

*PROPAGATION OF ONE-DIMENSIONAL SELF-SIMILAR PLASMA WAVES ALONG A MAGNETIC FIELD*

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A one-dimensional steady-state pulse propagating along a magnetic field in a cold plasma is considered. The calculations are carried out in the nonrelativistic single-particle approximation, neglecting collisions, and the plasma is assumed to be quasi-neutral. The equations are solved exactly. The pulse shape and velocity distribution are investigated. It is shown that the lines of force of the magnetic field and the trajectories of the particles in the pulse are helices. The dependence of Mach number on wave energy is determined. It is found that there is an appreciable acceleration of the electron component at the pulse maximum even for small oscillations of the magnetic field. Most of the wave energy is concentrated in the kinetic energy of the electrons.

THE stationary one-dimensional motion of a rare-field plasma perpendicular to a magnetic field has been investigated in detail by a number of authors.<sup>[1-2]</sup> In the present work we consider a stationary wave moving along the field.

We assume that a plane pulse is produced as a consequence of an initial perturbation of the plasma and then propagates along the field. The effect of the initial conditions disappears in the course of time and the pulse shape is then determined by nonlinear effects, dispersion, and dissipation. The last effect may be due to Coulomb scattering, the average electromagnetic acceleration of the particles in the wave by the unperturbed thermal velocity, or pulse instability.

Because of the complexity of a complete analysis of this problem the investigation is usually divided into two steps: the stationary shape of the pulse is first determined neglecting dissipation; then the effect of dissipative processes on this wave shape is considered.

It can be shown that with no dissipation the pulse shape remains unchanged at large distances from the source and depends (for a given unperturbed plasma state) on the velocity of the pulse; in turn, the pulse velocity is related uniquely to the wave energy  $\mathcal{E}$ .

We shall find the pulse shape as a function of  $\mathcal{E}$  under the following assumptions: a) the plasma density is low so that Coulomb scattering can be neglected; b) the thermal velocities of the unperturbed particles are small compared with the wave velocity so that the single-particle approxi-

mation can be used; c) the velocities of particles in the wave are small compared with the velocity of light.

1. BASIC EQUATIONS

In this formulation the problem is described by a system consisting of the equations of motion for the electrons and ions, the equation of continuity, and Maxwell's equations with self-consistent fields. We introduce the following notation:  $\mathbf{V}$  and  $\mathbf{v}$ ,  $N$  and  $n$  are respectively the velocities and densities of the ions and electrons;  $\mathbf{E}$  and  $\mathbf{H}$  are the electric and magnetic fields,  $m$  is the ion mass,  $\mu$  is the ratio of the electron mass to the ion mass,  $e$  is the absolute value of the electron charge and  $c$  is the velocity of light. To simplify the analysis we assume that the ions are singly charged.

The basic equations for the problems are

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\mathbf{V} = \frac{e}{m} \left\{ \mathbf{E} + \frac{1}{c} [\mathbf{V}\mathbf{H}] \right\},$$

$$-\mu \left\{ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} \right\} = \frac{e}{m} \left\{ \mathbf{E} + \frac{1}{c} [\mathbf{v}\mathbf{H}] \right\}, \tag{1}^*$$

$$\partial N / \partial t + \text{div } N\mathbf{V} = 0, \quad \partial n / \partial t + \text{div } n\mathbf{v} = 0, \tag{2}$$

$$\text{rot } \mathbf{H} = 4\pi e c^{-1} (N\mathbf{V} - n\mathbf{v}), \tag{3}^\dagger$$

$$\text{rot } \mathbf{E} = -c^{-1} \partial \mathbf{H} / \partial t, \tag{4}$$

$$\text{div } \mathbf{H} = 0, \tag{5}$$

$$\text{div } \mathbf{E} = 4\pi e (N - n). \tag{6}$$

The  $x$  axis is taken along the unperturbed field  $\mathbf{H}_0$ , which is assumed to be uniform. We write the

\* $[\mathbf{v}\mathbf{H}] = \mathbf{v} \times \mathbf{H}$ .

† $\text{rot} = \text{curl}$ .

solution in the form of a stationary plane wave traveling in the direction of positive  $x$  with velocity  $U$ . In this case all quantities depend on the single variable  $\xi = x - Ut$  so that  $\partial/\partial y = \partial/\partial z = 0$ ,  $\partial/\partial x = d/d\xi$  and  $\partial/\partial t = -Ud/d\xi$ . When  $\xi \rightarrow \infty$  (unperturbed plasma)  $\mathbf{E}$ ,  $H_y$ ,  $H_z$ ,  $\mathbf{V}$  and  $\mathbf{v}$  vanish and  $H_x = H_0$ ,  $n = N = N_0$ .

