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THE HEATING OF A GAS BY RADIATION

A. S. KOMPANEETS and E. Ya. LANTSBURG

Institute of Chemical Physics, Academy of Science, U.S.S.R.

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The problem considered is that of the radiative propagation in a cold gas of heat originally confined in some finite region of the gas itself. It is assumed that the temperature of the hot gas is so high that the gas inside the region is transparent for the radiation. The expansion of the heated region occurs on account of radiative heat exchange with the outer, nontransparent, layer adjacent to the cold gas, and the transfer of the energy of the radiation in this layer is assumed to be diffusive. The process of expansion is regarded as a quasi-stationary propagation of the boundary of the heated region in the form of a plane wave moving through the cold gas. A method is indicated for calculating the speed of propagation of the boundary, which takes into account the nonequilibrium state of the radiation in the internal, transparent part of the heated region in the gas.

1. INTRODUCTION. STATEMENT OF THE PROBLEM

At high temperatures, of the order of $10^5\,{}^\circ$K and above, energy transfer in a gas occurs mainly by radiation, although the fractional part of the energy that is in the form of radiation may be small (in comparison with the energy of the matter).

The problem of the propagation of heat from a point source can be solved simply in the case in which the range of the radiation and the internal energy of the matter depend on the temperature according to a power law.[1-3] A characteristic feature of the solution is a broad "plateau" of the temperature in the middle of the heated region and a steep drop at its edge. It is easy to understand this behavior of the solution, since in the middle, where the thermal conductivity is larger, the temperature is more easily equalized; it is obvious that a strict power-law dependence of the thermal conductivity on the temperature is not necessary for this result.

If the temperature dependence of the range of the radiation in the gas is strong enough, the solution obtained in this way can lead to the following internal contradiction: the range of the radiation inside the hot region is comparable with, or even larger than, the size of the region itself. It is obvious that the mechanism of energy transfer can then not be described in terms of thermal conductivity, whereas the basis of the solution in [1-3] was the thermal-conduction mechanism, i.e., the assumption of local thermodynamic equilibrium between the radiation and the matter.

In the present paper we consider the case in which this assumption does not hold, as can happen at temperatures $\sim 10^6\,{}^\circ$K and higher. In this case (even after a number of simplifying assumptions) a rigorous statement of the problem for the entire heated region is possible only by the use of the integral equation of radiative transfer. If, however, the range varies strongly enough with the temperature, the problem can be solved to good approximation by a simpler method.

Let the size of the hot region be R. Then the temperature T_0 at which the range is equal to R is given by the equation

$$l(T_0) = R. \qquad (1)$$

The definition of $l(T)$ will be given later. When l depends strongly on T, the dependence of the temperature T_0 on R is weak.

According to the definition (1) the whole heated region divides into two parts: an inner part which is transparent, and an outer part of small transparency bordering on the cold gas. Actually, on the low-temperature side the gas becomes transparent again at temperatures below $\sim 10^4\,{}^\circ$K.[4] Therefore there is a leakage of energy from the hot region "to infinity." If, however, the temperature in the inside region is sufficiently high, we can neglect the effect of this leakage in the overall energy balance of the entire heated region. Accordingly we assume that the temperature of the outer cold gas and the range of the radiation in it are equal to zero.

2. THE TRANSPARENT INNER REGION

Let us now consider the state of the radiation in the transparent region. Since the range of the radiation in this region is large in comparison with its size, we must suppose that the energy density U_1 of the radiation inside the hot region is constant over its cross section (but varies with time). Furthermore U_1 is much smaller than the equilibrium energy density aT^4 of radiation at the temperature T of the matter inside the region ($a = 7.55 \times 10^{-15}$ erg cm^{-3} deg^{-4}). It is obvious that everywhere inside the transparent region $T > T_0$. The drop of the temperature to T_0 near the boundary of the transparent region is not due to the radiation, but occurs in some other way (for example, by electronic thermal conductivity). Since the temperature drop takes place in a distance shorter than the range of the radiation, the energy density U of the radiation at the boundary of the transparent region is the same as inside the region, that is, it is equal to U_1.

