

MAGNETIC FLUX QUANTIZATION IN A SUPERCONDUCTING CYLINDER

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The problem of magnetic flux quantization is considered for the case of a superconducting cylinder of arbitrary wall thickness situated in an external field.

THE fact that the magnetic flux through a superconducting cylinder or ring should be quantized was pointed out some time ago^[1], but it is only quite recently that the corresponding experiments have been carried out.^[2,3] These experiments are of interest from several points of view, especially in connection with the fact that the "quantum of flux" was found to be $hc/2e = 2.07 \times 10^{-7}$ G-cm², and not, as London had predicted,^[1] hc/e .

It follows from the principles of gauge invariance^[1,4,5] that the equilibrium magnetic flux through an aperture in a bulky superconductor is $\Phi = hcn/e^*$, where n is an integer and e^* is the effective charge of the current-carrying particles.* The charge e^* need not depend upon pressure, temperature, amount of impurity, etc., (see^[6] for analogous considerations concerning the effective mass m^* in He II). The charge e^* is therefore universal; i.e., $e^* = ke$, where k is an integer and e is the electronic charge. The fact that in superconductors $e^* = 2e$ is attributable, as has previously been pointed out,^[4,5,7] to the formation of pairs.

The question of the total flux through an aperture in a massive superconductor is not the only one that arises from the analysis of experiments. The relatively small size of the "quantum of flux" makes it necessary to use comparatively thin cylinders^[2,3] whose wall thickness will be comparable with the penetration depth δ . The problem thus arises of finding the field, current, and derived quantities such as the magnetic flux, for a cylinder of arbitrary wall thickness in an external

field. It is necessary also to determine the values of the external field at which changes occur in the equilibrium flux through the cylinder. In order to answer these questions it is necessary to make use of the equations of electrodynamics for superconductors. We have done this for the case in which equations of the local type, as derived by Landau and the author,^[8] are applicable (see also^[7,9,10]). For finely-dispersed samples, among which are generally included sintered or evaporated layers and films, the local approach and the cited equations have an extremely broad range of applicability.^[7,10]

The free energy density for a superconductor in a field is, from^[8,9] (taking the energy density in the normal phase to be zero)

$$F_{sH} = F_s + \frac{H^2}{8\pi} = \frac{H_{cb}^2}{8\pi} \left\{ |\Psi_0|^4 - 2|\Psi_0|^2 + \frac{2\delta_0^2}{\kappa^2} \left| \nabla \Psi_0 + i \frac{e^*}{\hbar c} \mathbf{A} \Psi_0 \right|^2 \right\} + \frac{H^2}{8\pi} \quad (1)$$

where H_{cb} is the critical field for the bulk metal, δ_0 is the weak-field penetration depth (for $|\Psi| = 1$), and $\kappa = \sqrt{2} e^* H_{cb} \delta_0^2 / \hbar c$.

The equations for Ψ_0 and \mathbf{A} are obtained from the condition that $\int F_{sH} dV$ shall be a minimum (div \mathbf{A} is taken equal to zero)

$$(\nabla + i \frac{e^*}{\hbar c} \mathbf{A})^2 \Psi_0 = \frac{\kappa^2}{\delta_0^2} (1 - |\Psi_0|^2) \Psi_0, \quad (2)$$

$$\text{rot rot } \mathbf{A} = \frac{4\pi}{c} \mathbf{j}_s, \quad \mathbf{j}_s = -\frac{c}{4\pi\delta_0^2} \left\{ -\frac{i\hbar c}{2e^*} (\Psi_0^* \nabla \Psi_0 - \Psi_0 \nabla \Psi_0^* + |\Psi_0|^2 \mathbf{A}) \right\}. \quad (3)^*$$

Expressing Ψ_0 in the form $\Psi_0 = |\Psi_0| e^{-i\kappa}$ we arrive, after integrating about a closed contour in the superconductor under the condition $|\Psi_0| = \text{const}$, at the relation

$$\Phi = \frac{hcn}{e^*} - \frac{4\pi\delta_0^2}{c|\Psi_0|^2} \oint \mathbf{j}_s ds, \quad n = 0, \pm 1, \pm 2, \dots \quad (4)$$

*rot = curl.