Equations (2), (4) and (5) can be integrated directly:

$$N = N_0 U / (U - V_x), \quad n = N_0 U / (U - v_x), \quad (7)$$

$$E_y = U H_z / c, \quad E_z = -U H_y / c, \quad (8)$$

$$H_x = H_0 = \text{const.} \quad (9)$$

Substituting (7)–(9) in (1) we see that the  $y$  and  $z$  components of the current  $c(\mathbf{N}\mathbf{V} - n\mathbf{v})$  are total differentials. It then follows from (3) that

$$H_z = -\frac{H_0}{U} M^2 (V_z + \mu v_z), \quad H_y = -\frac{H_0}{U} M^2 (V_y + \mu v_y), \quad (10)$$

where

$$M = \sqrt{4\pi N_0 m U} / H_0 \quad (11)$$

is the magnetic Mach number.

We introduce the dimensionless variables

$$s = \frac{eH_0}{mcU} \xi, \quad \mathbf{h} = \frac{\mathbf{H}}{H_0}, \quad \varepsilon = \frac{c}{U} \frac{E_x}{H_0}, \quad \mathbf{W} = \frac{\mathbf{V}}{U}, \quad \mathbf{w} = \frac{\mathbf{v}}{U}$$

and set  $1 - W_x = \Theta$ ,  $1 - w_x = \vartheta$ .

After substitution of the integrals that have been obtained the equations are reduced to the system

$$\Theta \frac{d\Theta}{ds} = \varepsilon - M^2 \mu (W_y w_z - W_z w_y),$$

$$-\mu \Theta \frac{d\vartheta}{ds} = \varepsilon + M^2 (W_y w_z - W_z w_y), \quad (12)$$

$$\frac{dW_y}{ds} = -\left(\frac{1}{\Theta} - M^2\right) W_z + \mu M^2 w_z,$$

$$\mu \frac{dw_y}{ds} = \left(\frac{1}{\vartheta} - \mu M^2\right) w_z - M^2 W_z, \quad (13)$$

$$\frac{dW_z}{ds} = \left(\frac{1}{\Theta} - M^2\right) W_y - \mu M^2 w_y,$$

$$\frac{dw_z}{ds} = -\left(\frac{1}{\vartheta} - \mu M^2\right) w_y + M^2 W_y, \quad (14)$$

$$\beta^2 d\varepsilon/ds = \Theta^{-1} - \vartheta^{-1} \quad (\beta^2 = H_0^2 / 4\pi M N_0 c). \quad (15)$$

We seek a solution for which  $\Theta$  and  $\vartheta$  are everywhere positive (in which case particles are not reflected from the wave). We conclude from (15) that the neutrality condition  $\Theta - \vartheta \ll \Theta$  is satisfied in the nonrelativistic case ( $\beta^2 \ll 1$ ).<sup>\*</sup> We write  $\Theta - \vartheta = \beta^2 \psi$  and expand (12)–(15) in powers of  $\beta^2$ . We must write  $\Theta = \vartheta$  in the zeroth approximation. The quantity  $\varepsilon$  is eliminated by subtracting one equation in (12) from the other. The problem is then reduced to a system of five equations:

<sup>\*</sup>Since the small parameter  $\beta^2$  appears in the higher derivative in (14) this conclusion is subject to further verification; this will be given at the end of Sec. 2.

$$\Theta \frac{d\Theta}{ds} = -M^2 (W_y w_z - W_z w_y),$$

$$\frac{dW_y}{ds} = -\left(\frac{1}{\Theta} - M^2\right) W_z + \mu M^2 w_z,$$

$$\mu \frac{dw_y}{ds} = \left(\frac{1}{\vartheta} - \mu M^2\right) w_z - M^2 W_z,$$

$$\frac{dW_z}{ds} = \left(\frac{1}{\Theta} - M^2\right) W_y - \mu M^2 w_y,$$

$$\mu \frac{dw_z}{ds} = -\left(\frac{1}{\vartheta} - \mu M^2\right) w_y + M^2 W_y. \quad (16)$$

The electric field  $\varepsilon$  is defined by

$$\varepsilon = (1 - \mu) \Theta^2 d^2 \Theta^2 / ds^2,$$

while the correction is

$$\psi = -\frac{1}{2} \Theta^2 d^2 \Theta^2 / ds^2. \quad (17)$$

Introducing the complex variables

$$P = W_y + iW_z, \quad Q = w_y + iw_z, \quad (18)$$

we can write (16) in more compact form:

$$\Theta d\Theta/ds = \text{Im } PQ^*,$$

$$dP/ds = (\Theta^{-1} - M^2) iP - \mu M^2 iQ,$$

$$\mu dQ/ds = -(\Theta^{-1} - \mu M^2) iQ + M^2 iP. \quad (19)$$

When  $s \rightarrow +\infty$ , we have  $\Theta = 1$  and  $P = Q = 0$ .

