ISOTOPE SHIFT OF SPECTRAL LINES AND THE COMPRESSIBILITY OF DEFORMED NUCLEI

É. E. FRADKIN

A simple formula for the isotope shift constant, C_{theor}, is obtained which depends only on the mean square radius and its relative variation within the series of isotopes. Comparison with experiment leads to certain conclusions on the compressibility of deformed nuclei. Satisfactory agreement between the experimental and calculated values of the isotope shift for 23 isotope pairs of spherical and deformed nuclei can be attained by introducing the two empirical parameters of regular compressibility $\eta = 0.7$ and deformation compressibility $\xi = -5/8\pi$.

1. ISOTOPE SHIFT OF THE nlj LEVEL OF THE ELECTRON

THE volume isotope shift (i.s.) of the nlj level of an electron in the self-consistent field of the nucleus and the electron shells can be easily expressed in terms of the electron wave functions in the field of the nuclear isotopes. In the case of a spherically symmetric field the variables are separable, and the radial Dirac equations take the form (in relativistic units)

$$[\varepsilon_i + 1 - U_i(r)] F_i - dG_i/dr - kG_i/r = 0,$$
$$[\varepsilon_i - 1 - U_i(r)] G_i + dF_i/dr - kF_i/r = 0, \qquad (1)$$

where $i = 1, 2$ is the number of the isotope, and

$$k = -(j + 1/2) = -(l + 1) \quad j = l + 1/2,$$
$$k = +(j + 1/2) = l \qquad j = l - 1/2.$$

The potentials of the self-consistent field of the nucleus and the electron shells can be written in the form of a sum of the nuclear potential of the i-th isotope and the potential of the electron shells

$$U_i(r) = U_i^n(r) + U^e(r).$$

As we shall see in the following, we may restrict the solution to small values of r, for which the average potential of the electron shells $U^e(r)$ can be replaced by a constant potential U_0. An estimate of U_0 on the basis of the Thomas-Fermi model gives $U_0 = 1.9\,\alpha Z^{4/3}$. We can therefore assume that the Eqs. (1) are written in terms of the nuclear potentials $U_i^n(r)$* and the modified eigenvalues of the energy $\varepsilon_i' = \varepsilon_i - U_0$.

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*For deformed nuclei and electron states $s_{1/2}$ and $p_{1/2}$ ($k = \pm 1$), $U_i^n(r)$ is understood to denote the value of the potential averaged over the angles.

The electrostatic potentials $U_1(r)$ and $U_2(r)$ are subject to the natural conditions outside the nucleus

$$U_1(r) = U_2(r) = -e^2 Z/r, \qquad r \geqslant R_0. \qquad (2)$$

With condition (2) we obtain from (1) the following expression for the i.s.[1,2]

$$\delta T_k = (G_2 F_1 - G_1 F_2)_{R_0} \Big/ 2\pi \int_{R_\bullet}^{\infty} (F_1^* F_2 + G_1^* G_2)\, dr \quad \text{cm}^{-1}. \qquad (3)$$

δT_k does not depend on the particular value of R_0 as long as condition (2) is satisfied. It is advantageous to choose R_0 as small as possible, for example, equal to or somewhat larger than the largest of the radii of the nuclear isotopes. The numerator and denominator increase simultaneously, and the integral in the denominator can be replaced by unity to a good approximation.

We have therefore in this case

$$\delta T_k = \frac{1}{2\pi} (G_2 F_1 - G_1 F_2)_{R_0} \qquad (4)$$

$$= \begin{cases} \dfrac{G_2(R_0)\, G_1(R_0)}{2\pi}(K_1 - K_2), & k < 0 \\[2mm] -\dfrac{F_2(R_0)\, F_1(R_0)}{2\pi}(\chi_1 - \chi_2), & k > 0 \end{cases},$$

where the functions $K(r) = F(r)/G(r)$ and $\chi(r) = G(r)/F(r)$ satisfy the equations[3]

$$K(r) = -r^{2k} \int_0^r (\varepsilon - 1 - U(r'))$$
$$\times \left[1 + \frac{\varepsilon + 1 - U(r')}{\varepsilon - 1 - U(r')} (K(r'))^2\right](r')^{-2k}dr' \quad (k < 0),$$

$$\chi(r) = r^{-2k} \int_0^r (\varepsilon + 1 - U(r'))$$
$$\times \left[1 + \frac{\varepsilon - 1 - U(r')}{\varepsilon + 1 - U(r')}(\chi(r'))^2\right] r'^{2k}dr' \quad (k > 0). \qquad (5)$$

In the first approximation we find the solution in the form $(\alpha = e^2/\hbar c)$

$$\begin{Bmatrix} K(r) \\ \chi(r) \end{Bmatrix} = \mp \frac{\alpha Z}{2|k|} \pm \frac{\alpha Z}{2(1+2|k|)} \frac{\langle R^{2|k|} \rangle}{r^{2|k|}} + \frac{1 \mp \varepsilon}{1+2|k|},$$

(5a)

where

$$\langle R^{2|k|} \rangle = \frac{1}{eZ} \int_v \rho(r', \vartheta, \varphi)(r')^{2|k|+2} dr' \sin \vartheta d\vartheta d\varphi$$

is the moment of order $2|k|$ of the nuclear charge distribution $\rho(r, \vartheta, \varphi)$. The i.s. for s electrons $(k = -1)$ depends in this case on a single parameter, viz., the mean square radius of the nuclear charge distribution.

