photons (which has been treated somewhat differently in papers of Michel and Rouhaninejad[4,5]). Analogous expansions may be useful for describing β decay and other processes.

In deriving the formulas, we actually make use of the apparatus of the irreducible representations of the Lorentz group, whereas in most papers the authors use the infinitesimal rotations. In our method the concept of the little group (cf. [6-8,4]) is very important. In the book of Gel'fand et al[9] this group is called the stationary subgroup.

The little group L_q associated with the vector q (which we shall assume is always either timelike or lies on the light cone),* is that subgroup of improper Lorentz transformations which leaves the four-vector q fixed (in magnitude and direction). For example, any spacelike four-vector orthogonal to q transforms according to the representation L_q. An obvious example of such a vector is the spin of the particle (cf. [5,7]). Vectors (and tensors) orthogonal to a given timelike vector will also be used in the sequel.

From these remarks it is clear that if q is timelike, the little group L_q is isomorphic to the spherical symmetry group.

If the particle mass is zero (for example, a photon), then it is usually stated that one must in this case change the definition of the photon spin, because there is no rest frame. Actually the situation is rather the reverse: because of gauge invariance, the photon wave function can be made orthogonal to any timelike vector (cf. [11]), and therefore in this case all reference systems are simply equivalent. The specific choice of reference system is then dictated by convenience of computation. For example, in the multipole expansion one should choose as the vector determining the reference frame the 4-momentum of the center of inertia of the photon and the radiator, since uniform motion of the system has no effect on the properties of the radiation.

We note that if the momentum k lies on the light cone (zero rest mass), the little group L_k becomes isomorphic to the space group of the diatomic molecule, and consists of the two-dimensional rotations in the plane perpendicular to **k**

*We have no need here for the case of a spacelike vector which was treated in the paper of Yu. Shirokov[10] from the point of view of the representations of the inhomogeneous group. This case may be important for the treatment of more complicated diagrams (for example, for the Compton effect).

and the reflections in planes passing through **k**. The result is that the polarization of a particle with zero rest mass and arbitrary finite spin can be characterized by means of a scalar (the intensity) a pseudoscalar, and a two-dimensional vector (cf. the paper of Michel[4]). The total angular momentum of a particle with zero rest mass is analogous (in the sense of its transformation law) to the angular momentum of a diatomic molecule, while the spin of the particle corresponds to the quantum number Λ, the value of the projection of its angular momentum on the axis of the molecule.

2. COVARIANT STOKES PARAMETERS

The complex polarization four-vector for the photon satisfies the normalization condition

$$ee^* = \mathbf{ee}^* - e_0 e_0 = 1 \qquad (1)$$

and the four-dimensional transversality condition

$$ek = 0. \qquad (2)$$

Gauge invariance allows us to impose on the vector e the condition

$$eP = 0, \qquad (3)$$

where P is a four-momentum. Then, in the coordinate system in which $\mathbf{P} = 0$, relations (2) and (3) lead to the three-dimensional transversality condition

$$\mathbf{ek} = 0. \qquad (4)$$

The four-vector e, satisfying (1) and (2), can be given in terms of two scalars $e\chi^{(1)}$ and $e\chi^{(2)}$ by the formula

$$e = (e\chi^{(1)})\chi^{(1)} + (e\chi^{(2)})\chi^{(2)}, \qquad (5)$$

where $\chi^{(1)}$ and $\chi^{(2)}$ are two unit four-vectors satisfying relations (3) and (2). These vectors can be constructed from the momenta of other particles participating in the process.

We require that the three-dimensional transversality condition (4) be satisfied in any coordinate system. By the vector P we shall denote a momentum which determines the coordinate system (for example, the total momentum of the system). Clearly such a generalization of condition (3) can be meaningful only as a consequence of gauge invariance. Gauge invariance allows us to change the form of the Lorentz transformation for the photon. We shall assume that, in transforming to another system of reference, the polarization vector of the electromagnetic field is transformed by a gauge transformation and the usual Lorentz transformation, so that condition (4) is maintained in the new system. Thus the vector e transforms according to the formula

$$e' = G(P')L(P')e, \qquad (6)$$

where $L(P')$ is the matrix of the Lorentz transformation for a four-vector, while

$$G(P')e = e - k(eP')/kP'. \qquad (7)$$

We note that the gauge transformation (7) changes neither the four-dimensional transversality condition nor the normalization. From now on we shall refer to (6) as the Lorentz transformation for the four-vector of the photon potential.*