The system in (19) can be integrated by quadratures if the following substitutions are made:

$$P = p \exp\left\{i \int K(s) ds\right\}, \quad Q = (q + iq_1) \exp\left\{i \int K(s) ds\right\}, \quad (20)$$

where  $p$ ,  $q$ ,  $q_1$ , and  $K$  are real functions.

Substituting (20) in (19) and separating real and imaginary parts we have

$$\Theta d\Theta/ds = -q_1 p, \quad (21)$$

$$dp/ds = \mu M^2 q_1, \quad (22)$$

$$K = \Theta^{-1} - M^2 - \mu M^2 q/p, \quad (23)$$

$$\mu dq/ds = (\Theta^{-1} - \mu M^2) q_1 \quad (24)$$

$$\mu dq_1/ds + \mu K q = -(\Theta^{-1} - \mu M^2) q + M^2 p. \quad (25)$$

Writing  $\sqrt{\mu} M = \alpha$  and  $p = \alpha \varphi$  we have from (21) and (22)

$$\Theta = \sqrt{1 - \varphi^2}, \quad (26)$$

while (22)–(24) and (26) yield

$$q = (\text{arc sin } \varphi - \alpha^2 \varphi) / \mu \alpha, \quad (27)$$

$$K = \frac{1}{\sqrt{1 - \varphi^2}} - \frac{1}{\mu} \frac{\text{arc sin } \varphi}{\varphi}. \quad (28)$$

Substituting these results in (25) we determine  $q_1(\varphi)$ :

$$q_1(\varphi) = \pm \frac{1}{\mu \alpha} \left[ 2 \int_0^{\varphi} \frac{\text{arc sin}^2 \varphi}{\varphi} d\varphi - (1 + \mu) \text{arc sin}^2 \varphi \right. \\ \left. + 2\alpha^2 (1 + \mu) (1 - \sqrt{1 - \varphi^2}) \right]^{1/2}. \quad (29)$$

The spatial dependence of  $\varphi(s)$  is determined by quadratures from (29) and (22).

## 2. WAVES OF MEDIUM AND LOW INTENSITY

We consider the waves for which  $\alpha^2 \ll 1$  (i.e.,  $M^2 \ll 1/\mu$ ). In this case  $0 \leq \varphi^2 \leq \varphi_0^2 \ll 1$  so that  $\sin \varphi$  and  $\sqrt{1 - \varphi^2}$  can be expanded to second order. Limiting ourselves to zero order terms in  $\alpha$  and  $\mu$  in the coefficient for  $\varphi^4$  we have

$$q_1(\varphi) = \pm \varphi \mu^{-1} \Lambda(M) \sqrt{1 - \varphi^2/6\alpha^2 \Lambda^2(M)}, \quad (30)$$

where

$$\Lambda(M) = \sqrt{(M^2 - 1)/M^2 + \mu}. \quad (31)$$

To determine the spatial dependence of  $\varphi$  we choose the origin of coordinates in such a way that  $d\varphi/ds = 0$  when  $s = 0$ . We then have from (22) and (30)

$$\varphi = \sqrt{6} \alpha \Lambda / \text{ch} \frac{M \Lambda s}{\sqrt{\mu}}. \quad (32)^*$$

Keeping lowest order terms in  $\alpha$  and  $\mu$  we have

$$\begin{aligned} p &= \sqrt{6} M^2 \Lambda(M) \mu / \text{ch} \frac{M \Lambda}{\sqrt{\mu}} s, \\ q &= \frac{\sqrt{6} \Lambda}{\mu} / \text{ch} \frac{M \Lambda}{\sqrt{\mu}} s, \\ q_1 &= \sqrt{\frac{6}{\mu}} M \Lambda^2 \text{sh} \frac{M \Lambda}{\sqrt{\mu}} s / \text{ch}^2 \frac{M \Lambda}{\sqrt{\mu}} s, \\ K &= -1/\mu, \\ W_x &= 3\mu^2 M^4 \Lambda^4 / \text{ch}^2 \frac{M \Lambda}{\sqrt{\mu}} s. \end{aligned} \quad (33)$$

A numerical estimate shows that the error is no greater than 3% when  $M \leq 10$ .