Let us estimate the second order contribution to the i.s. of an s electron $(k = -1)$. It will depend on the details of the nuclear charge distribution. We choose the nuclear charge distribution and the corresponding potential in the form

$$\rho(r) = \frac{n+1}{4\pi} \frac{Ze}{R^3} \left(\frac{r}{R}\right)^{n-2},$$

$$\rho(r) = 0,$$

$$V(r) = \frac{n+1}{n}\left[1 - \frac{1}{n+1}\left(\frac{r}{R}\right)^n\right]\frac{Ze}{R}, \quad r \leqslant R;$$

$$V(r) = Ze/r, \qquad\qquad r > R \quad (6)$$

(R is the nuclear charge radius). In this case $\langle R^2 \rangle$ and the square of the electric charge radius R are simply related: $\langle R^2 \rangle = (n+1)(n+3)^{-1}R^2$.

Using this circumstance, we write the second order correction as a product of $\delta\langle R^2 \rangle$ and a small factor:

$$\delta K(R_0) = \frac{\partial K}{\partial \langle R^2 \rangle} \delta\langle R^2 \rangle = \frac{\alpha Z}{6} \frac{\delta\langle R^2 \rangle}{R_0^2}[1 - P(n,y)], \quad (7)$$

where

$$P(n,y) = (\alpha Z)^2(P_1(n) - P_2(y))$$
$$+ \alpha Z(1+\varepsilon)\langle R^2 \rangle^{1/2}(P_3(n) - P_4(y)),$$

$$P_1(n) = \frac{2(n+3)(n+1)^2}{3n^3}\left[\frac{1}{5} - \frac{n+9}{(n+1)(n+3)(n+5)}\right.$$

$$+ \frac{3(2n+9)}{(n+1)^2(n+3)^2(2n+5)} - \frac{9}{(n+1)^3(n+3)^2(3n+5)}\right]$$

$$+ \frac{1}{2} - \frac{3}{4}\frac{n+3}{n+1} + \frac{1}{12}\frac{n+1}{n+3} + \frac{1}{2}\ln\frac{n+3}{n+1},$$

$$P_2(y) = \frac{1}{6}y^2 - \ln y,$$

$$P_3(n) = \frac{(n+3)^{1/2}(n+1)^{1/2}}{n^2}\left[\frac{1}{5} - \frac{6}{(n+1)(n+3)(n+5)}\right.$$

$$+ \frac{9}{(n+1)^2(n+3)^2(2n+5)}\right]$$

$$+ \frac{1}{4}\left(\frac{n+1}{n+3}\right)^{1/2} + \frac{3}{2}\left(\frac{n+3}{n+1}\right)^{1/2} - \frac{3}{4}\left(\frac{n+3}{n+1}\right)^{1/2}.$$

$$P_4(y) = \frac{1}{3}y + \frac{1}{y},$$

$$y = \frac{\langle R^2 \rangle^{1/2}}{R_0} = \sqrt{\frac{n+1}{n+3}}\frac{R}{R_0} \leqslant \sqrt{\frac{n+1}{n+3}}.$$

The second order correction term P(n, y) gives a contribution to the structure terms in $\delta K(R_0)$. For a nucleus with surface charge $(n = \infty)$ we have

$$P(\infty, y) = -(\alpha Z)^2\left(\frac{1}{30} + \frac{1}{6}y - \ln y\right)$$

$$+ \alpha Z(1 + \varepsilon)R\left(\frac{6}{5} - \frac{y}{3} - \frac{1}{6}\right)$$

$$\approx -\frac{(\alpha Z)^2}{5} \qquad (R_0 = R, \ y = 1).$$

For a nucleus with a uniform volume charge distribution $(n = 2)$ the main terms of P(n, y) lead to

$$P(2, y) \approx -0.22(\alpha Z)^2 \qquad (R_0 = R, \ y = \sqrt{3/5}).$$

Thus the relative variation in δK in going from the model of a nucleus with a uniform volume charge distribution to one with a surface charge distribution amounts to only $0.02(\alpha Z)^2$.

We must now determine $G_1(R_0) G_2(R_0)$. For this purpose it suffices to find the solutions of (1) in the region of small r outside the nucleus, i.e., in the region of the Coulomb potential. The solutions have the form

$$F = \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}bA_k\left(\frac{2Zr}{a_H}\right)^\sigma\left(1 + B(R_0)\frac{c}{b}\left(\frac{R_0}{r}\right)^{2\sigma}\right), \quad r \geqslant R_0;$$

$$G = A_k\left(\frac{2Zr}{a_H}\right)^\sigma\left(1 + B(R_0)\left(\frac{R_0}{r}\right)^{2\sigma}\right), \quad (8)$$

where

$$b = \frac{N+k+n'+2\sigma}{N+k-n'-2\sigma}, \quad c = \frac{N+k+n'}{N+k-n'}, \quad n' = \varepsilon N - \sigma,$$

$$N = \alpha Z/\sqrt{1-\varepsilon^2}, \qquad \sigma = \sqrt{k^2 - (\alpha Z)^2}, \qquad a_H = \hbar^2/me^2,$$

$$\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}b \approx \frac{\alpha Z}{k-\sigma}, \qquad \frac{c}{b} \approx \frac{k-\sigma}{k+\sigma} \qquad (\varepsilon \approx 1).$$

