COLLECTIVE EXCITATIONS IN A SUPERCONDUCTOR

V. G. VAKS, V. M. GALITSKII, and A. I. LARKIN

Submitted to JETP editor June 15, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 1655-1668 (November, 1961)

We apply quantum field theory methods to find the collective excitation spectrum of a superconductor. We use the possibility to formulate superconductivity theory as a one-dimensional relativistic problem. We construct the Bethe-Salpeter equation for two-particle Green's functions, whose poles determine the excitation spectrum, in the weak coupling approximation. We obtain for a system of neutral particles a sound wave branch which does not terminate at momenta k of the order Δ, but continues into the large momentum region, and the excitation energy approaches 2Δ exponentially. In the case of charged particles the plasma oscillation dispersion is changed very little by superconductivity. We find a set of excitations with non-vanishing momenta; for small k their energy is close to 2Δ, differing by an amount quadratic in the coupling constant. For each non-vanishing component of the angular momentum along **k** there is one branch of the excitation spectrum which does not terminate for small k. In the large-momentum region, the energy of these excitations approaches 2Δ exponentially.

1. INTRODUCTION

THE recently developed theory of superconductivity is based upon the existence of a bound state of particles near the Fermi surface. The presence of this bound state leads to a gap in the single-particle excitation spectrum, since such excitations indicate the break-up of a bound pair. Apart from the single-particle excitations excited states corresponding to the motion of a pair can occur in the spectrum of superconductors. The interaction between the moving pair and the other pairs leads to the propagation of a collective excitation in the medium of the bound pairs. It is essential here that in excitations of this nature the particle pairs act as units in contradistinction to the situation in single-particle excitations. A study of the collective excitations is important to ascertain the stability of a state and the existence of superconductivity in it, for this requires that the appropriate criterion be satisfied by all branches of the spectrum. The collective excitations may also be important for the electrodynamics and the thermodynamics of superconductors.

The collective excitations of this kind may be considered to be bound states of two particles or quasi-particles with a non-vanishing total momentum. If the interaction between particles is non-vanishing only in the S-state, then only one kind of collective excitations with vanishing angular momentum is possible. This excitation is a sound wave in the electron system and changes into a plasma oscillation with a larger frequency when the Coulomb interaction is taken into account. Turning on an interaction with higher harmonics leads to the appearance of a number of branches of the spectrum with different angular momenta. Long-wavelength excitations of this kind were considered by Bogolyubov [1] and by Bardasis and Schrieffer. [2]

In the present paper we use Green's functions to study collective excitations. We consider the case of zero temperature. Treating the excitations as bound quasi-particle states enables us to determine their spectrum from the poles of the two-particle Green's function. We use for the evaluation of this function a method based upon the formal resemblance of our problem to a one-dimensional relativistic problem; the role of the mass is played by the magnitude of the gap, Δ, and that of the only spatial momentum is played by the nearness of the particle energy to the energy at the Fermi surface. We find the limiting frequencies and the dispersion of the oscillations with arbitrary angular momentum l in the long-wavelength region of the excitations; the results are the same as those obtained earlier by other methods for particular cases. We use the example of excitations with $l = 1$, $m = 0$ to study the spectrum $\omega(k)$ near its endpoint $\omega = 2\Delta$. We show that there are a number of branches of oscillations in the region of relatively large wave vectors. These branches may make an appreciable contribution to the electrodynamic and thermodynamic properties of superconductors.

2. THE RELATIVISTIC FORMULATION OF SUPERCONDUCTIVITY THEORY

We write the Lagrangian $\mathcal{L}(x)$ in the form

$$\int \mathcal{L}(x)\,dx = \int dx u^+(x)\left(i\frac{\partial}{\partial t} + \frac{1}{2m}\frac{\partial^2}{\partial \mathbf{x}^2} + \mu\right)u(x) + \frac{1}{2}\int dx\,dy\,u^+(x)\,u(x)\,D(x-y)\,u^+(y)\,u(y). \quad (1)$$

The spinor $u(x)$ is here the electron field operator, $x = (\mathbf{x}, t) \equiv (\mathbf{x}, x_0)$, $dx = d\mathbf{x}dx_0$, and μ is the chemical potential. In the case of an interaction through a potential, the function D is of the form $D(x-y) = V(\mathbf{x}-\mathbf{y})\,\delta(x_0 - y_0)$. For an electron interaction retarded by the exchange of phonons, D is the phonon Green's function,[3] which in the zeroth approximation is equal to

$$D^0_{ph}(x) = (2\pi)^{-4}\int dk d\omega e^{ikx}\frac{2\pi^2\lambda_0}{mp_F}\frac{\omega_k^2}{\omega_k^2 - \omega^2}, \quad (2)$$

where ω_k is the phonon frequency, p_F the momentum at the Fermi surface, and $\lambda_0 > 0$ Fröhlich's parameter.[4]

If we write the interaction in this form we can at the same time take into account both the direct Coulomb interaction between the electrons and the interaction through the phonons, so that for the complete problem we have

$$D(x-y) = D_{ph}(x-y) - \frac{e^2}{|\mathbf{x}-\mathbf{y}|}\delta(x_0 - y_0). \quad (3)$$

It is well known[5] that when there is superconductivity we must use three kinds of Green's functions:

$$\mathcal{G}_{\alpha\beta}(x-y) = \langle T u_\alpha(x)\,u^+_\beta(y)\rangle,$$

$$F_{\alpha\beta}(x-y) = \langle T u_\alpha(x)\,u_\beta(y)\rangle,$$

$$F^+_{\alpha\beta}(x-y) = \langle T u^+_\alpha(x)\,u^+_\beta(y)\rangle; \quad F^+_{\alpha\beta}(0^+) = -F_{\alpha\beta}{}^*(0^+). \quad (4)$$