A partially polarized beam is described by the two-row density matrix of the photon

$$\rho^{(ik)} = \overline{(e\chi^{(i)})(e^*\chi^{(k)})} \quad (i, k = 1, 2) \qquad (8)$$

or by the Stokes parameters

$$\varepsilon_1 = \mathrm{Sp}\,\sigma_1\rho, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$
$$\varepsilon_2 = \mathrm{Sp}\,\sigma_2\rho, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};$$
$$\varepsilon_3 = \mathrm{Sp}\,\sigma_3\rho, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (9)$$

In going to a new set of basis vectors $\chi^{(1)\prime}$ and $\chi^{(2)\prime}$ $(\chi^{(1)\prime}\chi^{(2)\prime} = 0)$, which corresponds to a Lorentz transformation, the density matrix is transformed according to the formula

$$\rho' = U\rho U^+; \qquad (10)$$

$$U = \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}; \qquad (11)$$

$$\alpha = \chi^{(1)\prime}\chi^{(1)} = -\chi^{(2)\prime}\chi^{(2)}, \qquad \beta = \chi^{(1)\prime}\chi^{(2)} = \chi^{(2)\prime}\chi^{(1)}. \qquad (12)$$

The matrix U establishes a homomorphic mapping of the proper Lorentz group on the group of two-dimensional rotations, in which every pure Lorentz transformation along the momentum **k** is put in correspondence with the unit matrix.

The representation (11) can be regarded as a representation of the proper little group L_k'. The invariants of the transformation (11) are the degree of circular polarization ϵ_2 and the degree of linear polarization $r = \sqrt{\epsilon_1^2 + \epsilon_3^2}$. A representation of the proper group L_k' of the form (11) is reducible and decomposes into one-dimensional representations. In this case the basis vectors of the one-dimensional representations are the states with complete right and left circular polarization.

With respect to the improper Lorentz group, the four-vector e transforms according to a two-dimensional irreducible representation of the im-

*A similar generalization of the Lorentz transformation was already essentially contained in the old paper of Heisenberg and Pauli.[12]

proper little group L_k, which is isomorphic to the space group of the diatomic molecule and where ϵ_2 behaves like a pseudoscalar.

For completeness, we mention that the Stokes parameters are simply related to the spin polarization and quadrupolarization of the photon. For a particle with zero rest mass,[7] the spin is determined by the four-vector $\Gamma = \epsilon_2 k$, where ϵ_2 is a pseudoscalar. In the case of the photon, ϵ_2 is the degree of circular polarization.

We define the three-dimensional spin vector in any coordinate system as follows:

$$\mathbf{S} = \mathbf{n}\Gamma_0/k_0 = \varepsilon_2 \mathbf{n} = i\,[\mathbf{e}\mathbf{e}^*], \quad \mathbf{n} = \mathbf{k}/k_0. \qquad (13)^*$$

For the quadrupolarization tensor (with x, y, z axes along $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{n}$, respectively), we have

$$\tfrac{1}{2}\ \langle S_\alpha S_\beta + S_\beta S_\alpha \rangle = \begin{pmatrix} {}^1/_2\,(1-\varepsilon_3) & {}^1/_2\varepsilon_1 & 0 \\ {}^1/_2\varepsilon_1 & {}^1/_2\,(1+\varepsilon_3) & 0 \\ 0 & 0 & 1 \end{pmatrix} (\alpha, \beta = 1, 2, 3). \qquad (14)$$

Quadrupolarization of the photon is associated only with linear polarization.

3. COVARIANT EXPANSION OF THE PHOTON IN ELECTRIC AND MAGNETIC MULTIPOLES

In three-dimensional notation, the electric and magnetic multipoles are expressed in terms of the spherical functions $Y_{LM}(n)$ in the following way ($\mathbf{n} = \mathbf{k}/|\mathbf{k}|$):[13]

$$\mathbf{E}_{LM}(\mathbf{k}) = L^{-1/2}(L+1)^{-1/2}|\mathbf{k}|\,\nabla_{\mathbf{k}} Y_{LM}(\mathbf{n}),$$
$$\mathbf{M}_{LM}(\mathbf{k}) = L^{-1/2}(L+1)^{-1/2}\,i\,[\mathbf{k}\nabla_k]\,Y_{LM}(\mathbf{n}). \qquad (15)$$