The quantity $B(R_0)$ is determined by joining $F/G = K$ near the boundary of the nucleus $(r = R_0)$:

$$B(R_0) = -\frac{\sqrt{(1-\varepsilon)/(1+\varepsilon)}\,b - K(R_0)}{\sqrt{(1-\varepsilon)/(1+\varepsilon)}\,c - K(R_0)}$$

$$\approx -\frac{k+\sigma}{k-\sigma}\frac{1 - (\alpha Z)^{-1}(k-\sigma)K(R_0)}{1 - (\alpha Z)^{-1}(k+\sigma)K(R_0)}, \quad (9)$$

where $K(R_0)$ is given by Eq. (5). In the most important case $k = -1$ we have in second approximation

$$K(R_0) \equiv K(y) = -\alpha Z\{\frac{1}{2} + \frac{1}{8}(\alpha Z)^2 - \frac{1}{6}y^2[1 - (\alpha Z)^2 P_0(n,y)]\},$$

$$P_0(n,y) = \frac{2}{3}\frac{(n+3)(n+1)^2}{n^3}\left[\frac{1}{5} - \frac{n+9}{(n+1)(n+3)(n+5)}\right.$$

$$+ \frac{3(2n+9)}{(n+1)^2(n+3)^2(2n+5)} - \frac{9}{(n+1)^3(n+3)^2(3n+5)}\right] \quad (9a)$$

$$- \frac{3}{4}\frac{n+3}{n+1} + \frac{1}{12}\frac{n+1}{n+3} + \frac{1}{2}\ln\frac{n+3}{n+1} + \ln y - \frac{1}{12}y^2.$$

A_k is a normalization constant. Its value depends on the behavior of the wave function in the entire region of r. If we replace the actual wave function of the electron in the self-consistent field of the atom by the relativistic Coulomb wave function, we find for A_k^2 the following expression:

$$A_k^2 = 4\pi R_\infty A_k' \left(\frac{N-n'-k}{\Gamma(2\sigma+1)}\right)^2 \frac{Z^2}{Z\alpha(n'+|k|)^3}$$

$$\approx 4\pi R_\infty A_k' \left(\frac{k-\sigma}{\Gamma(2\sigma+1)}\right)^2 \frac{Z^2}{Z\alpha n^3}, \quad (10)$$

where

$$A_k' = \frac{\Gamma(2\sigma+n'+1)\, n^3\, (1+\varepsilon)}{2\Gamma(n'+1)(N-k)\, N^{3+2\sigma}},$$

$R_\infty = 2\pi^2 me^4/h^3 c$ is the Rydberg constant and $n = n' + |k|$ is the principal quantum number. Neglecting the binding energy and the potential of the electron shells at the origin in comparison with the rest mass of the electron, we have $\varepsilon = 1$ and $n' \to \infty$, $N \to \infty$. In this approximation[4] we obtain

$$\lim_{n',\, N\to\infty} A_k = 1.$$

Using (4), (8), and (10), we can express the corresponding i.s. in the form*

$$\delta T_k = (Z^2/n^3)\, C_k;$$

$$C_k = 2R_\infty \left(\frac{k-\sigma}{\Gamma(2\sigma+1)}\right)^2 \left(\frac{2ZR_0}{a_H}\right)^{2\sigma}$$

$$\times\, [1 + B(R_0)]^2\, \frac{\delta K(R_0)}{\alpha Z} \quad (k < 0). \quad (11)$$

The isotope shift constants C_k are determined by the charge distributions of the nuclear isotopes and are practically independent of the state of the electron. In first approximation we have

$$C_k = R_\infty \left(\frac{\sigma+|k|}{\Gamma(2\sigma+1)}\right)^2 \left(\frac{2ZR_0}{a_H}\right)^{2\sigma} \frac{\delta\langle R^{2|k|}\rangle}{|k|(1+2|k|)R_0^{2|k|}} \quad (k < 0). \quad (12)$$

The ratio of the i.s. of the electron states $(n, j, |k|)$ and $(n, j, -|k|)$ is equal to

$$\frac{\delta T_{|k|}}{\delta T_{-|k|}} = \left(\frac{\alpha Z}{|k|+\sigma}\right)^2, \quad \gamma = \frac{\delta T_{np_{1/2}}}{\delta T_{ns}} = \left(\frac{\alpha Z}{1+\sigma}\right)^2.$$

For Z = 80 we have $\gamma = 0.104$ and the term $\delta T_{np_{1/2}}$ must be included in the calculation of the spectral lines.†

We consider separately the i.s. for the ns electron:

$$\delta T_{ns} = A^2 Z n^{-3} C \text{ cm}^{-1} \quad (A \equiv A_1').$$

The i.s. constant C has in second approximation the form

$$C_{\text{theor}} = \frac{R_\infty}{3}\left(\frac{1+\sigma}{\Gamma(1+2\sigma)}\right)^2 \left(\frac{2Z\langle R^2\rangle^{1/2}}{a_H}\right)^{2\sigma} f(n, y, Z) \frac{\delta\langle R^2\rangle}{\langle R^2\rangle}, \quad (13)$$

where

$$f(n, y, Z) = \left(1 - \frac{1-\sigma}{1+\sigma}\frac{\alpha Z + (1+\sigma)K(y)}{\alpha Z + (1-\sigma)K(y)}\right)^2 y^{2(1-\sigma)}\, [1 - P(n, y)],$$

$\sigma = \sqrt{1-(\alpha Z)^2}$, $P(n, y)$ is determined by (7) and $K(y)$ by (9a).