Here α and β are spinor indices; the average is taken over the ground state of the Lagrangian (1).* It is convenient to write the functions $\mathcal{G}$, F, and F^+ in the unified form $\langle T u^{1,2}(x)\,u^{1,2}(y)\rangle$, where $u^1(x) = u(x)$ and $u^2(x) = u^+(x)$. To do this it is natural to combine the operators u and u^+ into one operator $\psi(x)$ with components $\psi_1 = u_{1/2}$, $\psi_2 = u_{-1/2}$, $\psi_3 = -iu^+_{-1/2}$, and $\psi_4 = iu^+_{1/2}$, or, in "split" form

$$\psi(x) = \begin{pmatrix} u(x) \\ \sigma_y u^+(x) \end{pmatrix}, \quad (5)$$

where σ_y is a Pauli matrix. This enables us to give the theory a relativistic form. We introduce four-by-four matrices γ_i:

$$\gamma_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} \sigma_y & 0 \\ 0 & -\sigma_y \end{pmatrix} \quad (6)$$

and the operator $\bar\psi(x) = \psi^+(x)\gamma_4$. The matrices γ_i satisfy the usual anticommutation relations

$$\{\gamma_3, \gamma_4\} = \{\gamma_3, \gamma_5\} = \{\gamma_4, \gamma_5\} = 0, \quad \gamma_3^2 = \gamma_4^2 = \gamma_5^2 = 1. \quad (7)$$

We define the Green's function, as usually, by

$$G(x-y) = i\langle T\psi(x)\,\bar\psi(y)\rangle. \quad (8)$$

We can write the matrix G in expanded form:

$$G(x-y) = i\begin{vmatrix} F(x-y)\,\sigma_y & \mathcal{G}(x-y) \\ -\mathcal{G}(y-x) & \sigma_y F^+(x-y) \end{vmatrix}. \quad (9)$$

In the notation of (5) and (6), the Lagrangian (1) becomes

$$\int \mathcal{L}(x)\,dx = -\frac{i}{2}\int \bar\psi(x)\hat p\psi(x)\,dx - \frac{1}{8}\int dxdy\bar\psi(x)\,\gamma_3\psi(x)\,D(x-y)\,\bar\psi(y)\,\gamma_3\psi(y). \quad (10)$$

Here

$$\hat p = \gamma_3 p_3 + \gamma_4 p_4, \quad p_4 \equiv ip_0 = -\frac{\partial}{\partial t}, \quad p_3 = -\frac{1}{2m}\frac{\partial^2}{\partial \mathbf{x}^2} - \mu.$$

In the momentum representation the Green's function for non-interacting particles can be written as

$$G_0(p) = 1/i\hat p = -i\hat p/p^2, \quad (11)$$

where

$$p^2 = p_3^2 + p_4^2 = p_3^2 - p_0^2.$$

In the following we shall be interested in the values of all quantities only near the Fermi surface. In that region we can divide the integration over $d^4p \equiv (2\pi)^{-4}d\mathbf{p}dp_0$ into integrals over the angles of the vector $\mathbf{p}$ and a double integral over p_3 and p_0:

$$d^4p = \frac{p^2 dp\,d\Omega\,dp_0}{(2\pi)^4} \approx \frac{p_F m}{2\pi^2}\frac{d\Omega}{4\pi}\frac{dp_3\,dp_0}{2\pi} \equiv \rho\frac{d\Omega}{4\pi}d^2p \quad (12)$$

(where $\rho = (2\pi^2)^{-1}mp_F$ is the level density near the Fermi surface), and the integration over the angles can be left to the last. This is the form of a one-dimensional relativistic problem, so that we can use the well-developed technique of relativistic calculations.

We write Dyson's equation for the Green's function in the usual form

*Because we have introduced the term μu^+u into the Lagrangian (1), the functions F and F^+ depend only on the difference x–y, in contradistinction to[5].

$$G(p) = G_0(p) - G_0(p)\,\Sigma\,(p)\,G(p), \tag{13}$$

where Σ is the self-energy part. From (13) we have

$$G\,(p) = 1/(i\hat{p} + \Sigma). \tag{14}$$

In first approximation Σ is determined by the diagram of Fig. 1 and is expressed by the equation

$$\Sigma\,(p) = -\,i\int \gamma_3 G\,(p')\,\gamma_3 D\,(p - p')\,d^4p'. \tag{15}$$

The non-diagonal elements of Σ are in this approximation reduced to a renormalization of the chemical potential and the particle mass. We shall assume that this renormalization has taken place so that $p_3 = p^2/2m_* - u_*$, where m_* and μ_* are the effective mass and chemical potential. For what follows, the diagonal elements of the matrix Σ are important since they determine the gap in the energy spectrum. If we denote this diagonal part by $\hat{\Delta}$ we have from the definitions (9) and (4)

$$\hat{\Delta} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta^* \end{pmatrix}. \tag{16}$$

$\Sigma =$ 

FIG. 1

The phase constant in Δ can be chosen arbitrarily and below we shall assume Δ to be real. G (p) is then of the form

$$G(p) = \frac{1}{i\hat{p} + \Delta} = \frac{\Delta - i\hat{p}}{p^2 + \Delta^2}. \tag{17}$$

The single-particle excitation spectrum is determined by the pole $p_0 = (p_1^2 + \Delta^2)^{1/2}$ of G (p). From (15) we have for Δ the equation

$$\Delta = -\,i\int D\,(p - p')\,\frac{\Delta}{p'^2 + \Delta^2}\,d^4p'. \tag{18}$$

With logarithmic accuracy, the region near the Fermi surface is the important one in Eq. (18). In that region D depends only on the angle between p and p′: $D(p - p') = D(n \cdot n')$, $n = p/p$, $n' = p'/p'$. Assuming that the attraction in an S-state predominates in the interaction in D we find that Δ is independent of the angles and satisfies the relation

$$1 = -\,ig_0\int \frac{d^2p}{p^2 + \Delta^2} \qquad \left(g_0 = \rho\int D\,(nn')\,dn'/4\pi\right). \tag{19}$$

If we cut off, as is usually done,[6,1,5] the logarithmically diverging integral at the limiting phonon frequency ω_D, we get for Δ the well-known equation

$$1 = g_0 \ln(\omega_D/\Delta) \equiv g_0 L. \tag{20}$$

More complicated diagrams entering into Σ give terms of order $\omega_D/\mu \ll 1$ or terms in which the degree of the logarithm is less than the degree of the interaction constant, for instance, $g_0^2 L$,[7] so that if $g_0 \ll 1$ it is sufficient to limit oneself to the simplest diagram of Fig. 1. When the interaction is not too weak we must replace the dotted line in Fig. 1 by an irreducible four-pole (see, for instance, [8]).

