SOVIET PHYSICS JETP VOLUME 15, NUMBER 1 JULY, 1962

INELASTIC SCATTERING OF PHOTONS IN THE COULOMB FIELD OF A NUCLEUS

S. S. SANNIKOV

Physico-Technical Institute, Academy of Sciences, Ukrainian S.S.R.

Submitted to JETP editor August 21, 1961; resubmitted October 9, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **42**, 282-285 (January, 1962)

Dispersion relation methods are used to study the process where a high energy photon splits into two photons in the Coulomb field of a nucleus. We find expressions for the differential cross section for this process in the case where the energy of the second photon is small.

THE invariant form of dispersion relations [1] can be used to study scattering processes involving an external electromagnetic field. In the present paper we employ dispersion relations to study the inelastic scattering of a high-energy photon in the Coulomb field of a nucleus in the first non-vanishing perturbation-theory approximation, i.e., the splitting of a photon into two photons. The largest contribution to the cross section of this process is made by the region where the angle of scattering is small and the change in the energy of the incident photon is small. This is the case considered here.

1. The general expression for the matrix element for the splitting of a photon in the Coulomb field of a nucleus can be obtained simply from the matrix element for photon-photon scattering (see Eq. (8) in [2]). To do this we need only replace one of the photons in the initial state by the external Coulomb field. Denoting the frequency and polarization of that photon by ω_2 and $e_\nu^{(2)}$ we must perform in the photon-photon scattering matrix element the substitution

$$e_\nu^{(2)} / \sqrt{2\omega_2} \to -Ze\delta_{\nu 4}/2\pi^2 q^2, \tag{1}$$

where Ze is the charge of the nucleus and $\mathbf{q}$ the momentum transferred to the nucleus.

We write the matrix element for the splitting of the photon in the form

$$M = 4\pi^2 Ze^5 (2\omega_1\omega_3\omega_4)^{-1/2} \int \frac{d^3q}{q^2} A\delta(k_1 + \mathbf{q} - k_3 - k_4), \tag{2}$$

where A is the total amplitude of the process considered and ω_1, ω_3, and ω_4 are, respectively, the frequencies of the incident, scattered, and second photon. We use the unitarity condition and dispersion relation methods to determine the amplitude A.

First, we can write the amplitude A in the form

$$A = A_1 + A_2 + A_3 + A_{1e} + A_{2e} + A_{3e}, \tag{3}$$

where the partial amplitudes A_1, A_2, and A_3 describe the processes

$$(k_1, e_1) \to (k_3, e_3) + (k_4, e_4), \tag{4.I}$$
$$(-k_3, e_3) \to (-k_1, e_1) + (k_4, e_4), \tag{4.II}$$
$$(-k_4, e_4) \to (k_3, e_3) + (-k_1, e_1). \tag{4.III}$$

The partial amplitudes A_{1e}, A_{2e}, and A_{3e} correspond to the scattering processes involving the exchange of the photons (k_3, e_3) and (k_4, e_4); they are obtained from A_1, A_2, and A_3 through the substitution $(k_3, e_3) \longleftrightarrow (k_4, e_4)$.

The imaginary part of the amplitude A_1 is connected with the operator T occurring in the definition of the scattering matrix S = 1 + iT as follows:

$$i\langle k_3, e_3; k_4, e_4 | T^+ - T | k_1, e_1; \mathbf{q}\rangle = (4\pi^2 Ze^5/q^2)(2\omega_1\omega_3\omega_4)^{-1/2} \operatorname{Im} A_1 \delta(k_1 + \mathbf{q} - k_3 - k_4), \tag{5}$$