The dependence of the i.s. constant C_{theor} on the details of the nuclear charge distribution (6) for fixed values of $\langle R^2 \rangle$ is given by the functions $f(n, y, Z)$. We have carried out calculations for three cases: nonuniform charge distribution (n = 1), uniform volume charge distribution (n = 2), and uniform surface charge distribution (n = ∞). The results, given in Table I, indicate a very weak dependence of the i.s. on the details of the nuclear charge distribution (variations of less than 1%). The error in the determination of C from the experimental data on the i.s. is significantly larger. We can therefore assume in formula (13) that $f(n, y, Z) = f(Z)$ (see the figure).* Thus C_{theor} depends, apart from Z, only on the nuclear parameter $\langle R^2 \rangle$ and the relative variation $\delta\langle R^2\rangle/\langle R^2\rangle$ in a series of isotopes.

As is seen from Table I and the figure, $f(Z) \approx 1$ with an error not larger than 3.5% for a large region of nuclei with Z $\leq$ 80. In this region the formula for C_{theor} takes on the simplest form

$$C_{\text{theor}} = \frac{R_\infty}{3}\left(\frac{1+\sigma}{\Gamma(1+2\sigma)}\right)^2 \left(\frac{2Z\langle R^2\rangle^{1/2}}{a_H}\right)^{2\sigma} \frac{\delta\langle R^2\rangle}{\langle R^2\rangle}. \quad (13a)$$

In references [5] and [6] the i.s. constant C_{theor} has been determined for a definite radial charge distribution of the nucleus, given by the trapezoidal function

$$\rho(r) = \begin{cases} \rho_0 & 0 < r < c-z \\ \rho_0(c+z-r)/2z, & c-z < r < c+z, \\ 0 & c+z < r \end{cases}$$

where

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*The product $G_1(R_0)\, G_2(R_0)$ is replaced by $G_1^2(R_0)$, since the difference between $G_1(R_0)$ and $G_2(R_0)$ gives rise to a negligible contribution.

†A particularly large i.s. $\delta T_{np_{1/2}}$ will be observed for the spectral line emitted in the transition $(n+1)s \to np_{1/2}$.

*The i.s. constant should not depend on the value of R_0. i.e., on y. The dependence of $f(n,y,Z)$ on y indicates that the applicability of our approximation is limited to the region of small R_0 close to R, the radius of the nucleus. Since $R_0 > R = \sqrt{(n+3)/(n+1)}\, \langle R^2\rangle^{1/2}$, we choose, e.g., $y = \frac{2}{3}$ (column 3 of Table I and figure).

Table I. The function $f(n, y, Z)$ for different nuclear charge distributions

	$\rho_n(r)$	$y=^1/_3$	$y=^2/_3$		$\rho_n(r)$	$y=^1/_3$	$y=^2/_3$
$Z = 56$	$n = 1$	0.992	0.996	$Z = 90$	$n = 1$	0.863	0.934
	$n = 2$	0.993	0.998		$n = 2$	0.866	0.938
	$n = \infty$	0.995	1.000		$n = \infty$	0.869	0.942
$Z = 80$	$n = 1$	0.921	0.964	$Z = 100$	$n = 1$	0.781	0.885
	$n = 2$	0.923	0.967		$n = 2$	0.784	0.889
	$n = \infty$	0.927	0.971		$n = \infty$	0.788	0.894
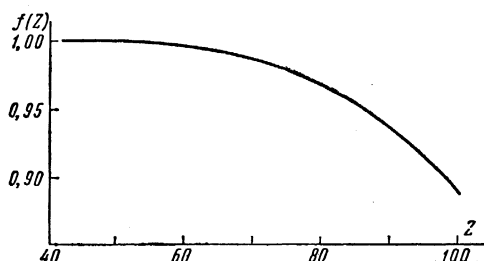
Behavior of the function $f(Z)$ in the region of $Z \leq 100$. For $Z \leq 80$, $f(Z) = 1$ with an error of less than 3.5%. The calculation was done for a uniformly charged nucleus ($n = 2$). The value of the parameter y was chosen as $y = 2/3$.

$$\rho_0 = \frac{3Ze}{4\pi c^3}\left[1 + \left(\frac{z}{c}\right)^2\right]^{-1}, \qquad c = 1.05 \cdot 10^{-13} A^{1/3} \text{ cm},$$

$$z = 1.53 \cdot 10^{-13} \text{ cm}.$$

It is asserted in the above-mentioned references that such a choice of a nonuniform nuclear charge distribution allows one to achieve agreement between theory and experiment for the i.s. without recourse to the hypothesis of the compressibility of the nucleus. However, it can be shown that formula (13a) gives the same results if we substitute in it $\langle R^2 \rangle$ for the trapezoidal distribution,

$$\langle R^2 \rangle = \frac{2}{5}(c+z)^2\left[1 + \left(\frac{c-z}{c+z}\right)^4\right] / \left[1 + \left(\frac{c-z}{c+z}\right)^2\right]$$

with the above-mentioned numerical values for the parameters c and z. The calculation of $\langle R^2 \rangle$ shows that the choice of numerical values for c and z of references [5] and [6] leads to values of $\langle R^2 \rangle$ which are too low by 10 to 15% for various nuclei, as compared to the values inferred from the experiments on mesic atoms and the scattering of fast electrons on nuclei. Thus the trouble does not lie in the shape of the radial distribution but in an incorrect choice of the numerical values of the parameters determining $\langle R^2 \rangle$ and $\delta\langle R^2 \rangle / \langle R^2 \rangle$.