3. TWO-PARTICLE GREEN'S FUNCTIONS

The collective excitation spectrum is determined by the poles of the two-particle Green's function which in our case can be written as a matrix

$$K\,(1,\,2;\,3,\,4) = i\,\langle T\psi\,(1)\,\overline{\psi}\,(2)\,\psi\,(3)\,\overline{\psi}\,(4)\rangle. \tag{21}$$

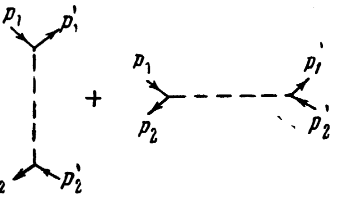
Dyson's equation for K is of the form

$$K = -\,iGG - iGG\int \Gamma K\,d\tau, \tag{22}$$

where Γ is an irreducible four-pole.[9,10] Instead of looking for a pole in the solution of the inhomogeneous integral Eq. (22) we can find the condition that the corresponding homogeneous equation be soluble.[10,11] The kernel of this equation is independent of the variables 3 and 4, and we shall therefore not write these variables out explicitly in K:

$$K\,(1,\,2) = -\,iG\,(1)\,G\,(2)\int \Gamma\,(1,\,2;\,1',\,2')\,K\,(1',\,2')\,d\tau_{1'}d\tau_{2'}. \tag{23}$$

In the weak-interaction approximation it is sufficient to limit oneself for the four-pole Γ to the lowest order diagrams:

$\Gamma\,(p_1, p_2; p_1', p_2') =$  (24)

If we change in the equation for $K_{\mu\nu}(p_1 p_2) = \langle \psi_\mu(p_1)\,\overline{\psi}_\nu(p_2)\rangle$ to the variables $k = p_1 - p_2$ and $p = (p_1 + p_2)/2$ we can write (23) and (24) in the form

$$K_{\mu\nu} = \frac{i}{2}\Big[\Big(G\Big(p + \frac{k}{2}\Big)\gamma_3\Big)_{\mu\rho}\Big(\gamma_3 G\Big(p - \frac{k}{2}\Big)\Big)_{\sigma\nu} + \Big(CG\Big(-p + \frac{k}{2}\Big)\gamma_3\Big)_{\nu\rho}\Big(\gamma_3 G\Big(-p - \frac{k}{2}\Big)\Big)_{\sigma\mu}\Big]$$
$$\times \int d^4p'\,[D\,(p - p')\,K_{\rho\sigma}(p',\,k) - \tfrac{1}{2}D\,(k)\,\gamma^3_{\rho\sigma}\,\mathrm{Sp}\,\gamma^3 K\,(p',\,k)]\,, \tag{25}$$

where C is given by (6). The integral equation (25) has a solution only if there are well-defined relations between the energy $k_0 = \omega$ and the momentum k of the excitation, and these determine the energy spectrum $\omega(k)$. The variable p characterizes the relative motion of the particles in the excitation.

One can expand the matrix $K_{\mu\nu}$ in terms of any arbitrary set of 16 linearly independent matrices. We write this expansion in the form

$$K_{\mu\nu} = K_i\gamma^i_{\mu\nu} + \mathbf{K}_i\boldsymbol{\gamma}^i_{\mu\nu}. \tag{26}$$

The γ^i are given here by Eqs. (6) and the $\boldsymbol{\gamma}^i$ are obtained from the γ^i by replacing the two-by-two unit matrices by the Pauli matrices $\boldsymbol{\sigma}$:

$$\gamma_1 = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}, \qquad \gamma_3 = \begin{pmatrix} 0 & i\sigma \\ -i\sigma & 0 \end{pmatrix},$$

$$\gamma_4 = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}. \tag{6'}$$

The K^i describe the excitations with spin zero and the $\mathbf{K}^i$ those with spin unity.

The presence of two G's in Eq. (25) leads to the fact that only values of the variables p and p′ near the Fermi surface are important in (25). We noted earlier that in that region

$$D(p-p') = D(\mathbf{nn}') \qquad (\mathbf{n} = \mathbf{p}/p,\ \mathbf{n}' = \mathbf{p}'/p'),$$

$$d^4p = \rho\,(d\Omega/4\pi)\,d^2p.$$

Integrating (25) over d^2p we get for the functions

$$K^i(\mathbf{n}, k) = \int d^2pK^i(p, k), \qquad K^{i\alpha}(\mathbf{n}, k) = \int d^2pK^{i\alpha}(p, k)$$

the equations

$$K^i(\mathbf{n}, k) = P^{i,r}(\omega, \mathbf{nk})\,\rho\int D(\mathbf{nn}')K^r(\mathbf{n}', k)\frac{d\mathbf{n}'}{4\pi} - 2P^{i,3}\rho D(\mathbf{k})\int K^3(\mathbf{n}', k)\frac{d\mathbf{n}'}{4\pi},$$

$$K^{i\alpha}(\mathbf{n}, k) = P^{i\alpha, r\beta}(\omega, \mathbf{nk})\,\rho\int D(\mathbf{nn}')K^{r\beta}(\mathbf{n}', k)\frac{d\mathbf{n}'}{4\pi} \tag{27}$$

$(\alpha, \beta = x, y, z)$. Here $P(\omega, \mathbf{n}\cdot\mathbf{k})$ is of the form

$$P^{i,r} = \frac{i}{8}\,\mathrm{Sp}\int d^2p\left[\gamma_iG\left(p+\frac{k}{2}\right)\gamma_3\gamma_r\gamma_3G\left(p-\frac{k}{2}\right) + C\tilde{\gamma}_iCG\left(-p+\frac{k}{2}\right)\gamma_3\gamma_r\gamma_3G\left(-p-\frac{k}{2}\right)\right],$$

$$P^{i\alpha,r\beta} = \frac{i}{8}\,\mathrm{Sp}\int d^2p\left[\gamma^{i\alpha}G\left(p+\frac{k}{2}\right)\gamma_3\gamma^{r\beta}\gamma_3G\left(p-\frac{k}{2}\right) + C\tilde{\gamma}^{i\alpha}CG\left(-p+\frac{k}{2}\right)\gamma_3\gamma^{r\beta}\gamma_3G\left(-p-\frac{k}{2}\right)\right]. \tag{27'}$$