They satisfy the relations

$$\mathbf{k}\mathbf{E}_{LM}(\mathbf{k}) = 0, \qquad \mathbf{k}\mathbf{M}_{LM}(\mathbf{k}) = 0, \qquad \nabla_k \mathbf{M}_{LM}(\mathbf{k}) = 0,$$
$$\mathbf{E}_{LM}(\mathbf{k})\,\mathbf{M}_{LM}(\mathbf{k}) = 0 \qquad (16)$$

and the orthogonality conditions

$$\int \mathbf{E}_{LM}(\mathbf{k})\,\mathbf{E}^*_{L'M'}(\mathbf{k})\,d\Omega_{\mathbf{k}} = \delta_{LL'}\delta_{MM'},$$
$$\int \mathbf{M}_{LM}(\mathbf{k})\,\mathbf{M}^*_{L'M'}(\mathbf{k})\,d\Omega_{\mathbf{k}} = \delta_{LL'}\,\delta_{MM'},$$
$$\int \mathbf{E}_{LM}(\mathbf{k})\,\mathbf{M}^*_{L'M'}(\mathbf{k})\,d\Omega_{\mathbf{k}} = 0. \qquad (16a)$$

Let us first consider the relativistic functions which go over into the normalized spherical functions in the system in which $\mathbf{P} = 0$ (where P is the total four-momentum of the system photon + radiator). To find the explicit form of such relativistic spherical functions, we introduce the four-vector n which satisfies the conditions

$$n^2 = 1, \qquad nP = 0. \qquad (17)$$

[e e] = e × e*.

It can be constructed uniquely from the photon momentum k ($k^2 = 0$) and P:

$$n = -(\sqrt{-P^2}/kP)\,(k - (Pk)P/P^2). \qquad (18)$$

Here $nk = -(kP)/\sqrt{-P^2}$, and in the system of the center of inertia

$$n = \{\mathbf{k}/k_0,\ 0\}.$$

Using the four-vector n, we construct a symmetric tensor of rank L, $T^{(L)}_{ik\ldots lm}$, which satisfies the conditions (we have $T_i P_i = \mathbf{TP} - T_0 P_0$),

$$T_{ik\ldots lm}P_i = 0, \qquad T_{ii\ldots lm} = 0. \qquad (19)$$

Under Lorentz transformations, this tensor obviously transforms according to a representation of the little group L_P, and has the form

$$T^{(L)}_{ik\ldots lm} = \frac{(2L-1)!!}{\sqrt{(2L)!}}\sqrt{\frac{2L+1}{4\pi}} \times \Big\{ n_i n_k \ldots n_l n_m - \frac{1}{2L-1}\Big(\delta_{ik} - \frac{P_i P_k}{P^2}\Big)\cdots n_l n_m - \cdots - \frac{1}{2L-1} n_i n_k \ldots \Big(\delta_{lm} - \frac{P_l P_m}{P^2}\Big) + \cdots \Big\}.$$

From the tensor $T^{(L)}_{ik\ldots lm}$, we go over to a relativistic spinor having L undotted and L dotted indices:[9]

$$Z_{s_1 s_2 \ldots s_L \dot r_1 \dot r_2 \ldots \dot r_L} = \Big(\prod_{i=1}^{L} \sigma_{\mu_i;\, s_i,\, \dot r_i}\Big) T^{(L)}_{\mu_1\mu_2\ldots\mu_L}, \qquad (20)$$

where $\sigma_{\mu_i;\, s_i \dot r_i} = (\sigma_{\mu_i})_{s_i \dot r_i}$ are the Pauli matrices for $\mu_i = 1, 2, 3$; $(-\sigma_0)$ is the unit matrix.

With respect to transformations of the rotation group, spinors with lower undotted indices and those with upper dotted indices behave in the same way. It is easily seen that in the system of the center of inertia, where all the time components of the tensor $T^{(L)}_{ik\ldots l}$ are equal to zero, the spinor $Z^{\dot r_1 \dot r_2 \ldots \dot r_L}_{s_1 s_2 \ldots s_L}$ goes over into a completely symmetric spinor of rank 2L. In the following we shall denote this spinor by Z_L.

With the spinor Z_L we associate the function

$$Y^{(i)}_{LM}(n) = (-1)^{L-M}\sqrt{\frac{(2L)!}{(L+M)!\,(L-M)!}}\,Z_L. \qquad (21)$$

Interchanging dotted and undotted indices of the spinor Z_L, we write

$$\tilde Y^{(i)}_{LM}(n) = (-1)^{L}\sqrt{\frac{(2L)!}{(L+M)!\,(L-M)!}}\,\tilde Z_L. \qquad (22)$$

Here $L + M$ is the number of indices which are equal to 1, $L - M$ is the number of indices which are equal to 2. The index i numbers the functions which correspond to the same values of L and M and which differ from one another in the distribu-