It is not hard to estimate the order of magnitude of U_1. Namely

$$U_1(T) \approx RaT^4/l(T). \tag{2}$$

Thus here the range $l(T)$ characterizes the emissive power per unit volume of the hot gas, which is equal to $caT^4/l(T)$ (c is the speed of light). The quantity $[l(T)]^{-1}$ is equal to the spectral absorption coefficient averaged over the Planck distribution with induced emission taken into account.[5] For our further work it is convenient to introduce the effective temperature T_1 of the radiation by the formula $U_1 \equiv aT_1^4$, or

$$T_1 = T\,[R/l(T)]^{1/4}. \tag{3}$$

In order for the heated region to expand by radiative energy transfer, it is necessary that each point of the nontransparent region receive radiation of a temperature T_1 somewhat higher than the local temperature of the matter. In our present problem the energy density of the radiation can considerably exceed the local equilibrium density aT^4 (in the approximation of radiative thermal conductivity this excess is regarded as infinitely small). In particular, at the boundary of the nontransparent region at $T = T_0$ we must also have the inequality

$$U_1 > aT_0^4. \tag{4}$$

Let us find the condition for this inequality to hold. By means of Eqs. (1) and (2) ratio U_1/aT_0^4 can be written in the form $l(T_0)aT^4/l(T)aT_0^4 = (T/T_0)^{4-n}$. For a completely ionized gas the power-law index n is $7/2$.[5,6] In an incompletely ionized gas it is smaller than $7/2$; that is, the inequality (4) is satisfied for $T > T_0$.

3. THE NONTRANSPARENT OUTER LAYER AND ITS SPEED OF PROPAGATION

The nontransparent region is that in which the temperature falls from T_0 to zero. The thickness of this region is much smaller than R, and therefore as an approximation it can be treated as a plane layer. In other words, at each instant we can assume that the transparent region occupies a half-space, and the state of the radiation in it is characterized by the temperature T_1. The boundary of the transparent region will then shift toward the colder gas at a constant speed $v(T_1, T_0)$. Any point of the nontransparent layer will shift at the same speed, in accordance with the fact that its thickness is much smaller than R.

Thus we have reduced the problem of the heating of a gas by radiation to that of finding the quasi-stationary mode of propagation of a plane thermal wave into the cold gas (cf. [7]). The radial distributions of the temperatures T of the matter and T_1 of the radiation are shown qualitatively in Fig. 1. The dashed area corresponds to the nontransparent layer.

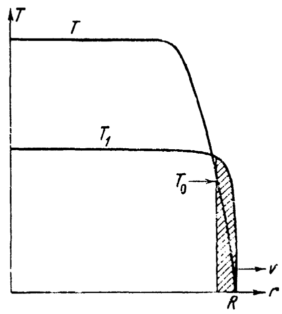

FIG. 1

Let us now write down the equation of radiative transfer of heat in the nontransparent region. For the energy density ϵ of the matter and the energy flux S we obviously have the relation

$$\partial(\varepsilon + U)/\partial t + \partial S/\partial x = 0. \tag{5}$$

We shall neglect all types of energy transfer other than the radiative transfer. At the high temperatures we are dealing with here the energy transfer by radiation occurs so rapidly that the gas does not have time to react to the change of pressure and is not set in motion; in other words, we can neglect the hydrodynamic energy transfer. We assume that $\epsilon = \rho c_V T$, where ρ is the density of the gas and c_V is the specific heat at constant volume, which we shall assume is independent of the temperature. Let us now write the energy-balance equation for the radiation alone:

$$\partial U/\partial t + \partial S/\partial x = c\,(aT^4 - U)/l\,(T). \tag{6}$$

The right member of Eq. (6) expresses the balance between the emission and absorption of radiative energy.

Since the range of the radiation decreases rapidly as the temperature falls, we can take it to be much smaller than the thickness of the region. Then we can treat the radiative transfer of energy in the diffusive approximation,[4] so that

$$S = -\tfrac{1}{3}\, l' c\, \partial U/\partial x. \tag{7}$$

In the approximation of radiative thermal conductivity one puts aT^4 instead of U in this equation. In the present problem this approximation is inadequate, since it leads to a zero speed of propagation. The range l' is not the same average over frequencies as the range $l\,(T)$ used earlier, which characterizes the emissive power of the transparent region of the hot gas. We shall, however, suppose for simplicity that in the nontransparent region the l of Eq. (6) and the l' of Eq. (7) are equal (cf.[4,8]).

