COVARIANT DERIVATION OF THE WEIZSÄCKER-WILLIAMS FORMULA

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We present a simple and manifestly covariant derivation of the Weizsäcker-Williams formula.

RECENTLY in a whole series of papers, starting with that of Chew and Low,[1] the contribution of pole diagrams to the cross sections for various processes has been calculated. Pomeranchuk and Shmushkevich [2] have pointed out that the cross section for inelastic processes occurring in a Coulomb field and calculated in the pole approximation using the "Coulomb photon" coincides with the well-known Weizsäcker-Williams formula (WW).[3] In calculating the invariant pole matrix element, Pomeranchuk and Shmushkevich,[2] in the spirit of the original quasi-classical derivation of Weizsäcker and Williams, used quantities measured in the rest system of the incident particles. In this note we give an explicitly covariant derivation of the WW formula.*

Let us calculate the cross section for the process shown in Fig. 1. Here k and p are the momenta of the colliding charged particles (for example, an electron and a proton), $k^2 = \mu^2$, $p^2 = m^2$; p′ and k′ are the momenta of the particles created, $p'^2 = p^2 = m^2$; q is the momentum of the virtual photon. We want to express the cross section associated with this graph in terms of the cross section for the photoprocess with a real photon q ($q^2 = 0$, $e_\mu q_\mu = 0$) shown in Fig. 2.

The cross section of the photoprocess for a photon of given polarization, integrated over the momenta of the created particles and summed over their polarizations, can be written in the form

$$\sigma_p^e = -e_\mu e_\nu T^0_{\mu\nu}. \quad (1)$$

For an unpolarized photon

$$\sigma_p = \tfrac{1}{2}\delta_{\mu\nu} T^0_{\mu\nu} = \tfrac{1}{2} T^0_{\mu\mu}. \quad (2)$$

In the expression for the cross section corresponding to the diagram of Fig. 1, after integration over the momenta of the particles k′ and summation over polarizations, there appears the tensor $T_{\mu\nu}$ depending on the vectors k and q, such that

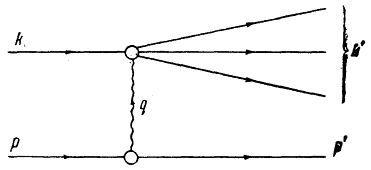

FIG. 1

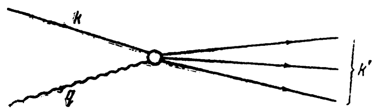

FIG. 2

$$T^0_{\mu\nu} = T_{\mu\nu}|_{q^2=0}. \quad (3)$$

The most general expression for the tensor $T_{\mu\nu}$ which satisfies the condition of gauge invariance

$$q_\mu T_{\mu\nu} = 0, \qquad q_\nu T_{\mu\nu} = 0, \quad (4)$$

has the form

$$T_{\mu\nu} = a\left(\frac{q^2}{kq} k_\mu k_\nu + kq\cdot\delta_{\mu\nu} - k_\mu q_\nu - k_\nu q_\mu\right) + b\,(q^2\delta_{\mu\nu} - q_\mu q_\nu). \quad (5)$$

The invariant functions a and b depend on k^2, q^2 and kq. Since the amplitude for the photoprocess has no singularity at $q^2 = 0$, this is also true for the functions a and b.* Substituting (5) in (2) we get

$$\sigma_p = a\,(kq). \quad (6)$$

The cross section for the process shown in Fig. 1, expressed in terms of the tensor $T_{\mu\nu}$, is equal to

$$d\sigma_{WW} = -\left[\frac{kq}{\sqrt{(kp)^2 - k^2p^2}}\right] \times e^2Z^2\frac{1}{q^4}(2p-q)_\mu(2p-q)_\nu T_{\mu\nu}\frac{dp'}{(2\pi)^3 2E'}. \quad (7)$$

*Arguments similar to our are contained in part in the work of Dalitz and Yennie[4] concerning the creation of pions in electron-proton collisions. See their paper for references to earlier work on the Weizsäcker-Williams method.