Values of the functions $Y^{(i)}_{LM}(n)$

M	L = 0	L = 1	L = 2
0	$\frac{1}{\sqrt{4\pi}}$	$Y^{(1)}_{10} = -\sqrt{2}\, Z^{\dot{1}}_{2} = \sqrt{\frac{3}{4\pi}}\,(n_z - n_0)$ $Y^{(2)}_{10} = -\sqrt{2}\, Z^{\dot{2}}_{1} = \sqrt{\frac{3}{4\pi}}\,(n_z + n_0)$	$Y^{(1)}_{20} = \sqrt{6}\, Z^{\dot{1}\dot{1}}_{22} = -\sqrt{\frac{45}{16\pi}}\Big\{n_x^2 + n_y^2 - \frac{2}{3} - 2n_0n_z + \frac{1}{3p^2}(p_x^2 + p_y^2 - 2p_0p_z)\Big\}$ $Y^{(2)}_{20} = \sqrt{6}\, Z^{\dot{1}2}_{21} = -\sqrt{\frac{45}{16\pi}}\Big\{n_x^2 + n_y^2 - \frac{2}{3} + \frac{1}{3p^2}(p_x^2 + p_y^2)\Big\}$ $Y^{(3)}_{20} = \sqrt{6}\, Z^{\dot{2}\dot{2}}_{11} = -\sqrt{\frac{45}{16\pi}}\Big\{n_x^2 + n_y^2 - \frac{2}{3} + 2n_0n_z + \frac{1}{3p^2}(p_x^2 + p_y^2 + 2p_0p_z)\Big\}$
1		$Y^{(1)}_{11} = Z^{\dot{1}}_{1} = \sqrt{\frac{3}{8\pi}}\,(n_x + in_y)$	$Y^{(1)}_{21} = -2\, Z^{\dot{1}\dot{1}}_{12} = \sqrt{\frac{15}{8\pi}}\Big\{n_z n_x + in_z n_y - n_0n_x - in_0n_y - \frac{1}{3p^2}(p_z p_x + ip_z p_y - p_0 p_x - ip_0 p_y)\Big\}$ $Y^{(2)}_{21} = -2\, Z^{\dot{1}2}_{11} = \sqrt{\frac{15}{8\pi}}\Big\{n_z n_x + in_z n_y + n_0 n_x + in_0n_y + \frac{1}{3p^2}(p_z p_x + ip_z p_y + p_0 p_x + ip_0 p_y)\Big\}$
−1		$Y^{(1)}_{1-1} = Z^{\dot{2}}_{2} = -Y^{(1)*}_{11}$	$Y^{(1)}_{2-1} = -2\, Z^{\dot{1}\dot{2}}_{22} = -Y^{(1)*}_{21}$ $Y^{(2)}_{2-1} = -2\, Z^{\dot{2}\dot{2}}_{12} = -Y^{(2)*}_{21}$
2			$Y^{(1)}_{22} = Z^{\dot{1}\dot{1}}_{11} = \sqrt{\frac{15}{32\pi}}\Big\{n_x^2 - n_y^2 - 2in_xn_y + \frac{1}{3p^2}(p_x^2 - p_y^2 - 2ip_xp_y)\Big\}$
−2			$Y^{(1)}_{2-2} = Z^{\dot{2}\dot{2}}_{22} = Y^{(1)*}_{22}$

tion of the index values 1 and 2 between dotted and undotted indices. The number of such functions for given L and M is $L - |M| + 1$.

In the system where $\mathbf{P} = 0$, the functions $Y^{(i)}_{LM}(n)$ are the same for all i:

$$Y^{(i)}_{LM}(n) = Y_{LM}(\mathbf{n}), \qquad \widetilde{Y}^{(i)}_{LM}(n) = Y^{*}_{LM}(\mathbf{n}),$$

where $Y_{LM}(\mathbf{n})$ are the ordinary spherical functions. We give the values of the tensor $T^{(L)}_{ik\ldots lm}$ for the first few values of L:*

$L =$	0	1	2
$T^{(L)}_{ik\ldots lm} =$	$1/\sqrt{4\pi}$	$\sqrt{3/8\pi}\; n_i$	$\sqrt{15/32\pi}\,\{n_in_k - {}^1/_3(\delta_{ik} - P_iP_k/P^2)\}$

The corresponding values of the functions $Y^{(i)}_{LM}(n)$ are given in the table.