2. COMPRESSIBILITY OF DEFORMED NUCLEI

Let us now determine the values of $\langle R^2 \rangle$ and $\delta\langle R^2 \rangle / \langle R^2 \rangle$ for deformed nuclei without assuming

a definite radial distribution of the nuclear charge. For this purpose we write the nuclear charge distribution in the form

$$\rho(r, \vartheta, \varphi) = \rho_0 f(r/r_1), \tag{14}$$

where ρ_0 is the charge density at the center of the nucleus and r_1 is the radius to the nuclear surface. For nuclei with ellipsoidal deformations [7]

$$r_1 = \bar{r}_1\left[1 + \sum_\mu \alpha_\mu Y_{2\mu}(\vartheta, \varphi)\right],$$

where $\bar{r}_1$ is the radius of the nucleus averaged over the angles. In a special coordinate system

$$\alpha_0 = \beta\cos\gamma, \qquad \alpha_1 = \alpha_{-1} = 0, \qquad \alpha_2 = \alpha_{-2} = 2^{-1/2}\beta\sin\gamma.$$

The radial charge distribution is given by the function

$$f(x) > 0 \text{ for } x < 1; \qquad f(x) = 0 \text{ for } x > 1.$$

For a uniform volume charge distribution, $f(x) = 1$ for $x < 1$. We shall not specify the form of the function $f(x)$.

Let us determine $\langle R^2 \rangle$ for a nuclear charge distribution chosen in the form (14):

$$\langle R^2 \rangle = \int \rho(r, \vartheta, \varphi) r^2 dv \Big/ \int \rho(r, \vartheta, \varphi) dv$$

$$= \bar{r}_1^2 s \oint [1 + \Sigma\alpha_\mu Y_{2\mu}]^5 d\Omega \Big/ \oint [1 + \Sigma\alpha_\mu Y_{2\mu}]^3 d\Omega$$

$$\approx \bar{r}_1^2 s\left[1 + \frac{7}{4\pi}\beta^2 + \frac{9}{14\pi}\sqrt{\frac{5}{4\pi}}\beta^3 f(\gamma) + \cdots\right],$$

where

$$s = \int_0^1 f(x) x^4 dx \Big/ \int_0^1 f(x) x^2 dx; \qquad f(\gamma) = \cos\gamma(1 - 4\sin^2\gamma).$$

We express $\bar{r}_1$ in terms of the radius of the volume-equivalent sphere, R_0:

$$\bar{r}_1^3\left\{1 + \frac{3}{4\pi}\beta^2 + \frac{1}{14\pi}\sqrt{\frac{5}{4\pi}}\beta^3 f(\gamma)\right\} = R_0^3. \tag{15}$$

The corresponding formula for $\langle R^2 \rangle$ takes the form*

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*The formula is also useful for dynamical deformations. For vibrational nuclei β^2 must be replaced by $\langle\beta^2\rangle$ and γ must be set equal to $\gamma = 30°$.

$$\langle R^2 \rangle = R_0^2 s \left[1 + \frac{5}{4\pi} \beta^2 + \frac{25}{42\pi} \sqrt{\frac{5}{4\pi}} \beta^3 f(\gamma) + \dots \right]. \quad (16)$$

Here s is a parameter of the radial charge distribution. For a uniform distribution $s = \frac{3}{5}$, and formula (15) takes on the usual form.[8]

We cannot separate the parameters R_0 and s without making specific assumptions about the radial distribution of the nuclear charge. Let us therefore introduce the radius of the equivalent uniform distribution

$$\frac{3}{5} R_{eq}^2 = R_0^2 s.$$

Choosing $R_{eq} = r_0 A^{1/3}$, where $r_0 = $ const, we thus regard the nucleus with the equivalent uniform charge distribution as incompressible.

Dropping the hypothesis of the incompressibility of the equivalent uniform charge distribution under deformations, we must assume that

$$R_{eq}^2 = R_{0\,eq}^2 f(\beta), \quad (17)$$

where $R_{0\,eq}$ does not depend on the deformation parameter β ($R_{0\,eq} = 1.20\,A^{1/3} \times 10^{-13}$ cm). The function $f(\beta)$ is chosen in the simplest form

$$f(\beta) = 1 + \xi \beta^2. \quad (17a)$$

The value of the coefficient ξ is determined by comparison with experiment. The dependence of R_{eq} on the deformations can be explained either by the compressibility of the actual nucleus under deformations [$R_0 = R_0(\beta)$] or by a change in the radial distribution under deformations of the nucleus [$s = s(\beta)$]. We shall not separate these two causes in our discussion.* Using (17) and (17a), we obtain for $\langle R^2 \rangle$†

$$\langle R^2 \rangle = \frac{3}{5} R_{0\,eq}^2 \left[1 + \left(\frac{5}{4\pi} + \xi \right) \beta^2 + \frac{25}{42\pi} \sqrt{\frac{5}{4\pi}} \beta^3 f(\gamma) \right]. \quad (18)$$

The final expression for the i.s. constant C_{theor} in the region of deformed nuclei has the form

$$C_{theor} = \frac{2}{3} R_\infty \left(\frac{3}{5} \right)^\sigma \left(\frac{1+\sigma}{\Gamma(1+2\sigma)} \right)^2 \left(\frac{2Z R_{0\,eq}}{a_H} \right)^{2\sigma} L(\beta, \gamma),$$

where

$$L(\beta, \gamma) = \left[1 + \left(\frac{5}{4\pi} + \xi \right) \left(\frac{\beta_1 + \beta_2}{2} \right)^2 \right] \frac{\delta R_{0\,eq}}{R_{0\,eq}} + \frac{5}{8\pi} [(1 + 0.8\pi\xi)$$

$$\times (\beta_1^2 - \beta_2^2) + 0.3\,(\beta_1^3 f(\gamma_1) - \beta_2^3 f(\gamma_2))]. \quad (20)$$

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*Ionesco-Pallas[9] has recently discussed the i.s. for a definite radial distribution of the nuclear charge depending on the deformation parameter β.