The boundary conditions for the system (5)—(7) are as follows: at the boundary of the transparent region $T = T_0$, $U = U_1$; at the forward boundary of the heated gas $T = 0$, $S = 0$, $U = 0$.

Since we are looking for a stationary mode of propagation, we must assume that all quantities depend on the coordinates and the time only through the argument $x - vt$.* In addition, it is convenient to introduce the optical thickness τ, whose differential is

$$d\tau = dx/l\,(T)$$

(the coordinate x is measured in the direction of decreasing temperature), and go over to dimensionless quantities

$$\gamma = \frac{aT_0^4}{\varepsilon\,(T_0)}, \quad \beta = \frac{v}{c}, \quad u = \frac{U}{aT_0^4}\left(\frac{\gamma}{\sqrt{3}\beta}\right)^{4/3}, \quad s = \frac{\sqrt{3}S}{caT_0^4}\left(\frac{\gamma}{\sqrt{3}\beta}\right)^{4/3},$$
$$u_1 = \frac{U_1}{aT_0^4}\left(\frac{\gamma}{\sqrt{3}\beta}\right)^{4/3}, \quad u_p = \left(\frac{T}{T_0}\right)^4\left(\frac{\gamma}{\sqrt{3}\beta}\right)^{4/3}, \quad u_{p_0} = \left(\frac{\gamma}{\sqrt{3}\beta}\right)^{4/3}. \tag{8}$$

Integration of Eq. (5) under the condition $T = S = U = 0$ (or $u_p = s = u = 0$) at the forward boundary of the heated gas at once leads to the equation

$$s = u_p^{1/4} + \sqrt{3}\beta u. \tag{9}$$

By eliminating $d\tau$ we can get instead of the two equations (6) and (7) one ordinary differential equation

$$\frac{ds}{du} = \sqrt{3}\beta + \frac{u - u_p}{s} = \sqrt{3}\beta + \frac{u}{s} - \frac{(s - \sqrt{3}\beta u)^4}{s}, \tag{10}$$

which, along with Eq. (9), connects the quantities s, u, and β (or the dimensional quantities S, U, and v). For a given value of the parameter β the last equation can be integrated in a unique way with the boundary condition $s = 0$, $u = 0$. At the boundary of the transparent region (with $T = T_0$, $U = U_1$, or $u_p = u_{p_0}$, $u = u_1$) this gives a quantity $s\,(u_1; \beta)$, which, according to Eq. (9), is to be equated to $u_{p_0}^{1/4} + 3^{1/2}\beta u_1$. From this one can determine the required value of the speed $\beta\,(u_1, u_{p_0})$ or $v\,(T_1, T_0)$.

Of course only those solutions for which $\beta = v/c \le 1$ have physical meaning. In a number of cases a necessary condition for this is the inequality $U \ll \epsilon$, which we shall take as a basis for our further calculations.

The omission of U in Eq. (5) corresponds to the neglect of the second term in the right member of Eq. (9). Equation (10) can then be rewritten in the form

$$\frac{ds}{du} = \sqrt{3}\beta + \frac{u}{s} - s^3, \tag{11}$$

with the initial condition $s = 0$ for $u = 0$.

The speed β of the wave is determined from the condition $U = U_1$ for $T = T_0$ (or $u = u_1$ for $u_p = u_{p_0}$), and the equation for the determination of β takes the form

$$s\,(u_1;\ \beta) = u_{p_0}^{1/4}. \tag{12}$$

Through the singular (saddle) point $u = 0$, $s = 0$ only one integral curve (the separatrix) goes into the region of positive values of the quantities (Fig. 2, curve 1). We have shown in this same diagram the curve $s = u^{1/4}$, which corresponds to Eq. (9) without the second term in the right member; this is curve 2 of Fig. 2 (i.e., curve 2 corresponds to equilibrium values of the energy density of the radiation, $u = u_p$). To the prescribed values u_1 and u_p (i.e., T_1 and T_0) there correspond the points I and II on curves 1 and 2. It is obvious that Eq. (12) will be satisfied if the points I and II have the same ordinate, and from this it is clear that the speed β can be found in a unique way.