We now proceed to the expansion of the free electromagnetic field in multipoles. In k-space, on the hypersurface $k^2 = 0$, we define the electric multipoles by the conditions

$$k_m E^{(i)}_{mLM}(k) = 0, \qquad P_m E^{(i)}_{mLM}(k) = 0 \quad (m = 1, 2, 3, 4), \quad (23)$$

and the magnetic multipoles by

$$k_m M^{(i)}_{mLM}(k) = 0, \qquad P_m M^{(i)}_{mLM}(k) = 0,$$
$$\partial M^{(i)}_{mLM}(k) / \partial k_m = 0. \quad (23a)$$

Here

$$\partial M_m/\partial k_m = \nabla_{\mathbf{k}}\mathbf{M} + \partial M_0/\partial k_0,$$
$$E^{(i)}_{mLM}(k)\, M^{(i)}_{mLM}(k) = 0. \quad (24)$$

We note that the tensors $T^{(L)}$ considered above satisfy the relation $k_m \partial T^{(L)}/\partial k_m = 0$, since $\partial n_s/\partial k_m = \delta_{ms}/kP - P_m k_s/(kP)^2$ and $k_m \partial n_s/\partial k_m = 0$. Using this and conditions (23), (23a), and (24), we can by means of the relativistic functions (21) obtain for the covariant multipoles the expressions

*The spacelike functions $Z(\alpha, \theta, \varphi)$ introduced by Dolginov[3] (where α, θ, φ are the angles of the vector n in four-space), in the system where $P = 0$, do not go over into spherical functions with a definite L, since in general the tensor corresponding to them is not orthogonal to any constant vector P. It should be noted that a correspondence between Dolginov's functions and the functions (21) exists only for $L = 1$, when they coincide to within a factor.

$$E^{(i)}_{mLM}(k) = L^{-1/2}(L+1)^{-1/2}$$

$$\frac{1}{\sqrt{-P^2}}\left[k_m\left(P_n\frac{\partial}{\partial k_n}\right)-(kP)\frac{\partial}{\partial k_m}\right]Y^{(i)}_{LM}(n), \qquad (25)$$

$$M^{(i)}_{mLM}(k) = \frac{1}{i\sqrt{-P^2}}L^{-1/2}(L+1)^{-1/2}\varepsilon_{mlst}k_l\left(\frac{\partial}{\partial k_s}Y^{(i)}_{LM}(n)\right)P_t. \qquad (26)$$

In the system for which $\mathbf{P} = 0$, formulas (25) and (26) go over into (15), since $\epsilon_{1230} = 1$.

The multipole expansion of the Fourier components of the electromagnetic field

$$A_m(x) = \int A_m(k)e^{ikx}\frac{d^3k}{|\mathbf{k}|} + \text{c.c.}$$

will have the form

$$A_m(k) = \sum_{LMi} e^{(i)}_{LM}E^{(i)}_{mLM}(k) + \sum_{LMi} m^{(i)}_{LM}M^{(i)}_{mLM}(k) \qquad (27)$$

or, changing to the spinor Z_L, we get

$$A_m(k) = \sum_{L=1}^{\infty} e_L\frac{1}{\sqrt{-P^2}}L^{-1/2}(L+1)^{-1/2}$$

$$\times\left[k_m\left(P_n\frac{\partial}{\partial k_n}\right)-(kP)\frac{\partial}{\partial k_m}\right]Z_L$$

$$+\sum_{L=1}^{\infty} m_L\frac{1}{\sqrt{-P^2}}\frac{1}{i}L^{-1/2}(L+1)^{-1/2}\varepsilon_{mlst}k_l\left(\frac{\partial}{\partial k_s}Z_L\right)P_t. \qquad (28)$$

Here e_L and m_L are spinors with L dotted and undotted indices, by means of which the products e_LZ_L and m_LZ_L, respectively, are contracted. The quantities $e^{(i)}_{LM}$ and $m^{(i)}_{LM}$ in (27) transform like the components of the spinors e_L and m_L. In the system of the center of inertia, they are equal for all i, for the same values of L and M.

4. COVARIANT EXPANSION OF THE CURRENT OF CHARGED PARTICLES IN MULTIPOLE MOMENTS

For a classical system of charges, the current density $j_k(\mathbf{x}, t)$ is given by the formula[14]

$$j_k(\mathbf{x}, t) = \sum_i e_{(i)}v_{(i)k}\delta^3(\mathbf{x} - \mathbf{x}_{(i)}(t)), \qquad (29)$$

$$\partial j_k(\mathbf{x}, t)/\partial x_k = 0. \qquad (29a)$$

Here $\mathbf{x}_{(i)}(t)$ is the trajectory of the i-th charge and $v_i = \{\mathbf{v}_{(i)}, 1\}$. We use the symbol P for the energy-momentum 4-vector of the system of charges, and first treat the problem in the system where $\mathbf{P} = 0$.