It is evident from (33) and (17) that quasi-neutrality holds if the following inequality is satisfied:

$$6\mu\beta^2 M^6 \ll 1. \quad (34)$$

The nonrelativistic condition for electrons requires that

$$\beta M \Lambda \ll \mu. \quad (35)$$

Comparison of (34) and (35) shows that the requirement  $\mu M^2 \ll 1$  is satisfied automatically as a consequence of our assumptions of nonrelativistic velocities and quasi-neutrality.

Returning to the original variables  $\mathbf{V}$ ,  $\mathbf{v}$  and  $\mathbf{H}$  we write the dependence of these quantities on  $\xi$ :

$$V_x = v_x = 3\mu^2 M^5 \Lambda^4 U_0 / \text{ch}^2 \frac{\Lambda}{\sqrt{\mu}} \frac{\xi}{\xi_0}. \quad (36)$$

$$V_y + iV_z = \sqrt{6} M^3 \Lambda \mu U_0 \exp \left\{ -i \frac{\xi}{M \mu \xi_0} \right\} / \text{ch} \frac{\Lambda}{\sqrt{\mu}} \frac{\xi}{\xi_0}. \quad (37)$$

$$\begin{aligned} v_y + iv_z &= \sqrt{6} \Lambda M U_0 \exp \left\{ -i \frac{\xi}{M \mu \xi_0} \right\} \\ &+ i \sqrt{\mu} M \Lambda \text{th} \frac{\Lambda}{\sqrt{\mu}} \frac{\xi}{\xi_0} / \mu \text{ch} \frac{\Lambda}{\sqrt{\mu}} \frac{\xi}{\xi_0}, \end{aligned} \quad (38)$$

$$\begin{aligned} H_y + iH_z &= -M^2 H_0 \sqrt{6} \Lambda \exp \left\{ -i \frac{\xi}{M \mu \xi_0} \right\} \\ &+ i \sqrt{\mu} M \Lambda \text{th} \frac{\Lambda}{\sqrt{\mu}} \frac{\xi}{\xi_0}, \end{aligned} \quad (39)^\dagger$$

\*ch = cosh; sh = sinh.

†th = tanh.

where  $\xi_0 = mcU_0/eH_0$  and  $U_0 = H_0/\sqrt{4\pi N_0 m}$  are independent of  $M$ .

The pulse width

$$\delta = \frac{\sqrt{\mu} mc U_0}{e H_0} \frac{M}{[M^2(1+\mu) - 1]^{1/2}} \quad (40)$$

approaches a constant limit at large values of  $M$ ; this is the geometric mean of the Larmor radii for electrons and ions with velocity  $U_0$  in a field  $H_0$ . The pulse spreads for weak waves: when  $M = 1$  the width is equal to the ion Larmor radius and when  $M^2 \rightarrow 1/(1+\mu)$  the width  $\delta \rightarrow \infty$ .

It is evident from (36)–(39) that the magnetic field and the particle trajectories inside the pulse traverse a helix of pitch  $M\mu\xi_0$ . The helix pitch increases with increasing wave intensity and is a minimum at low wave intensities. We note, as is evident from these results, that the linear solutions for which  $\mathbf{V}$  and  $\mathbf{H}$  lie in the same plane everywhere (Alfvén waves), can only exist for a bounded time interval. Nonlinear effects then lead to curvature of the lines of force and the particle trajectories.

It follows from (36) that there is a concentration of particle density at the pulse maximum. Equation (38) shows that the electrons lead the ions in phase at the leading edge of the pulse and lag at the trailing edge.

The particle is accelerated in the plane perpendicular to  $\mathbf{H}_0$  as the pulse maximum is approached. The maximum transverse ion velocity is

$$V_0 = \sqrt{6} M^3 \Lambda \mu U_0$$

and does not reach the value  $U_0$  if  $\mu M^2 \ll 1$ . At the same time the maximum electron velocity

$$v_0 = \sqrt{6} \Lambda M U_0 / \mu \quad (41)$$

exceeds  $U_0$  by a factor  $\mu^{-1/2}$  even at relatively small values of  $M$  ( $M = 1$ ). Consequently, in contrast with transverse waves [1] the wave considered here accelerates electrons predominantly. When  $M \rightarrow 1 - \mu$  the ion and electron accelerations approach zero because of the factor  $\Lambda$ .

Acceleration occurs because individual ions moving with velocity  $U$  in the field being considered pass through the wave freely whereas the electrons behave as though effectively reflected from it. On the other hand, to maintain the neutrality condition the ions pull the electrons through the potential barrier, communicating to them the necessary kinetic energy. It is evident from (38) and (39) that under these conditions the electrons move along the force lines of the perturbed field. Since the current is basically due to the motion of the electrons the field is force free ( $\mathbf{H} \parallel \text{curl } \mathbf{H}$ ).