*Some doubt arises concerning the generality of the treatment by Byers and Yang:^[4] the possibility of finding energy levels, assuming $\mathbf{B} = \text{curl } \mathbf{A} = 0$ everywhere, is not obvious, owing to the presence of boundary conditions. If one were to follow Byers and Yang, then quantization of flux should also occur in a ferromagnetic medium for $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M} = 0$ (for a cylinder with spontaneous magnetization M_0 , induced magnetization being neglected, $\mathbf{B} = 0$ in the metastable but, in principle, possible case $\mathbf{H} = -4\pi M_0$).

(the flux $\Phi = \int \mathbf{H}d\mathbf{S} = \oint \mathbf{A}_s ds$, and, due to the constancy of Ψ_0 , we have $\oint \nabla \chi ds = 2\pi n$).

Within the thickness of the superconductor $j_s = 0$ and $\Phi = hcn/2e$ (here and subsequently, $e^* = 2e$, while the sign has already been so chosen that for an electron $e > 0$). This last result is somewhat more general than Eq. (4), since $j = 0$ automatically for equilibrium within the thickness of the superconductor. In the weak-field region [with $H \ll H_{Cb}$ for the bulk metal, with accuracy to terms of order $\sim (H/H_{Cb})^2$] one can also set $|\Psi_0| = 1$. In the experiments performed by Deaver and Fairbank^[2] and by Doll and Näbauer,^[3] as well as in most other possible cases (among them those involving thin-walled cylinders), the fields can be regarded as weak.

Let us consider a circular cylinder (with radii r_1 and $r_2 > r_1$) in a homogeneous external field H_2 parallel to its axis: the field within the cylinder $H_1 = H (r \leq r_1)$. For the axially-symmetric case $j_r = j_z = 0$, $j_\varphi = j(r)$, $\Psi_0 = |\Psi_0(r)| \exp(-in\varphi)$, and one can take

$$A_r = A_z = 0, \quad A_\varphi = A(r), \quad H = H_2 = \frac{1}{r} \frac{\partial}{\partial r} (rA);$$

in accordance with Eq. (3), therefore

$$\frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} (rA) \right] = \frac{1}{\delta_0^2} \left(A - \frac{hcn}{2er} \right) |\Psi_0|^2.$$

The solution of this equation has been obtained for a vortex line in^[6,11]. Here, we shall confine ourselves to the weak-field case ($|\Psi_0| = 1$). Then, for a hollow cylinder

$$A = \frac{hcn}{2e\delta_0\xi} + A', \quad A' = \delta_0 \{ aI_1(\xi) + bK_1(\xi) \};$$

$$H = aI_0(\xi) - bK_0(\xi),$$

$$a = f(\xi_2, \xi_1) [H_2 K_0(\xi_1) - H_1 K_0(\xi_2)];$$

$$b = f(\xi_1, \xi_2) [H_2 I_0(\xi_1) - H_1 I_0(\xi_2)],$$

$$f(\xi_2, \xi_1) = [I_0(\xi_2) K_0(\xi_1) - I_0(\xi_1) K_0(\xi_2)], \quad r = \delta_0 \xi, \quad (5)$$

where I_n and K_n are the familiar Bessel functions. At the inner boundary (where $r_1 = \delta_0 \xi_1$) the potential $A(\xi_1)$ must be equal to the potential $A(\xi_1) = H_1 \delta_0 \xi_1 / 2$ in the interior region, which imposes a specific condition upon the field $H_1 = H_1(H_2, n, \xi_2, \xi_1)$. The magnetic flux through a contour of radius ξ is

$$\Phi = 2\pi \delta_0 \xi A(\xi) = \frac{hcn}{2e} + 2\pi \delta_0 \xi A'(\xi).$$

We shall give here only the formulas corresponding to the case in which $\xi_1 \gg r_1/\delta_0 \gg 1$ and one can set

$$K_n = \sqrt{\pi/2\xi} e^{-\xi} \quad \text{and} \quad I_n = \sqrt{1/2\pi\xi} e^\xi.$$

Then (hereafter, $d = \xi_2 - \xi_1$)

$$H_1 = \left[\frac{hcn}{e\delta_0^2 \xi_1^2} + \frac{2\sqrt{\xi_1 \xi_2} H_2}{\xi_1^2 \text{sh } d} \right] \left[1 + \frac{2}{\xi_1} \text{cth } d \right]^{-1}, \quad (6)^*$$

$$\Phi(\xi) = \frac{hcn}{2e} + 2\pi \delta_0^2 \sqrt{\xi} \frac{H_2 \sqrt{\xi_2} \text{ch}(\xi - \xi_1) - H_1 \sqrt{\xi_1} \text{ch}(\xi_2 - \xi)}{\text{sh } d}.$$