Using the equations $(i = 1, 3, 5)$

$$C\tilde{\gamma}_iC = \gamma_i, \qquad C\tilde{\gamma}_{i\alpha}C = -\gamma_{i\alpha},$$

$$C\tilde{\gamma}_4C = -\gamma_4, \qquad C\tilde{\gamma}_{4\alpha}C = \gamma_{4\alpha}$$

and noting that the trace in $P^{i\alpha,r\beta}$ is proportional to $\delta_{\alpha\beta}$ so that we can replace the matrices $\boldsymbol{\gamma}_i$ under the trace sign by γ_i, we find

$$P^{i,r} = \tfrac{1}{2}\left(\Pi^{ir}(\mathbf{n}) + \Pi^{ir}(-\mathbf{n})\right),$$

$$P^{i\alpha,r\beta} = \delta_{\alpha\beta}\tfrac{1}{2}\left(\Pi^{ir}(\mathbf{n}) - \Pi^{ir}(-\mathbf{n})\right) \qquad (i = 1, 3, 5);$$

$$P^{4,r} = \tfrac{1}{2}\left(\Pi^{4r}(\mathbf{n}) - \Pi^{4r}(-\mathbf{n})\right),$$

$$P^{4\alpha,r\beta} = \delta_{\alpha\beta}\tfrac{1}{2}\left(\Pi^{4r}(\mathbf{n}) + \Pi^{4r}(-\mathbf{n})\right), \tag{28}$$

where

$$\Pi^{ir}(\omega, \mathbf{nk}) = \frac{i}{4}\,\mathrm{Sp}\int\gamma^iG\left(p+\frac{k}{2}\right)\gamma^3\gamma^r\gamma^3G\left(p-\frac{k}{2}\right)d^2p. \tag{28'}$$

The parity of the function $P(\mathbf{n})$ determines the parity of the angular momentum of the corresponding excitation. In the following we shall for the sake of simplicity drop the index α of the functions $K^{i,\alpha}$ and denote all functions simply by K^i. The spinor structure of the excitation for the functions K^i with $i = 1, 3, 5$ is then given by the first term in (26) (spin 0) for even angular momentum, and by the second term (spin 1) for odd angular momentum. For the function K^4, on the other hand, even angular momenta correspond to spin 1 and odd angular momenta to spin 0.

The quantities Π^{ir} are evaluated in the appendix and given by Eqs. (A.9). In order to change from the integral Eq. (28) to an algebraic equation we expand $K^i(\mathbf{n})$ and $D(\mathbf{n}\cdot\mathbf{n}')$ in powers of spherical harmonics:*

$$K^i(\mathbf{n}) = \sum_{lm}K^i_{lm}Y_{lm}(\mathbf{n});$$

$$\frac{1}{4\pi}\rho D(\mathbf{nn}') = \sum_l g_l\left(\frac{2l+1}{8\pi^2}\right)^{1/2}P_l(\mathbf{nn}') = \sum_{lm}g_lY_{lm}(\mathbf{n})Y^*_{lm}(\mathbf{n}'). \tag{29}$$

If we choose the z axis along $\mathbf{k}$, the z component m of the angular momentum will be an integral of motion. If we substitute into the set of Eqs. (28) the explicit form of the quantities

$$\Pi^{ik}_{ll_1m} = \int Y^*_{lm}(\mathbf{n})\,\Pi^{ik}Y_{l_1m}(\mathbf{n})\,d\mathbf{n},$$

we get

$$K^5_{lm} = \sum_{l_1}g_{l_1}\left[(L+\beta^2f)_{ll_1m}K^5_{l_1m} + \frac{q_4}{2\Delta}f_{ll_1m}K^3_{l_1m} + \frac{1}{2\Delta}(q_3f)_{ll_1m}K^4_{l_1m}\right] - 2\delta_{m0}\rho D(k)\frac{q_4}{2\Delta}f_{l00}K^3_{00},$$

*We use the normalization $\int_{-1}^{1}P^2_{lm}(x)\,dx = 1$.

$$K^3_{lm} = \sum_{l_1} g_{l_1}\left[\frac{q_4}{2\Delta} f_{ll_1m} K^5_{l_1m} - \left(f + \frac{q_3^2 - q_3^2 f}{q^2}\right)_{ll_1m} K^3_{l_1m} + q_4\left(\frac{q_3 - q_3 f}{q^2}\right)_{ll_1m} K^4_{l_1m}\right] + 2\delta_{m0}\rho D(k)\left(f + \frac{q_3^2 - q_3^2 f}{q^2}\right)_{l00} K^3_{00},$$

$$K^4_{lm} = \sum_{l_1} g_{l_1}\left[-\frac{1}{2\Delta}(q_3 f)_{ll_1m} K^5_{l_1m} - q_4\left(\frac{q_3 - q_3 f}{q^2}\right)_{ll_1m} K^3_{l_1m} - \left(\frac{q_3^2 - q_3^2 f}{q^2}\right)_{ll_1m} K^4_{l_1m}\right] + 2\delta_{m0}\rho D(k)\, q_4\left(\frac{q_3 - q_3 f}{q^2}\right)_{l00} K^3_{00}.$$

$$K^1_{lm} = \sum_{l_1} g_{l_1}(L - f + \beta^2 f)_{ll_1m} K^1_{l_1m}. \tag{30}$$

Here we have (see appendix)

$$q_3 = \mathrm{k n}v, \quad q_4 = i\omega, \quad q^2 = q_3^2 + q_4^2,$$
$$\beta^2 = -q^2/4\Delta^2, \quad f(\beta) = \frac{\arcsin\beta}{\beta\sqrt{1-\beta^2}}. \tag{31}$$

Since Π^{55}, Π^{53}, Π^{33}, and Π^{44} are even functions of $(\mathbf{k}\cdot\mathbf{n})$, and Π^{45} and Π^{43} are odd, Eq. (30) connects the even harmonics K^5_l and K^3_l with the odd harmonic K^4_l and vice versa, so that we can separate the equations for even and odd l. It is also clear from (30) that one can separate the equation for K^1 from those for K^3, K^4, and K^5. One verifies easily that when $k \neq 0$ the equation for K^1 has no non-vanishing solutions so that we shall not consider K^1 in what follows.