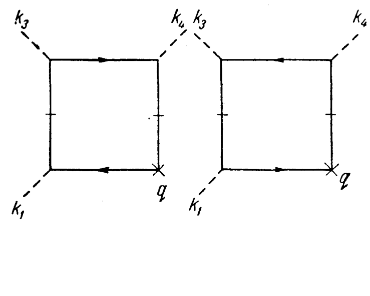
where T^+ is the Hermitean conjugate operator with respect to T. From the unitarity condition which for the operator T is of the form

$$T - T^+ = iT^+T, \tag{6}$$

it follows that in the first perturbation theory approximation (of order Ze^5) Im A_1 corresponds to the Feynman diagrams given in the figure (dashed lines denote free particles). These diagrams are characterized by the following invariants:

$$s = -(k_3 + k_4)^2, \quad t = (k_1 - k_3)^2, \quad u = (k_1 - k_4)^2, \tag{7}$$

where we have from the conservation law



$$-s + t + u = q^2. \quad (8)$$

The crossing symmetry property enables us to express all other partial amplitudes in terms of the amplitude A_1 through the following substitutions

$$A_1 \to A_2 \text{ when } (k_1, e_1) \leftrightarrow (-k_3, e_3);\ s \to -u,\ t \to t.$$
$$A_1 \to A_3 \text{ when } (k_1, e_1) \leftrightarrow (-k_4, e_4);\ s \to -t,\ u \to u.$$
$$A_1 \to A_{1e} \text{ when } (k_3, e_3) \leftrightarrow (k_4, e_4);\ s \to s;\ t \to u.$$
$$A_1 \to A_{2e} \text{ when } (k_1, e_1) \to (-k_4, e_4) \to (-k_3, e_3) \to (k_1, e_1);\ s \to -t \to -u \to s.$$
$$A_1 \to A_{3e} \text{ when } (k_1, e_1) \to (-k_3, e_3) \to (-k_4, e_4) \to (k_1, e_1);\ s \to -u \to -t \to s. \quad (9)$$

We now write down an explicit expression for Im A_1, using the notation

$$p_1 - p_2 = v,\ k_3 + k_4 = k,\ k_1 - q = p,\ k_3 - k_4 = p'$$

[p_1 and p_2 are the four-momenta of the free electron and positron in the intermediate state in the unitarity condition (6)],

$$\operatorname{Im} A_1 = \frac{1}{4\pi^2}\int d^4v\cdot\delta(kv)\,\delta(k^2 - s + 4m^2) \times \frac{S}{(vp - s + q^2)(vp' - s)},$$

$$S = \operatorname{Sp}\left(\frac{i}{2}(\hat{k} + \hat{v}) - m\right) \times \gamma_\mu\left(\frac{i}{2}(\hat{v} - \hat{p}) - m\right)\gamma_4\left(\frac{i}{2}(\hat{k} - \hat{v}) + m\right) \times \gamma_\rho\left(\frac{i}{2}(\hat{v} - \hat{p}') - m\right)\gamma_\sigma e_\mu^{(1)} e_\sigma^{(3)} e_\rho^{(4)} \quad (10)$$

($\hat{k} = k_\mu\gamma_\mu$, γ_μ are the Dirac matrices, and m the electron mass).

2. We consider in detail the case where the following inequalities hold:

$$s \gg t \approx q^2 \gg 4m^2, \quad t \gg s - u. \quad (11)$$

They are equivalent to the conditions

$$2m/\sqrt{\omega_1\omega_3} \ll \theta_3 \ll 2\sqrt{\omega_4/\omega_1}\sin\frac{\theta_4}{2},$$
$$\omega_4 \sin\frac{\theta_4}{2} \ll \sqrt{\omega_1\omega_3}\,\frac{\theta_3}{2} \qquad \sqrt{\omega_3} \gg \sqrt{\omega_4}, \quad (11')$$

where θ_3 and θ_4 are respectively the angles between the momenta of the incident and the scattered and those of the incident and the second photon. The emission angle of the photon (k_4, e_4) must therefore satisfy the condition

$$\frac{\theta_3}{2}\sqrt{\frac{\omega_1}{\omega_4}} \ll \sin\frac{\theta_4}{2} \ll \frac{\theta_3}{2}\,\frac{\sqrt{\omega_1\omega_3}}{\omega_4}. \quad (12)$$

It is clear from (11′) that in the case considered the photon (k_3, e_3) is scattered over a small angle and carries away the largest part of the energy.