The significance of these expressions is clear. In particular, for a thick-walled cylinder ($d \gg 1$) the flux

$$\Phi(\xi_1) \approx \frac{hcn}{2e} \left(1 - \frac{2\delta_0}{r_1} \right)$$

and, with exponential accuracy, $\Phi(\xi) = hcn/2e$ (for $|\xi - \xi_{1,2}| \gg 1$). For a thin cylinder ($d \ll 1$) the quantity $\xi_1 d$ is also of importance; as $\xi_1 d \rightarrow 0$ the field $H_1 \rightarrow H_2$, and, of course $\Phi(\xi_1) \rightarrow \pi \delta_0^2 \xi_1^2 H_2$.

A more interesting case is that for which $d = \Delta r/\delta_0 \ll 1$, $\xi_1 d \approx r \Delta r/\delta_0^2 \gg 1$. Here, for $H_2 = 0$ the flux $\Phi = (hcn/2e)(1 - 2\delta_0^2/r \Delta r)$; i.e., the flux changes but little, despite the thinness of the cylinder. The result (6) may be extended to a more general form of cylinder. In the simplest case, with $H_2 = 0$ and wall thickness $\Delta r \ll \delta_0$, the density $j_s = cH_1/4\pi \Delta r$, and one readily obtains, directly from Eq. (4), the formula ($H_2 = 0$, $|\Psi_0| = 1$)

$$\Phi = \frac{hcn}{2e} \left(1 + \frac{\delta_0^2 l}{S \Delta r} \right)^{-1}, \quad (6a)$$

where S is the area and l the perimeter of a cross-section of the cylinder (for $S = \pi r^2$, $l = 2\pi r$ and $\delta_0^2/r \Delta r \ll 1$, Eq. (6a) reduces to the form given above).

All of the solutions for various values of n correspond to relative minima in the thermodynamic potential, but the lowest state is of especial importance. This state is the one which must be established by the slow cooling of a non-superconducting cylinder in a magnetic field. The thermodynamic potential, which, at equilibrium, is a minimum for the given temperature T and external field H_2 , may be written in the form (see^[9])

$$G(T_1, H_2) = \int \left\{ F_s + \frac{(H - H_2)^2}{8\pi} \right\} dV;$$

in the present case

$$G(T_1, H_2) = \frac{(H_1 - H_2)^2}{8\pi} \pi r_1^2 + \frac{1}{8\pi} \int_{r_1}^{r_2} \left\{ (H - H_2)^2 + \frac{(A')^2}{\delta_0^2} \right\} 2\pi r dr. \quad (7)$$

For a thick-walled cylinder [$(r_2 - r_1)/\delta_0 = d \gg 1$] with $r_1/\delta_0 \gg 1$, only the dependence of the first term in Eq. (7) upon H_1 is of significance, in which event $H_1 = hcn/2e\pi r_1^2$. It can readily be seen that in this case the potential G is a minimum for a

*ch = cosh, sh = sinh, cth = coth.

given $n \geq 0$, provided that $hc(n - 1/2)/2e < \Phi_2 < hc(n + 1/2)/2e$. In other words, the state with a given value of n arises in the presence of a flux

$$\Phi_2 = \pi r_1^2 H_2 = \frac{hc}{2e} (n - 1/2) \quad (8)$$

of the external magnetic field through the opening.

This result is found to be in agreement with experiment^{[2,3]*} and was obtained by Byers and Yang^[4] for a specific thin-cylinder model. As a matter of fact, Eq. (8) is simply related to considerations of magnetic energy, and is, under the indicated conditions ($d \gg 1$, $r_1 \gg \delta_0$), of a general character; i.e., it does not depend upon specific equations for the field in the superconductor.

Note added in proof (7 December, 1961). Subsequent to the submission to press of the present note, two letters to the editor^[12,13] appeared presenting a treatment similar to that given above.

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*According to ^[2], the first jump in the field H_1 for cylinder No. 1 occurred for $\Phi_2(r_2) = \pi r_2^2 H_2 = hc/2e$. In this case, however, to approximately the same degree of accuracy, the flux through the aperture was, in accordance with Eq. (8), $\Phi_2(r_1) = hc/4e$.

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