4. THE CASE k = 0

When $k = 0$ the coefficients Π^{ik} are independent of $\mathbf{n}$ so that

$$\Pi^{ik}_{ll_1m} = \delta_{ll_1}\Pi^{ik}_{llm}$$

and the equations for different l become separate ones. We consider first S excitations. In the case of neutral particles $D(k)$ is finite when $k = 0$. The quantities Π^{45}, Π^{43}, Π^{44}, and hence also K^4, vanish so that (30) becomes

$$g_0\frac{\omega^2}{4\Delta^2} f K^5_{00} + \frac{i\omega}{2\Delta} f\,(g_0 - 2\rho D(\omega, 0))\, K^3_{00} = 0,$$
$$g_0\frac{i\omega}{2\Delta} f K^5_{00} - (1 + g_0 f - 2f\rho D(\omega, 0))\, K^3_{00} = 0. \tag{32}$$

Equations (32) have a solution $K^3_{00} = 0$, $K^5_{00} \neq 0$ when $\omega = 0$. This solution corresponds to the usual sound wave excitations [1,12,13] which are discussed in the next section.

We find now the limiting frequencies of the oscillations with $l \neq 0$. We can put Eq. (30) in the form

$$K^5_{lm} = g_l\left(L + \frac{\omega^2}{4\Delta^2} f\right) K^5_{lm} + g_l\frac{i\omega}{2\Delta} f K^3_{lm},$$
$$K^3_{lm} = g_l\frac{i\omega}{2\Delta} f K^5_{lm} - g_l f K^3_{lm}. \tag{33}$$

If we put the determinant of (33) equal to zero, we find [1]

$$(1 - g_l L)(1 + g_l f) - g_l \omega^2 f / 4\Delta^2 = 0. \tag{34}$$

If $g_l^2(g_0 - g_l)^{-1} \ll 1$ the value of ω is close to 2Δ and $f(\omega/2\Delta) \approx \frac{1}{2}\pi(1 - \omega^2/4\Delta^2)^{-1/2}$, and thus

$$\omega_l^2(0) = 4\Delta^2(1 - \alpha_l^2), \tag{35}$$

where

$$\alpha_l = \tfrac{1}{2}\pi g_l^2 (g_0 - g_l)^{-1}. \tag{35'}$$

Equation (35) shows that for any small g_l, regardless of its sign, a bound state of two quasiparticles exists with a binding energy $\sim \Delta\alpha_l^2$. One can understand the existence of this level as follows. For the α_l considered, which are very small compared to unity, the energy of the system is close to 2Δ so that the problem becomes a one-dimensional non-relativistic one. It is well known, however, that in that case there is always a bound state for the particles, however weak the interaction [14]; to be sure, it does not follow from this simple picture that the result is independent of the sign of g_l.

If one of the g_l is very close to g_0 so that $g_l^2(g_0 - g_l)^{-1} \gg 1$, then ω is close to zero, $f = 1$, and

$$\omega^2(0) = 4\Delta^2(g_0 - g_l)(1 + g_l)/g_0 g_l. \tag{36}$$

If $g_l > g_0$ the value of $\omega^2(0)$ given by (36) becomes negative. The presence of excitation with an imaginary frequency in the system means that the original state is unstable with respect to these excitations. In the present case this indicates the instability of a state with S-pairing. The stable state will be the one in which pairing occurs with angular momentum l. [2,15]

We note that the instability of the normal, non-superconducting state $(\Delta = 0)$ can also be ascertained from the form of K. [16] Putting $\Delta = 0$ in (30) we find that the equations for K^4_{00} and K^5_{00} are the same and have a solution when

$$1 = \frac{g_0}{2}\ln\frac{\omega_D^2}{(-\omega^2)}. \tag{37}$$

The change in the state caused by this instability leads to the appearance of the gap Δ in the single-particle excitation spectrum, while the frequency of the collective excitations becomes equal to zero [see (32)].

5. SOUND WAVE OSCILLATIONS ($l = 0$)

We consider excitations with $l = 0$. In that case we find that in the first of Eqs. (30) the quantity K^5_{00} occurring on the left-hand side cancels the logarithmic term $g_0 L K^5_{00}$ on the right-hand side. The subsequent considerations are different for systems of neutral or of charged particles. For neutral particles $D(k)$ is finite for small k and in the first of Eqs. (30) we can drop all terms apart from those involving K^5_{00}, if our accuracy is to terms of the order g_0 or g_1.* The dispersion equation becomes of the form

$$\int_{-1}^{1} (\omega^2 - k^2v^2x^2)\, f(\beta)\, dx = 0. \tag{38}$$

Using (A.10a) and (38) we get for $kv/2\Delta \ll 1$ [1,12,13]

$$\omega^2 = \tfrac{1}{3} v^2k^2. \tag{39}$$

It turns out unexpectedly that Eq. (38) has also a solution when $kv/2\Delta \gg 1$. This is connected with the fact that if ω is near to 2Δ the integral of the first term in (38) is logarithmically large when we take the region of small x into account. Using the limiting expressions (A.10b) and (A.10c) for f we find

$$\frac{\pi\Delta}{2vk} \ln \frac{4\Delta^2}{4\Delta^2 - \omega^2} - \left(\ln \frac{kv}{\Delta} - 1\right) = 0, \tag{40}$$

and hence

$$2\Delta - \omega = \Delta \exp\left(-\frac{2kv}{\pi\Delta} \ln \frac{kv}{\Delta e}\right). \tag{41}$$

It is not difficult to find the corrections of the order of magnitude of the interparticle interaction g. For instance, in the region of small k we can rewrite Eq. (39) as follows, if we take these corrections into account,

$$\omega^2 = \tfrac{1}{3} k^2v^2 (1 + g_0 - 2\rho D(0) + g_1). \tag{39'}$$

If the interaction is non-vanishing only in the S state, then $g_1 = 0$, $\rho D(0) = g_0$, and (39′) is the same as Anderson's result.[12]