If condition (11′) is satisfied Im A_1 is determined from the equation

$$\operatorname{Im} A_1 \approx \frac{1}{4\pi^2}\Big[(k_1)_4(k_4e_1)(e_3e_4)\,\alpha + (k_1)_4(k_4e_3)(e_1e_4)\,\beta + (k_4)_4(k_1e_4)(e_1e_3)\,\gamma + \frac{(k_1)_4}{s}(k_1e_4)(k_4e_1)(k_4e_3)\,\delta\Big]. \quad (13)$$

The calculation of the trace in (10) leads to the following expression for the coefficients α, β, γ, and δ:

$$\alpha = -\beta \approx 2(\pi/s)\ln(t/s),\ \gamma \approx -(\pi/s)\ln(t/s),\ \delta \approx -(\pi/s)\ln(t/s) \quad (14)$$

(the imaginary parts are found with logarithmic accuracy. We have here already taken it into account that $s \gg q^2$).

For $t > 0$ only the amplitudes A_1, A_2, A_{1e}, and A_{3e} give contributions while one can verify immediately that the contributions of the imaginary parts of the amplitudes A_{1e} and A_{3e} do not contain logarithmic terms. The total amplitude is thus of the form

$$A \approx \frac{1}{4\pi^2}\left[\omega_1(k_4e_1)(e_3e_4)\,a + \omega_1(k_4e_3)(e_1e_4)\,b + \omega_4(k_1e_4)(e_1e_3)\,c + (\omega_1/s)(k_1e_4)(k_4e_1)(k_4e_3)\,d\right], \quad (15)$$

where

$$\operatorname{Im} a \approx \alpha - \beta_c \approx 0,\ \operatorname{Im} b \approx \beta - \alpha_c \approx 0,$$
$$\operatorname{Im} c \approx \gamma - \gamma_c \approx -2(\pi/s)\ln(t/s),$$
$$\operatorname{Im} d \approx \delta + \delta_c \approx -2(\pi/s)\ln(t/s). \quad (16)$$

The functions with index c take into account the imaginary part of the amplitude A_2 and are obtained from the corresponding functions α, β, γ, and δ without an index by using (9) and the substitution $s \to -u$, $t \to t$ (where $t \approx q^2$, $u \approx s$).

Substituting the functions α, β, γ, and δ into the dispersion relations, which have for the channel (4.I) the form

$$F_1(s, t) = \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{\operatorname{Im} F_1(s', t)}{s' - s}\,ds', \quad (17)$$

where Im F_1 is any of the functions α, β, γ, and δ, we find expressions for the real parts of these functions. We write down directly the expressions for the real parts of the functions a, b, c, and d occurring in the total amplitude A:

$$\operatorname{Re} a = \operatorname{Re} b \approx 0,\ \operatorname{Re} c \approx -(1/s)\ln^2(t/s),$$
$$\operatorname{Re} d \approx -(1/s)\ln^2(t/s). \quad (18)$$

3. The differential cross section for the inelastic scattering of a photon in the Coulomb field is connected with the matrix element through the equation

$$d\sigma = (2\pi)^{-7}\,|M|^2\,\omega_3^2\,\omega_4^2\,d\omega_4\,do_3\,do_4. \quad (19)$$