*The functions a and b for the Compton scattering of pseudophotons were found by A. Badalyan.[5]

The factor in square brackets is the ratio of the invariant fluxes for the reactions $k + q = k'$ and $k + p = k' + p'$. The expression $Ze(2p - q)$ is the photon vertex part for the spinless nucleus p. Changing to the variables q^2, $\omega^2 = (k + q)^2$ and φ (where φ is the angle between $\mathbf{p}'$ and $\mathbf{k}'$ in the laboratory system) we easily find

$$d\mathbf{p}'/2E' = d\omega^2\, d(-q^2)\, d\varphi/8\sqrt{(kp)^2 - k^2p^2}. \tag{8}$$

Integrating (7) over the azimuthal angle φ, substituting in (5) and using the fact that $2pq = q^2$, we get the invariant formula

$$d\sigma_{WW} = \frac{Z^2\alpha}{2\pi}\frac{(kp)^2(kq)}{(kp)^2 - k^2p^2}\left\{a\left[1 + \frac{(kq)^2p^2}{(kp)^2q^2} - \frac{(kq)}{(kp)}\right] + b\left[p^2 - \frac{q^2}{4}\right]\frac{(kq)}{(pk)^2}\right\}\frac{d\omega^2}{(kq)}\frac{dq^2}{q^2}. \tag{9}$$

Using (6), we have

$$d\sigma_{WW} = \frac{Z^2\alpha}{\pi}\sigma_P\left(1 - \frac{k^2p^2}{(kp)^2}\right)^{-1}\left\{\left[1 + \frac{(kq)^2p^2}{(kp)^2q^2} - \frac{(kq)}{(kp)}\right] + \frac{b}{a}\frac{(p^2 - q^2/4)(kq)}{(pk)^2}\right\}\frac{dq^2}{q^2}\frac{d\omega^2}{2kq}. \tag{10}$$

For large electron energies ($kp \gg kq$, $(kp)^2 \gg k^2p^2$), this formula simplifies:

$$\sigma_{WW} = \frac{Z^2\alpha}{\pi}\sigma_P\left(1 + \frac{(kq)^2p^2}{(kp)^2q^2}\right)\frac{dq^2}{q^2}\frac{d\omega^2}{\omega^2 - q^2 - k^2} \tag{11}$$

and coincides with the result of WW. As for the last term in (10), for the case of large electron energy it is small, since $p^2(kq)^2/(pk)^2q^2 \leqslant 1$, and $q^2/kq \ll 1$. This term is missing from the original result of WW, but appears in that of Pomeranchuk and Shmushkevich.[2] In the terminology of WW,[3] it apparently corresponds to the small contribution from the photoprocess due to longitudinal "pseudophotons."

In conclusion, we note in a recent paper of Badalyan and Smorodinskii,[6] what appears to be an incorrect assertion that they have given a new derivation of the Weizsäcker-Williams formula and that the relation found by them enables one to get the cross section for photoproduction, for a given polarization of the photon, from the differential cross section for electric production. Actually the derivation which they present is just the usual classical calculation of the spectrum of pseudophotons, since the expression for the energy-momentum tensor does not depend on the choice of gauge. It is therefore natural that the expression obtained by them does not contain the term which was found in [2] and is contained in formula (10). The expression given in [6] for the polarization vector of the pseudophoton is correct only to the extent that one can neglect this term. The correct expression is obtained if the vector $P \equiv 2p - q$, which enters for the lower vertex of the diagram in Fig. 1, is represented in the form

$$P_\mu = Ae_\mu + \frac{(Pk)}{(kq)^2 - k^2q^2}\left[(kq)\, q_\mu - q^2k_\mu\right].$$

Here $(eq) = (ek) = 0$. The vector e_μ is spacelike, and if we normalize it so that $e = -1$, it will be the polarization vector of the pseudophoton, while

$$A^2 = \frac{(Pk)^2}{(kq)^2 - k^2q^2}\left[-q^2 - \frac{P^2\left[(kq)^2 - k^2q^2\right]}{(Pk)^2}\right].$$

The spectrum of pseudophotons is proportional to A^2. The second term in the expression for P_μ gives the additional term which was mentioned above.

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[1] G. F. Chew and F. E. Low, Phys. Rev. **113**, 1640 (1959).

[2] I. Ya. Pomeranchuk and I. M. Shmushkevich, Nuclear Phys. **23**, 452 (1961).

[3] K. F. Weizsäcker, Z. Physik **88**, 612 (1934). E. J. Williams, Phys. Rev. **45**, 729 (1934). Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **13**, #4, 1935. A. I. Akhiezer and V. B. Berestetskii, Kvantovaya Elektrodinamika (Quantum Electrodynamics) second edition, Fizmatgiz, 1959, p. 376.

[4] R. Dalitz and D. Yennie, Phys. Rev. **105**, 1598 (1957).

[5] A. Badalyan, JETP **41**, 1315 (1961), Soviet Phys. JETP **14**, 935 (1962).

[6] A. Badalyan and Ya. A. Smorodinskii, JETP **40**, 1231 (1961), Soviet Phys. JETP **13**, 865 (1961).

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