The branch found here is the sound wave branch in the system of electrons. This branch goes therefore over into plasma waves in a system of charged particles. In that case the function $D(k) \to -4\pi e^2/k^2 \to -\infty$ as $k \to 0$. One must therefore also take into account terms in K^3_{00} in the set (30). Dropping terms of order g_l we get

$$K^5_{00} = (1 + g_0\beta^2 f)_{00} K^5_{00} + \frac{i\omega}{2\Delta} f_{00} (g_0 - 2\rho D(k)) K^3_{00},$$
$$K^3_{00} = g_0 \frac{i\omega}{2\Delta} f_{00} K^5_{00} + (2\rho D(k) - g_0)\left(f - \frac{(\mathbf{kn}v)^2 (1 - f)}{\omega^2 - (\mathbf{kn}v)^2}\right)_{00} K^3_{00}. \tag{42}$$

Solving (42) for $kv/\omega \ll 1$ and taking into account that then also $\Delta/\omega \ll 1$ we get the following expression for the frequency

$$\omega^2(k) = \tfrac{8}{3} \pi e^2 \rho v^2 + \tfrac{3}{5} k^2v^2. \tag{43}$$

If we take g_l into account the equation for $\omega^2(0)$ will be of the form

$$\omega^2(0) = \frac{8}{3} \pi e^2 \rho v^2 (1 + g_1) = \frac{4}{3\pi} e^2 p_F m_* \cdot p_F^2 m_*^{-2} (1 + g_1)$$

or, if we use the expression for the effective mass $m_* = m(1 + g_1)$, [17]

$$\omega^2(0) = 4\pi n e^2/m = \omega^2_{\mathrm{pl}}. \tag{44}$$

The plasma oscillation frequency (44) has thus the same value as for a free electron gas.[12,1] This result is physically clear since the frequency of the long-wavelength oscillations which occur because of the long-range Coulomb forces cannot be changed by the presence of finite-range forces in the system. We neglect the influence of these forces on the dispersion (43), for when $\Delta/\omega \ll 1$ the corresponding corrections will be the same in superconducting and in non-superconducting systems.

6. EXCITATIONS WITH NON-VANISHING ANGULAR MOMENTUM AT SMALL k ($l \neq 0$, $kv \ll \alpha_1\Delta$)

When $k \neq 0$ the degeneracy with respect to the component m of the angular momentum along the direction of motion is lifted. When $kv \ll \alpha_1\Delta$ the splitting is small and the distance between the levels having different m but the same l is small compared with the distance between levels with different l. In that case l is a good quantum number and the set (30) can be solved, as before, as a set of independent equations for different l. One verifies easily that the influence of neighboring harmonics leads to corrections in the dispersion which are small either in the interaction constant or in the parameter $(kv/\alpha_1\Delta)^2 \ll 1$. When there is Coulomb interaction we must consider more carefully the case $l = 2$, $m = 0$, but one can verify that also in that case the correction is proportional to g. Taking into account the fact that ω is close to 2Δ and neglecting terms k^2v^2 compared to terms k^2v^2/α^2 we get

*For large $kv \gg \Delta$ the corrections to Eq. (38) are of the magnitude $\sim g \ln(kv/\Delta)$. In accordance with the basic approximations in the theory we restrict ourselves to the region $kv \ll \omega_D$ when $g \ln(kv/\Delta) \ll 1$; one can find the correction terms using perturbation theory.

$$K_{lm}^5 = g_l(L + f_{llm}) K_{lm}^5 + i g_l f_{llm} K_{lm}^3,$$
$$K_{lm}^3 = i g_l f_{llm} K_{lm}^5 - g_l f_{llm} K_{lm}^3. \quad (45)$$

If we use (A.10b) we get from the condition that (45) be soluble for $\omega^2(k)$ the following expression

$$\omega_{lm}^2(k) = 4\Delta^2(1-\alpha_l^2) + \tfrac{1}{3}k^2v^2(1 + 2C_{20,}^{l0}\ {}_{l0}C_{20,lm}^{lm}), \quad (46)$$

where C are Clebsch-Gordan coefficients. In particular, we get from (46) when $l = 1$

$$\omega_{10}^2(k) = \omega_1^2(0) + \tfrac{3}{5}k^2v^2, \quad \omega_{1,\pm1}^2(k) = \omega_1^2(0) + \tfrac{1}{5}k^2v^2, \quad (47)$$

which is the same as Bardasis and Schrieffer's result.[2]

For large l Eq. (46) gives

$$\omega_{lm}^2(k) = \omega_l^2(0) + \frac{k^2v^2}{2}(1 - m^2l^2) \quad (l \gg 1). \quad (48)$$

7. EXCITATIONS WITH NON-VANISHING ANGULAR MOMENTUM AT LARGE k

When the wave vector k increases the magnitude of the angular momentum l of the excitation ceases to be a good quantum number, and each excitation is a superposition of harmonics with different values of l (but one value of m). For a certain value of k which is about equal to $\Delta a/v$ the excitation energy becomes, generally speaking, equal to 2Δ, and after that the excitation ceases to exist since as $\omega > 2\Delta$ it is unstable with respect to a break-up into two single-particle excitations. We determine now the shape of the spectrum near its endpoint. Let us, for example, consider the case $l = 1$, $m = 0$, and for the sake of simplicity assume that all g_l with $l > 1$ vanish. Up to terms of order g_0 we can neglect in Eqs. (30) the quantity K^4 so that this set becomes

$$K_{10}^5 = g_1(L + f_{110})K_{10}^5 + i f_{110} K_{10}^3, \quad K_{10}^3 = i g_1 f_{110} K_{10}^5 - f_{110} K_{10}^3. \quad (49)$$

Using Eq. (A.10b) from the appendix we get from (49)

$$1 = 3\alpha_1 \int_0^1 x^2 dx \left(1 - \frac{\omega^2}{4\Delta^2} + \frac{k^2v^2x^2}{4\Delta^2}\right)^{-1/2}, \quad (50)$$

where α_1 is given by (35). The end of the spectrum k_{max} is then determined by the value $\omega(k_{max}) = 2\Delta$, and thus

$$k_{max} = 3\alpha_1\Delta/v. \quad (51)$$

Expanding Eq. (50) near k_{max} we get

$$(4\Delta^2 - \omega^2)\ln\frac{4\Delta^2}{4\Delta^2-\omega^2} - \frac{v^2}{2}(k_{max}^2 - k^2) = 0. \quad (52)$$

It is clear from Eq. (52) that the curve $\omega(k)$ is tangent to the horizontal line $\omega = 2\Delta$ near k_{max} so that the complete $\omega(k)$ curve is of the form given in Fig. 2.