†Corrections for the compressibility of the nucleus under deformations are introduced only in the term $\sim\beta^2$. However, this should not lead to a great error, since the term $\sim\beta^3$ is small. Its relative contribution to <R²> is $\leq$ 1% and to δ <R²> not more than 10%.

For spherical nuclei $L = \delta R_{0\,eq}/R_{0\,eq}$.

Let us now turn to the determination of the relative change in the equivalent radius $\delta R_{0\,eq}/R_{0\,eq}$ under variations of the number of neutrons, N, in the nucleus. We have

$$\frac{\delta R_{0\,eq}}{R_{0\,eq}} = \frac{\delta N}{R_{0\,eq}} \frac{\delta R_{0\,eq}}{\delta N} = \frac{\delta N}{R_{0\,eq}} \frac{\delta R_{0\,eq}}{\delta A} \eta = \frac{\delta A}{3A} \eta \zeta, \quad (21)$$

where

$$\eta = \frac{\delta R_{0\,eq}}{\delta N} \bigg/ \frac{\delta R_{0\,eq}}{\delta A}, \qquad \zeta = \frac{3A}{R_{0\,eq}} \frac{dR_{0\,eq}}{dA}.$$

For the model of the nucleus as an incompressible drop we have $\eta = 1$ and $\zeta = 1$.

The data on the scattering of fast electrons on nuclei speak in favor of $\rho(r) = $ const, except in the surface region, where $\rho(r)$ goes smoothly to zero.[7] The existence of a surface region has practically no effect on the relation $R_{0\,eq} \sim A^{1/3}$, but changes the value of ζ. In evaluating the experimental data with the help of the Fermi model, ζ is expressed in the following form:[2]

$$\zeta = \frac{3A}{R_{0\,eq}} \frac{dR_{0\,eq}}{dA} = \frac{1 - 1.929 A^{-2/3} + 4.687 A^{-4/3}}{1 + 1.929 A^{-2/3} - 1.562 A^{-4/3}}, \quad (22)$$

which implies $\zeta = 0.88 \pm 0.02$ in the region $125 < A < 218$. If the experimental data are evaluated on the basis of the trapezoidal model, one also obtains $\zeta = 0.87 \pm 0.03$ for this region of values of A. Thus the value of ζ does not depend on the specific shape of $\rho(r)$ but is determined by the experimental data on the dimensions of the nuclear surface region. The coefficients of regular compressibility, η,[2] and deformation compressibility, ξ, of the nucleus are determined by comparing C_{theor} and C_{exp} in the region of spherical and deformed nuclei. We assume that η and ξ depend weakly on A and can be regarded as constant in the region $134 < A < 210$.

In order to be able to use formula (20), we must know the values of the deformation parameters β and γ. The parameter β can be calculated from the data on the intrinsic, quadrupole moment of the nucleus Q_0, which has been measured with an accuracy of better than 5% by Coulomb excitation. For this purpose we give the formula expressing the dependence of Q_0 on β with an accuracy up to terms of order β^2:

$$Q_0 = \int \rho(r, \vartheta, \varphi)\, r^2 (3 \cos^2\vartheta - 1)\, dv$$

$$= 4 \sqrt{\frac{\pi}{5}} \int \rho(r, \vartheta, \varphi)\, r^4 Y_{20}(\vartheta)\, dr \sin\vartheta\, d\vartheta\, d\varphi.$$

Choosing $\rho(r, \vartheta, \varphi)$ in the form (15), we obtain with the help of (16)

Table II. Deformation parameter β

Nucleus	$Q_0, 10^{-24}$ cm^2	β	Nucleus	$Q_0, 10^{-24}$ cm^2	β	Nucleus	$Q_0, 10^{-24}$ cm^2	β
Sm^{150}	3.65 [11]	0.180	Gd^{158}	7.41 [11]	0.325	Hf^{180}	6.61 [15]	0.243
Sm^{152}	5.86 [11]	0.278	Gd^{160}	7.65 [11]	0.332	W^{182}	6.34 [15]	0.226
Sm^{154}	6.83 [11]	0.316	Yb^{170}	7.48 [11]	0.290	W^{184}	6.04 [15]	0.215
Eu^{151}	2.73 *	0.134	Yb^{171}	7.98 [14]	0.306	W^{186}	5.99 [15]	0.212
Eu^{153}	6.94 [12]	0.317	Yb^{172}	7.72 [11]	0.296	Re^{185}	6.28 [16]	0.219
Gd^{154}	5.88 [11]	0.268	Yb^{173}	7.77 [14]	0.296	Re^{187}	5.60 [16]	0.196
Gd^{155}	6.50 [13]	0.292	Yb^{174}	7.72 [11]	0.294	Os^{186}	5.50 [16]	0.191
Gd^{156}	6.79 [11]	0.303	Yb^{176}	7.60 [11]	0.287	Os^{188}	5.30 [16]	0.184
Gd^{157}	6.60 [13]	0.294	Hf^{178}	6.85 [15]	0.253	Os^{190}	5.06 [16]	0.174
						Os^{192}	4.54 [16]	0.156

*Computed from the ratio Q_0 (Eu^{151})/Q_0 (Eu^{153}) = 0.393.[17]

$$Q_0 = \frac{3}{\sqrt{5\pi}} ZR^2_{eq}\beta \left\{ \cos\gamma + \frac{2}{7}\sqrt{\frac{5}{\pi}}\beta(1 - 2\sin^2\gamma) + \ldots \right\}$$

$$= \frac{3}{\sqrt{5\pi}} ZR^2_{0eq}\beta \left\{ \cos\gamma + 0.36\beta(1 - 2\sin^2\gamma) + \ldots \right\}. \quad (23)$$