We go over to time and three-space variables (which is convenient because of the explicit asymmetry of (29) in the coordinates and time). We denote the coordinates of the center of inertia by x. Then

$$\mathbf{x}_{(i)}(t) = \mathbf{x}_c + \boldsymbol{\xi}_{(i)}(t). \qquad (30)$$

In the system where $\mathbf{P} = 0$, $\mathbf{x}_c$ is obviously independent of t and we may set $\mathbf{x}_c = 0$. Expanding (29) in powers of $\xi_{(i)}$, we can formally write

$$j_k(\mathbf{x}, t) = \sum_i e_{(i)}\dot{\xi}_{(i)k}\exp[-(\boldsymbol{\xi}_{(i)}\nabla)]\,\delta^3(\mathbf{x}) \qquad (31)$$

or, in components ($\dot{\xi}_{(i)} = \partial\xi_{(i)}/\partial t$),

$$\mathbf{j} = \sum_i\{\quad + e_{(i)}\dot{\boldsymbol{\xi}}_{(i)} \quad - e_{(i)}\dot{\boldsymbol{\xi}}_{(i)}(\boldsymbol{\xi}_{(i)}\nabla) \quad + \cdots\}\,\delta^3(\mathbf{x}), \quad (32)$$

$$\rho = \sum_i\left\{e_{(i)} - e_{(i)}(\boldsymbol{\xi}_{(i)}\nabla) + \frac{1}{2}e_{(i)}(\boldsymbol{\xi}_{(i)}\nabla)^2 - \cdots\right\}\delta^3(\mathbf{x}). \quad (33)$$

In these formulas we have placed above one another terms which together satisfy condition (29a); in this sense we may call them terms of the same order. We use the notation

$$\sum_i e_{(i)} = Z, \quad \sum_i e_{(i)}\xi_{(i)\alpha} = Q^{(1)}_\alpha(t), \quad \frac{1}{2}\sum_i e_{(i)}\xi_{(i)\alpha}\xi_{(i)\beta} = Q^{(2)}_{\alpha\beta}(t)$$

and, in general,

$$\sum_i\frac{1}{L!}\xi_{(i)\alpha}\xi_{(i)\beta}\ldots\xi_{(i)\gamma} = Q^{(L)}_{\alpha\beta\ldots\gamma}(t). \qquad (34)$$

In this notation, the expansion of the charge density is given simply in the form

$$\rho = \sum_L(-1)^L Q^{(L)}\nabla^L\delta^3(\mathbf{x}), \qquad (35)$$

where ∇^L denotes the tensor $\underbrace{\nabla_\alpha\nabla_\beta\ldots}_{L\text{ times}}$; it is contracted with the tensor $Q^{(L)}$.

Formula (35) is the expansion of the charge density in multipole moments.

In the expansion (32) for the current, the first term obviously gives the derivative of the dipole moment. In the second term, we separate the symmetric and antisymmetric parts. The symmetric part, $\frac{1}{2}(\dot{\boldsymbol{\xi}}(\boldsymbol{\xi}\nabla) + \boldsymbol{\xi}(\dot{\boldsymbol{\xi}}\nabla))$, together with the quadrupole term in (33) satisfies the continuity equation. The antisymmetric part is solenoidal and corresponds to the magnetic moment, since it can be written in the form $-\frac{1}{2}[\boldsymbol{\xi}\times\dot{\boldsymbol{\xi}}]\times\nabla$. (In the following, we shall omit the charge factors.)

In general, we have for the L-th term in the expansion (32),

$$\mathbf{B}_L = \frac{(-1)^{L-1}}{(L-1)!}\dot{\boldsymbol{\xi}}(\boldsymbol{\xi}\nabla)^{L-1}\delta^3(\mathbf{x}). \qquad (36)$$

In formula (33), there corresponds to the vector $\mathbf{B}_L$ the scalar

$$B_L = (-1)^L Q^{(L)}\nabla^L\delta^3(\mathbf{x}). \qquad (37)$$

The symmetric part of $\mathbf{B}_L$ is equal to

$$\mathbf{B}_{Ls} = (-1)^{L-1}\frac{1}{L!}\frac{\partial}{\partial t}\boldsymbol{\xi}(\boldsymbol{\xi}\nabla)^{L-1}\delta^3(\mathbf{x})$$

$$= (-1)^{L-1}\frac{\partial}{\partial t}Q^{(L)}\nabla^{L-1}\delta^3(\mathbf{x}) \qquad (38)$$

and satisfies the condition

$$B_L + \nabla \mathbf{B}_{Ls} = 0. \quad (39)$$

The antisymmetric part is

$$\mathbf{B}_{La} = (-1)^{L-1} \frac{L-1}{L!} \{\dot{\boldsymbol{\xi}}(\boldsymbol{\xi}\nabla) - \boldsymbol{\xi}(\dot{\boldsymbol{\xi}}\nabla)\} (\boldsymbol{\xi}\nabla)^{L-2} \delta^3(\mathbf{x}) \quad (40)$$

and satisfies the condition

$$\nabla \mathbf{B}_{La} = 0 \quad (41)$$

and is related to the 2^L-pole magnetic moment of the system (after summation over the charges).