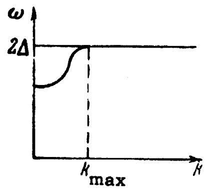


FIG. 2

It turns out, however, that for each $m \neq 0$ there is one excitation branch which does not end, even at large k. Indeed, if l and m have the same parity, the matrix elements of f for ω near to 2Δ are logarithmically large because of the small x. This logarithm compensates for the fact that the coupling constant in the right-hand sides of Eqs. (30) is small. Using Eq. (A.13) we can write the set (30) in the form

$$K_{lm}^5 = gLK_{lm}^5 + \frac{2\pi\Delta}{kv}P_{lm}(0)\ln\frac{\widetilde{kv}}{\sqrt{4\Delta^2-\omega^2}}\sum_{l_1} g_{l_1}P_{l_1m}(0)(K_{l_1m}^5 + iK_{l_1m}^3),$$
$$K_{lm}^3 = i\frac{2\pi\Delta}{kv}P_{lm}(0)\ln\frac{\widetilde{kv}}{\sqrt{4\Delta^2-\omega^2}}\sum_{l_1} g_{l_1}P_{l_1m}(0)(K_{l_1m}^5 + iK_{l_1m}^3). \quad (53)$$

Introducing the notation

$$\chi_m = \frac{2\pi\Delta}{kv}P_{lm}(0)\ln\frac{\widetilde{kv}}{\sqrt{4\Delta^2-\omega^2}}\sum_{l_1} g_{l_1}P_{l_1m}(0)(K_{l_1m}^5 + iK_{l_1m}^3), \quad (54)$$

we get for the excitation considered here

$$K_{lm}^5 = \frac{1}{1-g_lL}P_{lm}(0)\chi_m, \quad K_{lm}^3 = iP_{lm}(0)\chi_m. \quad (55)$$

Substituting (55) into (54) we find

$$1 = \frac{4\Delta}{kv}\ln\frac{\widetilde{kv}}{\sqrt{4\Delta^2-\omega^2}}\sum_l \alpha_l P_{lm}^2(0), \quad (56)$$

and hence

$$4\Delta^2 - \omega^2 = \min\{k^2v^2, 4\Delta^2\}\cdot\exp\left[-\frac{kv}{2\Delta}\left(\sum_l \alpha_l P_{lm}^2(0)\right)^{-1}\right]. \quad (57)$$

Equation (57) is valid apart from a possible numerical factor in front of the exponential, and is applicable in the region

$$kv \gg \Delta\sum_l \alpha_l P_{lm}^2(0). \quad (58)$$

The difficulty of considering the region $k \sim \alpha\Delta/v$ makes it impossible for us to follow in detail the behavior of these branches, starting from $k = 0$. It is, however, clear from the theorem on the non-intersection of terms of the same symmetry[14] that the branch extending into the large k region is the one with the smallest energy for a given m, that is, the branch with maximum g_l.

When m = 0 an annihilation term with $2\rho D(k)$ is added to the right-hand side of (53). Restricting ourselves to the region $\alpha_l\Delta \ll kv \ll \Delta$ we get

$$K_{l0}^5 = g_l L K_{l0}^5 + \frac{2\pi\Delta}{kv} P_{l0}(0) \ln \frac{kv}{\sqrt{4\Delta^2-\omega^2}}$$

$$\times \left[\sum_{l} g_{l_1} P_{l_1 0}(0)(K_{l_1 0}^5 + iK_{l_1 0}^3) - 2i\rho D(k) K_{00}^3\right],$$

$$K_{l0}^3 = \frac{2\pi\Delta}{kv} P_{l0}(0) \ln \frac{kv}{\sqrt{4\Delta^2-\omega^2}}$$

$$\times \left[\sum_{l_1} g_{l_1} P_{l_1 0}(0)(K_{l_1 0}^5 + iK_{l_1 0}^3) - 2i\rho D(k) K_{00}^3\right]. \quad (59)$$

Putting $l = 0$ in the first of Eqs. (55) we see that the expression in square brackets vanishes and that thus $K_{10}^3 = K_{10}^5 = 0$. The equations do therefore not have a solution with ω close to 2Δ in the region $\alpha\Delta \ll kv \ll \Delta$. All excitation branches with m = 0, $l \neq 0$ which are close to 2Δ for small k stop thus for $kv \sim \alpha_l\Delta$.

We note in conclusion that the results of this paper have been obtained assuming an isotropic model of a metal. Deviations from isotropy [18] may turn out to be important when ω is close to 2Δ. The results are, apparently, little changed at small k, if the relative anisotropy of Δ is less than or of the order of g^2. The case of large k needs separate consideration.

The authors are grateful to A. B. Migdal, S. T. Belyaev, and L. P. Gor'kov for interesting discussions.