Substituting Eq. (15) instead of A into (2) we get the following expression for the differential cross section (we neglect the contribution from the imaginary parts):

$$d\sigma \approx \frac{Z^2\alpha^5}{\pi^2}\frac{\omega_4 d\omega_4}{\omega_1^4}\frac{do_3}{\theta_3^4}\frac{do_4}{\sin^4(\theta_4/2)}\Big[(\mathbf{e}_1\mathbf{e}_3)^2(\mathbf{n}_1\mathbf{e}_4)^2 + \frac{1}{2\sin^2(\theta_4/2)}$$

$$\times (\mathbf{e}_1\mathbf{e}_3)(\mathbf{n}_4\mathbf{e}_1)(\mathbf{n}_4\mathbf{e}_3)(\mathbf{n}_1\mathbf{e}_4)^2$$

$$+ \frac{1}{16\sin^4(\theta_4/2)}(\mathbf{n}_4\mathbf{e}_1)^2(\mathbf{n}_4\mathbf{e}_3)^2(\mathbf{n}_1\mathbf{e}_4)^2\Big]\ln^4\frac{4\omega_4\sin^2(\theta_4/2)}{\omega_1\theta_3^2}, \quad (20)$$

where $\mathbf{n}_i = \mathbf{k}_i/\omega_i$.

We see that the cross section for the inelastic scattering of a photon accompanied by a small change in energy is proportional to the energy of the second photon ω_4, while inelastic processes with the emission of a soft photon contain as usual an infra-red singularity of the type $d\omega/\omega$.[3] Such a behavior is connected with the fact that usually the matrix element of the basic process corresponding to elastic scattering is nonvanishing. In the case considered by us, however, the matrix element of the basic scattering process corresponds to Feynman diagrams with three photon ends and vanishes by Furry's theorem.

Averaging over the polarizations we get for the cross-section for the splitting-up of a photon

$$d\sigma \approx 8\frac{Z^2\alpha^5}{\pi^2}\frac{\omega_4 d\omega_4}{\omega_1^4}\frac{do_3}{\theta_3^4}\cot^2\frac{\theta_4}{2}\,do_4$$

$$\times\Big[1+\cos^2\frac{\theta_4}{2}\Big(1+\frac{1}{2}\cos^2\frac{\theta_4}{2}\Big)\Big]\ln^4\frac{4\omega_4\sin^2(\theta_4/2)}{\omega_1\theta_3^2}. \quad (21)$$

Integrating this expression over the angular variables of the second photon $(\mathbf{k}_4, \mathbf{e}_4)$ with lower and upper limits on $\sin(\theta_4/2)$ of the order $\sqrt{\omega_1/\omega_4}\,\theta_3/2$ and $\sqrt{\omega_1\omega_3}\,\theta_3/2\omega_4$ we get

$$d\sigma \approx 16\frac{Z^2\alpha^5}{\pi}\frac{\omega_4 d\omega_4}{\omega_1^4}\frac{do_3}{\theta_3^4}\ln^5\frac{\omega_3}{\omega_4}. \quad (22)$$

If we integrate this expression over the angular variables of the $(\mathbf{k}_3, \mathbf{e}_3)$ photon with a lower limit on θ_3 equal to $2m/\omega_1$ we get the following expression for the cross-section:

$$d\sigma \approx 4r_0^2Z^2\alpha^3\frac{\omega_4 d\omega_4}{\omega_1^2}\ln^5\frac{\omega_1}{\omega_4}, \quad (23)$$

where r_0 is the classical electron radius. The cross section decreases with increasing energy of the initial photon ω_1. The maximum value of the cross-section (23) occurs when $\omega_4 \approx \omega_1 e^{-5}$. To estimate the magnitude of the total cross-section we integrate (23) over ω_4 from 0 to $\omega_1 e^{-5}$. As a result we obtain

$$\sigma \approx 8\cdot 5^5 e^{-10} r_0^2 Z^2\alpha^3.$$

At $Z \sim 40$ the total cross-section has a magnitude of the order of magnitude $\sigma \sim 10^{-30}$ cm^2.

In conclusion I express my deep gratitude to Professor A. I. Akhiezer for advice and discussions.

[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

[2] S. S. Sannikov, JETP **41**, 467 (1961), Soviet Phys. JETP **14**, 336 (1962).

[3] A. I. Akhiezer and V. B. Berestetskii, Kvantovaya elektrodinamika (Quantum Electrodynamics), 2d. ed., Fizmatgiz, 1959.

Translated by D. ter Haar
42