We now rewrite (32), using (38) and (40) ($\alpha, \beta, \gamma = 1, 2, 3$)

$$j_\alpha = \Big\{ \sum_{L=1}^{\infty} (-1)^{L-1} \frac{\partial}{\partial t} Q^{(L)}_{\alpha\beta\gamma\ldots} \underbrace{\nabla_\beta \nabla_\gamma \ldots}_{L-1} + \sum_{L=2}^{\infty} \sum_i \frac{L-1}{L!} e_{(i)} (\boldsymbol{\xi}_{(i)}\nabla)^{L-2} (-1)^L [[\dot{\boldsymbol{\xi}}_{(i)} \boldsymbol{\xi}_{(i)}] \nabla]_\alpha \Big\} \delta^3(\mathbf{x}). \quad (42)$$

Relations (35) and (42) can be combined as follows:

$$\rho = -\operatorname{div} \mathbf{N}, \qquad \mathbf{j} = \partial \mathbf{N} / \partial t + \operatorname{rot} \mathbf{M}, \quad (43)^*$$

where

$$N_\alpha = \Big\{ \sum_{L=1}^{\infty} (-1)^{L-1} Q^{(L)}_{\alpha\beta\gamma\ldots\delta} \nabla_\beta \nabla_\gamma \ldots \nabla_\delta \Big\} \delta^3(\mathbf{x}),$$

$$M_\alpha = \Big\{ \sum_{L=1}^{\infty} (-1)^{L-1} M^{(L)}_{\alpha\beta\gamma\ldots\delta} \nabla_\beta \nabla_\gamma \ldots \nabla_\delta \Big\} \delta^3(\mathbf{x}). \quad (44)$$

Here

$$M^{(L)}_{\alpha\beta\gamma\ldots\delta} = \sum_i \frac{L}{(L+1)!} e_{(i)} \xi_{(i)\beta} \xi_{(i)\gamma} \ldots \xi_{(i)\delta} [\dot{\boldsymbol{\xi}} \boldsymbol{\xi}]_\alpha; \quad (45)$$

N corresponds to the electric, and **M** to the magnetic part of the current.

In order to write the expansion (32), (33) in relativistic form, we associate with the multipole moments symmetric four-dimensional tensors orthogonal to P, in the same way as was done in Sec. 3. These tensors must satisfy the relations†

$$Q^{(L)}_{mn\ldots l} P_m = 0, \qquad M^{(L)}_{mn\ldots l} P_m = 0, \quad (46)$$

i.e., they must transform according to a representation of the little group L_P.

The tensors Q and M are calculated in the system where $\mathbf{P} = 0$ or, what is the same thing, on the spacelike hypersurface orthogonal to P. To get a covariant expression, the vector product $[\dot{\boldsymbol{\xi}} \times \boldsymbol{\xi}]_\alpha$ in formula (45) must be replaced by $-(-P^2)^{-1/2} \epsilon_{klmp} \dot{\xi}_l \xi_m P_p$, where the dot now denotes differentiation with respect to the proper time of the system.

We can now write for the electric part of the current density four-vector*

$$j_i^{\mathrm{el}} = \Big[Z P_i + \frac{\partial}{\partial x_k} (P_k N_i - P_i N_k) \Big] P_0^{-1} \delta^3 [\mathbf{x} - \mathbf{x}_c(t)], \quad (47)$$

where $\mathbf{x}_c(t) = \mathbf{P}t/P_0$, and where N_i in the system with $\mathbf{P} = 0$ is given by the first of formulas (44). In an arbitrary system, the definition is analogous to the definition of the dipole moment (condition $PN = 0$!).

For the magnetic part of the current we find

$$j_i^{\mathrm{M}} = -\varepsilon_{iklm} M_k P_l \frac{\partial}{\partial x_m} P_0^{-1} \delta^3 [\mathbf{x} - \mathbf{x}_c(t)], \quad (48)$$

where M is defined when $\mathbf{P} = 0$ by the second of formulas (44) and the condition $PM = 0$.

Formulas (47) and (48) are the relativistic generalization of relations (43) and (44). They can be applied to the treatment of the electromagnetic properties of systems of charges.