It can be seen from Fig. 2 that the actual energy density of the radiation is always larger than the equilibrium value u_p; as has been remarked,

*We note that the solution of the form $f(x - vt)$ for the equations of radiative heat transfer has also been considered earlier by a number of authors.[4,9,10]

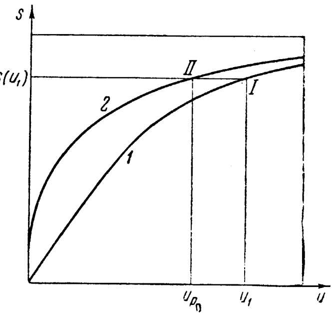

FIG. 2

this is necessary for the propagation of the thermal wave.

4. SOME SPECIAL CASES

In the general case Eq. (11) cannot be integrated in terms of known functions. Therefore we shall consider limiting cases.

1) $(T_1 - T_0)/T_1 \ll 1$. This case corresponds to the part of the curve 1 of Fig. 2 that belongs to large values of u, where the curves 1 and 2 are close to each other. For values of the argument the solution of Eq. (11) is of the form

$$s(u, \beta) \approx u^{1/4}(1 - 1/16u^{3/2}). \quad (13)$$

Here we have already dropped a term proportional to β, since β turns out to be a small quantity. Substituting the value of $s(u, \beta)$ for $u = u_1$ from Eq. (13) in Eq. (12), we get

$$\beta = \frac{v}{c} = \frac{4}{\sqrt{3}}\gamma\left(\frac{T_1 - T_0}{T_1}\right)^{1/2}. \quad (14)$$

It can be seen from Eq. (14) that if the original inequality $(T_1 - T_0) \ll T_1$ is satisfied, then even for $\gamma \sim 1$ we have $\beta \ll 1$. Therefore it is legitimate to use the more exact equation (10). This gives a more accurate value of the speed:

$$\beta = \frac{v}{c} = \frac{4}{\sqrt{3}}\frac{\gamma}{\sqrt{1+\gamma}}\left(\frac{T - T_0}{T_1}\right)^{1/2}. \quad (14a)$$

2) $T_1 \gg T_0$, but $U_1 \ll \epsilon(T_0)$. Small values of u are then important, and this corresponds to the approximate solution of Eq. (11)

$$s(u, \beta) \approx u\left(1 + \frac{\beta\sqrt{3}}{2}\right). \quad (15)$$

For $\beta \ll 1$ we get the expression

$$\beta = U_1/\sqrt{3}\,\varepsilon(T_0). \quad (16)$$

3) $\gamma \ll 1$, and T_1/T_0 is of the order of several units. We then get $\beta \ll 1$. Neglecting β in Eq. (11), we reduce this equation to a Riccati equation, which can be integrated in terms of Bessel functions of pure imaginary argument.[11] Equation (12) for the determination of the speed is then written in the form

$$I_{2/3}\left(\tfrac{4}{3}u_1^{3/2}\right) / I_{-1/3}\left(\tfrac{4}{3}u_1^{3/2}\right) = (T_1/T_0)^2. \quad (17)$$

Since $T_1 > T_0$, this equation always has a solution. Here we have assumed that $T_1 - T_0$ is not small in comparison with T_1. Then u_1 is obviously of the order of several units, and the speed is found from the equation

$$\beta = \frac{v}{c} = \frac{1}{\sqrt{3}}\frac{\gamma}{u_1^{3/4}}\left(\frac{T_1}{T_0}\right)^3. \quad (18)$$

After the speed v of the boundary of the heated gas has been determined as a function of T_1 and T_0, the problem of the propagation of the thermal wave is solved from considerations of balance. Neglecting the energy contained in the region where the temperature falls, we have the energy equation

$$\tfrac{4}{3}\pi R^3 [U_1(T) + \varepsilon(T)] = E, \quad (19)$$

which, together with the equation

$$dR/dt = v(T_1, T_0) \quad (20)$$

and the relations (1) and (2), determines R and T as functions of the time t (here T is the actual temperature of the matter in the transparent region, and E is the total energy in the wave).

The problem we have considered has meaning only as long as $T_1 > T_0$. If it turns out in the course of the solution of a given system that T_1 and T_0 are comparable, the further work of solution can be conducted in the approximation of radiative thermal conduction, since when this is true the radiation has come into equilibrium with the matter (cf. [1-3]). If the time at which this occurs is sufficiently large, there may have been time for a rather large hydrodynamical expansion of the hot region.

In conclusion we express our gratitude to Yu. P. Raĭzer for helpful discussions.

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