The fact that formulas (17) and (17a) violate the condition of incompressibility of the nucleus under deformations affects only terms of order β^3, which have been omitted in (23). Within our limits of accuracy* we can thus in (23) replace R^2_{eq} by R^2_{0eq}, a quantity which does not depend on the deformation parameter β. Within the framework of our model it is not possible to determine the terms of order β^3 more precisely. For axially symmetric prolate nuclei we have†

$$Q_0 = \frac{3}{\sqrt{5\pi}} ZR^2_{0eq}\beta \left\{ 1 + 0.36\beta + \ldots \right\}. \quad (23a)$$

In Table II we list the values of the deformation parameter β for 28 nuclei, as computed with the help of formula (23a) ($R_{0\,eq} = 1.20 A^{1/3} \times 10^{-13}$ cm).

3. COMPARISON WITH EXPERIMENT

The constant C is found from the experimental data on the i.s. with the help of the formula

$$C_{exp} = \delta T_{ns} Z / A^2 \pi a^3_H \psi^2_s(0), \quad (24)$$

where $\psi_s(0)$ is the value of the nonrelativistic wave function at the position of the nucleus.

Setting $A^2 = 1$, we obtain the expression given by Rosenthal and Breit,[4] where the relativistic

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*The error in the estimate of β due to the neglect of terms $\sim\beta^3$ in (23) is not larger than 3 to 5%, i.e., of the same order as the experimental error in the determination of the intrinsic quadrupole moment of the nucleus.

†In a number of papers ([10] and others) the following formula is given:

$$Q_0 = (3/\sqrt{5\pi})ZR^2_{0\,eq}\beta\{1 + 0.16\beta + \ldots\} \quad (23b)$$

The discrepancy between (23a) and (23b) amounts to a few percent for strongly deformed nuclei. It can be shown that the term $\sim\beta^2$ has been determined incorrectly in the derivation of formula (23b).

wave function is normalized by its asymptotic form. The function $\psi_s(0)$ can be treated as a wave function in some effective field of the nucleus and the electron shells and the semi-empirical Fermi-Segrè-Goudsmit formula[18] can be used. Assuming $\epsilon < 1$, we find the following expression for $A^2(n')$ in the Coulomb field:

$$A^2(n') = \frac{(n' + 1)^3\Gamma(n' + 2\sigma + 1)(N + 1 - n')^2(N + n' + \sigma)}{\Gamma(n' + 1)(1 + \sigma)^2 2 N^{3 + 2\sigma}(N + 1)}.$$

In the effective field of the nucleus and the electron shells the number n' can be regarded as an effective radial quantum number equal to $n_a - 1$. The calculations show that for 56 < Z < 82 the coefficient A^2 varies within the limits 0.80 < A < 1.10 as n' changes from 0 to 7. Changing n' from 0.5 to 7 induces variations of A^2 within the limits 1 < A < 1.10. It follows that a sufficiently accurate estimate will be $A^2 = 1 \pm 0.1$.

The coefficient in (24) can also be determined in a different manner, by using the dependence of the hyperfine structure constant of the s electron term of an odd isotope on the magnetic moment of the nucleus:[18]* $a_s = GH(n')\pi a^3_H\psi^2_s(0)/Z$, where

$$G = \frac{8}{3} R_\infty \alpha^2 Z \frac{m}{M} \frac{\mu}{I} F_r(j, Z)(1 - \delta)(1 - \epsilon),$$

$$H(n') = (n' + 1)^3 [2(n' + \sigma) + N]/3N^4.$$

The formula for C_{exp} can be rewritten

$$C_{exp} = \frac{\delta T_s}{\alpha(n')a_s/G},$$

$$\alpha(n') = \frac{A^2}{H}$$

$$= \frac{3}{2} \frac{\Gamma(n' + 2\sigma + 1)}{\Gamma(n' + 1)(1 + \sigma)^2} \frac{(N + n' + \sigma)(N + 1 - n')^2}{[N + 2(n' + \sigma)](N + 1)N^{2\sigma - 1}}. \quad (24a)$$

The calculations show that the quantity $\alpha(n')$ increases smoothly from 0.95 to 1 (Z = 55) and

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*Usually one leaves out the coefficient H(n') in the expression for the hyperfine structure constant. This corresponds to the approximation $\epsilon = 1$, $n' \to \infty$, $N \to \infty$. Then $\lim H(n') = 1$. Actually, $\epsilon < 1$ and $H(n')$ differs from unity by terms of order $(\alpha Z)^2$, which may introduce errors of the order of a few per cent for heavy nuclei.