APPENDIX

EVALUATION OF THE COEFFICIENTS Π^{ir}

It is convenient to introduce for the evaluation of the quantities Π^{ir} which are defined by Eqs. (28) instead of the 4-vector $k = k_1 - k_2$ the two-dimensional vector q with components

$$q_4 = k_4 = i\omega; \qquad q_3 = (k_1 - k_2)_3 = \mathbf{kp}/m_* = \mathbf{kn}v, \quad (A.1)$$

where v is the particle velocity on the Fermi surface. Using Eq. (17) and writing

$$t_{ir} = -\tfrac{1}{4}\,\mathrm{Sp}\,\gamma_i(\Delta - i\hat{p}_1)\gamma_3\gamma_r\gamma_3(\Delta - i\hat{p}_2),$$

$$p_1 = p + \tfrac{1}{2}q, \quad p_2 = p - \tfrac{1}{2}q, \quad (A.2)$$

we rewrite (28′) in the form

$$\Pi^{ir} = -i\int \frac{d^2p\, t_{ir}}{(p_1^2+\Delta^2)(p_2^2+\Delta^2)}. \quad (A.3)$$

Dropping the odd terms in p in t_{ir} which vanish when we integrate over p, we get for the t_{ir} the following expressions:

$$t_{11} = p_1p_2 - \Delta^2, \qquad t_{15} = t_{13} = t_{14} = 0,$$
$$t_{55} = p_1p_2 + \Delta^2, \qquad t_{53} = t_{35} = q_4\Delta, \qquad t_{54} = -t_{45} = q_3\Delta,$$
$$t_{33} = (p_1)_3(p_2)_3 + (p_1)_4(p_2)_4 - \Delta^2, \qquad t_{44} = t_{33} + 2\Delta^2,$$
$$t_{34} = -t_{43} = 2(p_1)_3(p_2)_4. \quad (A.4)$$

The relativistic integrals which we obtained,

$$I(q) = -i\int \frac{\Delta^2 d^2p}{(p_1^2+\Delta^2)(p_2^2+\Delta^2)},$$

$$I_{\alpha\beta}(q) = -i\int \frac{(p_1)_\alpha (p_2)_\beta\, d^2p}{(p_1^2+\Delta^2)(p_2^2+\Delta^2)} \quad (A.5)$$

$(\alpha, \beta = 3, 4)$ can be evaluated simply using Feynman's method.[19] Applying this method to evaluate I(q) we find

$$I = -i\int_0^1 dx \int \frac{\Delta^2 d^2p}{[p_1^2(1-x) + p_2^2 x + \Delta^2]^2}$$

$$= -i\int_0^1 dx \int \frac{\Delta^2 d^2 f}{[f^2 + \Delta^2 + q^2(x-x^2)]^2}$$

$$= \frac{1}{2}\int_0^1 \frac{\Delta^2 dx}{\Delta^2 + q^2(x-x^2)} \quad (f = p + (q - qx)/2). \quad (A.6)$$

Introducing $\beta^2 = -q^2/4\Delta^2 = [\omega^2 - (\mathbf{k}\cdot\mathbf{n}v)^2]/4\Delta^2$, we find

$$I(q) = \frac{1}{2}\frac{\arcsin\beta}{\beta\sqrt{1-\beta^2}} \equiv \frac{1}{2} f(\beta). \quad (A.7)$$

The integral $I_{\alpha\beta}(q)$ contains a logarithmic divergence for large p. We must therefore evaluate instead of $I_{\alpha\beta}(q)$ the convergent quantity $I_{\alpha\beta}(q) - I_{\alpha\beta}(0)$, and find the constant $I_{\alpha\beta}(0)$ by direct "non-covariant" integration, with cutoff at a frequency ω_D taken into account, as when deriving (20). As a result we find

$$I_{\alpha\beta}(q) = I_{\alpha\beta}(0) + \tfrac{1}{2}\delta_{\alpha\beta}(1 - f + \beta^2 f)$$

$$- \tfrac{1}{2} q_\alpha q_\beta q^{-2}(1 - f). \quad (A.8)$$

We must bear in mind that the interaction in the l-th harmonic may be cut off at a frequency ω_1 different from ω_D. The resultant logarithm $L(l) = \ln(\omega_1/\Delta)$ may therefore, strictly speaking, be different from $L = \ln(\omega_D/\Delta) = g_0^{-1}$. To simplify our formulae we put henceforth $\omega_l = \omega_D$; if necessary, we can easily make the appropriate corrections. Substituting Eqs. (A.4) to (A.8) into (A.3) we get the following values for the Π^{ir}:

$$\Pi_{11} = L - f + \beta^2 f,\ \Pi_{55} = L + \beta^2 f,\quad \Pi_{53} = \Pi_{35} = q_4 f/2\Delta,$$
$$\Pi_{54} = -\Pi_{45} = q_3 f/2\Delta,\qquad \Pi_{33} = -f - q_3^2 q^{-2}(1-f),$$
$$\Pi_{44} = q_3^2 q^{-2}(f-1),\qquad \Pi_{34} = -\Pi_{43} = q_3 q_4 q^{-2}(1-f). \tag{A.9}$$

We give the expressions for the function $f(\beta)$ given by (A.6) and (A.7) in the limiting cases:

1) $\beta \to 0 \qquad f(\beta) \to 1,$ (A.10a)

2) $\beta^2 \to 1 - 0 \qquad f(\beta) \to \pi/2\sqrt{1-\beta^2},$ (A.10b)

3) $\beta^2 \to -\infty \qquad f(\beta) \to -\frac{1}{2}\beta^{-2}\ln(-4\beta^2).$ (A.10c)

We find also with logarithmic accuracy the form of the matrix element f_{ll_1m} for large k. In the integral

$$f_{ll_1m} = \int_{-1}^{1} P_{lm}(x) P_{l_1m}(x) f(\beta)\, dx \tag{A.11}$$

the small x region is important. With $\omega \to 2\Delta$ and small x, Eq. (A.11) can, according to (A.10b) be written in the form

$$f_{ll_1m} \approx P_{lm}(0) P_{l_1m}(0) \int_0^{x_{max}} \frac{2\pi\Delta dx}{\sqrt{4\Delta^2 - \omega^2 + k^2v^2x^2}}\,. \tag{A.12}$$

If $kv < 2\Delta$, we have $x_{max} \sim 1$; if, however, $kv > 2\Delta$ the function f increases fast according to Eq. (A.10c) starting at $kvx \sim 2\Delta$ so that the expression for $f(\beta)$ which was used in (A.12) becomes inapplicable. The matrix element f_{ll_1m} is thus with logarithmic accuracy equal to

$$f_{ll_1m} = \frac{2\pi\Delta}{kv} \ln \frac{\widetilde{kv}}{\sqrt{4\Delta^2 - \omega^2}} P_{lm}(0) P_{l_1m}(0), \tag{A.13}$$

where

$$\widetilde{kv} = \min\{kv, 2\Delta\}.$$

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Translated by D. ter Haar
276