### 3. MACH NUMBER AS A FUNCTION OF WAVE ENERGY

The results above give the field  $H$  and the velocities  $V$  and  $v$  as functions of the Mach number  $M$ . The Mach number has a clear physical meaning for longitudinal waves, in which case the direction of propagation of the wave is the same as the direction of motion. The Mach number is then related to the ratio of the source velocity to the velocity of the small perturbation. This simple correspondence between the Mach number and the source velocity does not hold for transverse waves (including the pulse being considered above). A more physically meaningful characteristic of the wave is the wave energy.

In stationary motion there is a unique relation between  $M$  and the energy  $\mathcal{E}$  per square centimeter of wave front. It is evident from (37) and (38) that the kinetic energy is concentrated basically in the electron component so that

$$d\mathcal{E}_K = \frac{3\Lambda^2 M^2 U_0^2 m N_0}{\mu c h^2 (\Lambda \xi / \sqrt{\mu \xi_0})} d\xi. \quad (42)$$

The electrical energy  $\mathcal{E}_E$  is small compared with the magnetic energy  $\mathcal{E}_H$  and  $d\mathcal{E}_H = \mu M^2 d\mathcal{E}_K \ll d\mathcal{E}_K$ .

Thus, most of the energy of a nonrelativistic quasi-neutral pulse is concentrated in the kinetic energy of the electrons.

Integrating, we have

$$\mathcal{E} = 6\Lambda M^2 \mathcal{E}_0 / \sqrt{\mu}, \quad (43)$$

where  $\mathcal{E}_0 = H_0^2 m c U_0 / 8\pi e H_0$  is the magnetic energy of the unperturbed field in a column with a cross section of one square centimeter and a length  $\xi_0$ .

Using the definition of  $\Lambda$  (31) we have

$$M^2 = 1 + \sqrt{1 + \frac{\mu}{36} \left(\frac{\mathcal{E}}{\mathcal{E}_0}\right)^2} / 2(1 + \mu). \quad (44)$$

Our formulas hold for  $\mu M^2 \ll 1$  and, consequently,  $\mathcal{E} \ll 6\mathcal{E}_0 / \mu^{3/2}$ . The Mach number is a weak function of  $\mathcal{E}$ : thus, when  $\mathcal{E} \rightarrow 0$ ,  $M = 1/(1 + \mu)$ ; the value  $M = 1$  is reached when  $\mathcal{E} = 12\mathcal{E}_0$ . It may be as-

sumed over reasonably wide limits that  $M = 1$  while  $\Lambda(M) = \sqrt{\mu}$ . Under these conditions the electron energy at the maximum is of order  $MU_0^2$  where  $m$  is the ion mass and  $U_0$  is the Alfvén velocity. At the same time the perturbed field  $H$  is of order  $\sqrt{\mu} H_0 \ll H_0$ . Thus we see that very weak field fluctuations are capable of accelerating electrons to high energies.

If the system is large enough, initially weak pulses of long duration can form intense short pulses which provide still stronger electron acceleration. An instability can then arise as a consequence of the relative motion of the electrons and ions.<sup>[3]</sup> If the instability develops in a time  $\tau$ , which is much smaller than the time for the pulse to travel its own length, then the waves considered above must cause strong heating of the electron component of the plasma without heating of the ion component.

These effects are of interest in connection with the theory of the origin of the outer radiation belt of the earth, in which only fast electrons have been observed at the present time.<sup>[4]</sup>

In conclusion I wish to express my deep gratitude to Academician M. A. Leontovich for discussion of this work.

<sup>1</sup>R. Z. Sagdeev, *Fizika plazmy in problema upravlyaemykh termoyadernykh reaktiv* (Plasma Physics and the Problem of Controlled Thermonuclear Reactions), Vol. 4, p. 384 AN SSSR, 1958.

<sup>2</sup>Gardner, Goertzel, Grad, Morawets, Rose and Rubin, Geneva Conference on the Peaceful Uses of Atomic Energy. Paper No. 374.

<sup>3</sup>Vedenov, Velikhov, and Sagdeev, *UFN* **73**, 701 (1961), *Soviet Phys. Uspekhi* **4**, 332 (1961).

<sup>4</sup>N. S. Vernov and A. E. Chudakov, *UFN* **70**, 585 (1960), *Soviet Phys. Uspekhi* **3**, 230 (1960).