5. MULTIPOLE RADIATION

As an example we consider the radiation of electromagnetic waves by a system of charges. To a source of the electromagnetic field with current density $j_i(y)$ there corresponds the vector potential

$$A_i(x) = 4\pi \int D_R(x-y) j_i(y) d^4y, \quad (49)$$

where $D_R(x-y)$ is the retarded Green's function. Expanding the current in multipole moments, we can write

$$A_i(x) = \sum_{L=1}^{\infty} A_i^{(L)\mathrm{el}} + \sum_{L=1}^{\infty} A_i^{(L)\mathrm{M}}, \quad (50)$$

$$A_i^{(L)}(x) = 4\pi \int D_R(x-y) j_i^{(L)}(y) d^4y. \quad (51)$$

In momentum space, Eq. (51) can be written as

$$A_i^{(L)}(k) = 4\pi (2\pi)^4 j_i^{(L)}(k) D_R(k); \quad (52)$$

$$j_i^{(L)}(k) = \frac{1}{(2\pi)^4} \int j_i^{(L)}(y) e^{iky} d^4y, \quad (53)$$

$$D_R(k) = -\frac{1}{(2\pi)^4} \Big\{ \mathrm{P} \frac{1}{k^2} + i\pi\delta(k^2) \varepsilon(k_0) \Big\} \quad (54)$$

(P denotes the principal value; $\epsilon(k_0) = k_0/|k_0|$).

In formula (54), the term containing $\delta(k^2)$ obviously describes the radiation of electromagnetic waves. We integrate this term over the energy.

*rot = curl.

†In addition, the equations

$$\frac{\partial}{\partial x_n} Q^{(L)}_{\ldots n \ldots}(xP) = 0, \qquad \frac{\partial}{\partial x_n} M^{(L)}_{\ldots n \ldots}(xP) = 0$$

are valid.

*$P_0^{-1} \delta^3 [\mathbf{x} - \mathbf{x}_c(t)]$ is an invariant. If all the multipole moments of the system are equal to zero, then $j_i = Z P_i P_0^{-1} \delta^3 [\mathbf{x} - \mathbf{x}_c(t)]$, which coincides with the formula for the current density of a point charge.

We then get

$$A_i^{(L)}(x) - P\int j_i^{(L)}(k)\frac{e^{-ikx}}{k^2}d^4k = \int a_i^{(L)}(k)e^{ikx}\frac{d^3k}{k_0} + \text{ c.c.}, \quad (55)$$

where $k_0 = |\mathbf{k}|$, i.e., $k^2 = 0$. As usual we impose on $a^{(L)}(k)$ the condition $a_i^{(L)}(k)P_i = 0$.

The calculations give the following values for $a^{(L)}(k)$ (where we use the long-wave approximation in the system in which $\mathbf{P} = 0$):

$$a_i^{(L)\mathrm{el}}(k) = B\frac{(kP)^L}{(-P^2)^{L/2+1/2}}\left\{(kP)\frac{\partial}{\partial k_i}T_{lm\ldots n}^{(L)}(k) - k_i\left(P_k\frac{\partial}{\partial k_k}T_{lm\ldots n}^{(L)}(k)\right)\right\}Q_{lm\ldots n}^{(L)}(x), \quad (56)$$

$$a_i^{(L)\mathrm{M}}(k) = B\frac{(kP)^L}{(-P^2)^{L/2+1/2}}\frac{1}{i}\varepsilon_{istr}k_s\left(\frac{\partial}{\partial k_t}T_{lm\ldots n}^{(L)}(k)\right)P_rM_{lm\ldots n}^{(L)}(x), \quad (57)$$

where $x = kP/(-P^2)^{1/2}$; $T_{lm\ldots n}^{(L)}(k)$ are the tensors (19) of Sec. 3; $Q^{(L)}(x)$ and $M^{(L)}(x)$ are the Fourier coefficients with respect to x of the tensors of the electric and magnetic multipole moments, in which the traces over any pair of indices are made equal to zero.

If we go over from the four-tensors $T^{(L)}$, $M^{(L)}$, and $Q^{(L)}$ to relativistic spinors, we arrive at formula (28), where

$$e_L(x) = -2^{-L}B(kP)^L(P^2)^{-L/2}Q_L(x), \quad (58)$$

$$m_L(x) = 2^{-L}B(kP)^L(P^2)^{-L/2}M_L(x), \quad (59)$$

$$B = \frac{i^{L+1}}{L(2L-1)!!}\sqrt{\frac{(2L)!\,L(L+1)}{(2L+1)4\pi}}. \quad (60)$$

Formulas (56) and (57) express the connection between the expansion of the radiation in multipoles, (25) and (26), and the expansion of the current density of the radiator in multipole moments.

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