Table III. Isotope shifts

Ele-ment	z	$A_1 - A_2$	C_{theor}, 10^{-3} cm^{-1}				βC_{exp}, 10^{-3} cm^{-1}
			$\zeta=1, \eta=1, \xi=0$	ζ (22), $\eta=0.7$ $\xi=0$	$\eta=0, \xi=0$	ζ (22), $\eta=0$ $\xi=-\frac{5}{8\pi}$	
Ba	56	134—136	102.2	62		62	44±9 [24]
		136—138	101.7	62		62	67±13 [24]
Sm	62	150—152	486	428	334	275	312±65 [25]
		152—154	327	268	173	189	170±35 [25]
Eu	63	151—153	818	754	657	458	450±50 [25]
		154—156	353	285	177	203	162±26 [26]
		155—157	185	118		110	109±18 [26]
Gd	64	156—158	300	233		174	121±19 [26]
		158—160	218	152		128	125±20 [26]
Yb	70	170—172	312	215		185	110±10 [27]
		171—173	175	78		109	112±12 [27]
		172—174	238	142		144	99±8 [27]
		174—176	196	100		121	94±5 [27]
Hf	72	178—180	206	97		132	144±13 [25]
W	74	182—184	234	111		152	132±23 [28]
		184—186	301	179		188	117±20 [28]
Re	75	185—187	149	18		106	157±40 [25]
Os	76	186—188	316	177		200	176±27 [25]
		188—190	291	152		186	150±23 [25]
		190—192	235	98		157	130±20 [25]
Hg	80	198—200	479.7	301		301	243±27 [25]
		200—202	477.6	300		300	270±30 [25]
		202—204	475.5	299		299	267±30 [25]
Pb	82	204—206	548.0	345		345	280±30 [25]
		206—208	546.2	344		344	315±35 [25]

from 0.92 to 1 (Z = 80) as n' is increased from 0 to 6.

Thus we may estimate $\alpha(n')$ to be equal to $\alpha(n') = 0.95 \pm 0.05$. If sufficiently accurate experimental data are used for a_s and μ (error ~ 1%), formula (24a) is more reliable than (24). The i.s. constant C_{exp} is determined from the i.s. of the spectral line $\delta \nu$ up to a factor β which allows for the contribution from the electron shells to the i.s. Numerous computations [18-23] indicate that $\beta \approx 1$ (with an error of 10 to 15%, usually in the direction of a lower value of β).

The total error in the determination of C_{exp} from the experimental data on the i.s. of the spectral lines $\delta \nu$ is equal to 15 to 25%. This estimate is supported by the fact that the values of C_{exp} obtained from different spectral lines of an element show a spread of the same size. [18]

In Table III we make a comparison of the i.s. constants C_{exp} and C_{theor} for 25 pairs of isotopes. C_{theor} is computed from formulas (19), (20), and (21) with different assumptions about the values of the nuclear compressibility parameters η, ζ, and ξ. It follows from the analysis of Table III that it is necessary to introduce, in addition to ζ [formula (22)], the coefficient of regular compressibility $\eta = 0.7$ (column 5) in order to achieve agreement between the calculated and experimental values of C in the region of spherical (Ba, Hg, Pb) and equally deformed nuclei (Cd$^{155-157}$, Yb$^{172-174}$). [2,21] However, for isotope pairs with

large differences in the deformation it is not sufficient to decrease the regular part of the i.s. in order to obtain agreement between theory and experiment. We must also decrease the deformation part of the i.s. Illustrative examples of this are the i.s. of the nuclei Eu$^{151-155}$ and others (column 6), where only the deformation isotope shifts are given. The latter are not smaller than the total experimental i.s. C_{exp}.

It is therefore necessary to introduce the compressibility constant of the nucleus for deformations $\xi < 0$. If we set $\eta = 0.7$ and $\xi = -5/8\pi$, we obtain agreement between the calculated and experimental values in almost all cases (23 pairs of isotopes, column 7).*

The computations leading to Table III have all been carried out for axially symmetric nuclei ($\gamma = 0$). According to the model of Davydov and Filippov, [29] some of the nuclei considered by us are not axially symmetric. However, the effect of the deviations from axial symmetry on the i.s. is small (not larger than 10%) and is not observable in practice in view of the experimental errors in the i.s. itself as well as in the values of the deformation parameters. We illustrate this on the example of the relative i.s. of the isotopes of Os (Table IV).

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*The calculated values C_{theor} for the isotope pairs Yb^{170}, Yb^{172} and W^{184}, W^{186} are too high, apparently owing to the use of incorrect values for the deformation parameter β for the nuclei Yb^{170} and W^{186}.

Table IV. Relative isotope shifts of the isotopes of Os*

$(\eta = 0.7, \quad \xi = 0.894)$

Isotope of Os	γ	Relative i. s.		γ	Relative i. s.			Relative i. s.		Experiment[25]
		$\xi = -\frac{5}{8\pi}$	$\xi=0$		$\xi = -\frac{5}{8\pi}$	$\xi=0$		$\xi = -\frac{5}{8\pi}$	$\xi=0$	
186	0			15.9						
		1.27	1.81		1.31	2.00	Vibrational Nuclei	1.36	2.30	1.35
188	0			18.5						
		1.18	1.55		1.26	1.78		1.22	1.83	1.15
190	0			21.9						
		1	1		1	1		1	1	1
192	0			24.5						

*The relative i. s. were computed by formulas (20) and (21) for three nuclear models: axially symmetric nucleus ($\gamma=0$), non-axially symmetric nucleus,[29] and vibrational nucleus, with different assumptions about the value of the nuclear compressibility constant for deformations: $\xi=0$ and $\xi=-5/8\pi$. The i. s. of the pair Os^{190}, Os^{192} was taken as unity.

Using the value $\xi = -5/8\pi$ we obtain for the mean square radius of the deformed nuclei

$$\langle R^2 \rangle = \tfrac{3}{5} R_{0\,eq}^2 \left[1 + \tfrac{5}{8\pi} \beta^2 + \tfrac{25}{42\pi} \sqrt{\tfrac{5}{4\pi}} \beta^3 f(\gamma) \right], \quad (18a)$$

which exhibits a slower increase of $\langle R^2 \rangle$ with β than is the case for incompressible nuclei with a radial charge distribution which is independent of the deformation. This can be interpreted as an increase in the concentration of protons at the center of the deformed nucleus as compared to spherical nuclei.

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