MOTION OF CHARGED QUASIPARTICLES IN A VARYING INHOMOGENEOUS ELECTROMAGNETIC FIELD

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We consider the motion of charged quasiparticles in electric and magnetic fields varying slowly in time and space. We derive and investigate "averaged" equations of motion for various geometries of the equal-energy surfaces. Properties of the motion in the vicinity of saddle-points in **p** space are studied. It is found that for such points there arises a peculiar type of scattering which is not connected with the presence of a force center in configuration space. We calculate the probabilities of the particle arriving in various regions for different types of motion.

1. INTRODUCTION

ALL statistical, thermodynamical, and kinetic properties of metals and semiconductors are related to the dynamics of charged quasiparticle motion, the charged quasiparticles being the current carriers in such substances. The energy $\epsilon(\mathbf{p})$ of a quasiparticle is a complicated periodic function of the quasimomentum **p**, the period being that of the reciprocal lattice multiplied by $2\pi\hbar$. In the ideal-gas approximation, which is good enough for describing most of the phenomena, the dynamics of the motion is determined by the dispersion law of $\epsilon(\mathbf{p})$, for which it is important to know the geometry of the isoenergetic surfaces. Anisotropy in $\epsilon(\mathbf{p})$ leads to several important peculiarities in the quasiparticle motion as compared with the motion of free electrons. These peculiarities are manifested in macroscopic properties of metals and semiconductors when the mean free path is much greater than lengths of the order of the trajectory in **r** space. This is a condition which is fulfilled for sufficiently low temperatures.

The equations of motion of noninteracting particles in electric and magnetic fields are the usual Lorentz equations

$$\dot{\mathbf{p}} = e\mathbf{E} + (e/c)[\mathbf{vH}], \qquad \mathbf{v} = \partial\varepsilon/\partial\mathbf{p}, \tag{1*}$$

where **E** is the electric field, **H** the magnetic field, and **v** the velocity of the particle.

The motion of a quasiparticle in a homogeneous constant magnetic field is well understood. For this case the trajectory of the particle in **p** space is given by ϵ = const, p_H = const, where p_H is the component of the momentum parallel to the magnetic field. Further, the motion in **p** space and **r** space depends strongly on the topological properties of this trajectory curve ϵ = const, p_H = const. For a closed trajectory the motion in **p** space is periodic with period $T_0 = m^*c/eH$, where m^* is the effective mass for the region enclosed by the trajectory;[1] the motion in **r** space is unbounded only in the direction of the magnetic field. For an open trajectory, the motion in **r** space is unbounded also in directions perpendicular to the magnetic field.

If the magnetic field is allowed to vary in space and time, and also if there exists an electric field parallel to **H**, the energy and the component of the momentum along the magnetic field are no longer integrals of the motion. The nonconservation of ϵ and p_H, as well as significant anisotropies in the dispersion law lead to unique peculiarities in the quasiparticle motion.

The present article is a study of quasiparticle motion in electric and magnetic fields varying "slowly" in time and space. Such fields satisfy the conditions

$$\gamma_t = T_0/t_0 \ll 1, \qquad \gamma_l = R_0/L \ll 1, \qquad \gamma_E = cE/vH \ll 1. \tag{2}$$

Here t_0 and L are, respectively, a characteristic time and length for the variation of the electromagnetic field, and R_0 is the radius of curvature of the trajectory in **r** space. In practice these conditions are fulfilled up to very large field gradients and frequencies (for instance for H of the order of 10^3 oersteds, gradients of the order of

*[vH] = v × H.

$\nabla H/H \sim 10^2\ cm^{-1}$ and frequencies of the order of $\omega_0 \sim 10^8\ sec^{-1}$ are permissible). Conditions (2) mean that the true motion of the particle can be treated as the sum of two motions, namely a smooth variation of the "averaged" quantities $\mathbf{R} = \bar{\mathbf{r}}$, $P_H = \bar{p}_H$, and $\mathcal{E} = \bar{\epsilon}$, and rapid oscillations which depend on $\mathbf{R}$, P_H, and $\mathcal{E}$ as parameters.

For the closed-trajectory case, the motion in $\mathbf{p}$ space can be thought of as a combination of the drift, rotation, and deformation of the "current leaf," or of the curve $\epsilon = \mathcal{E}(t)$, $p_H = P_H(t)$ along which the particle keeps rapidly rotating.

If the equal-energy surface is not everywhere convex, the particle will be "scattered" by saddle points of the surface. This peculiar scattering does not depend on the existence of any force center in $\mathbf{r}$ space, but is related to the fact that a saddle point is a singular point (stationary point) for motion of a quasiparticle in a constant and homogeneous magnetic field. When dealing with an electromagnetic field satisfying (2), this point divides $\mathbf{p}$ space into several regions of essentially different types of motion. When the $\epsilon = \mathcal{E}(t)$, $p_H = P_H(t)$ surface passes through such a singular point, the type of motion of the particle changes abruptly, and the region it ends up in depends on the exact initial conditions. In our discussion it is the probability for scattering into these regions which is of physical interest. We shall calculate these probabilities for various transitions from one type of motion to another.

2. CLOSED TRAJECTORIES IN p SPACE

In studying the motion of quasiparticles in fields satisfying (2), we shall use the coordinates $\mathbf{r}$, p_H, ϵ, and τ (where τ is the angle variable which defines the position of the particle on the trajectory given by $\epsilon = \text{const}$, $p_H = \text{const}$). We shall write the functions $\mathbf{r}(t)$, $p_H(t)$, and $\epsilon(t)$ in the form

$$\mathbf{r}(t) = \mathbf{R}(t) + \boldsymbol{\rho}(t), \quad p_H(t) = P_H(t) + \tilde{p}_H(t), \quad \varepsilon(t) = \mathcal{E}(t) + \tilde{\varepsilon}(t), \tag{3}$$

where the "averaged" variables $\mathbf{R} = \bar{\mathbf{r}}$, $P_H = \bar{p}_H$, and $\mathcal{E} = \bar{\epsilon}$ define the smooth motion of the particle;* $\mathbf{R}$ can then be thought of as the coordinate of the "center of the orbit," while P_H, $\mathcal{E}$, and $\boldsymbol{\xi}(\mathbf{R}, t) = \mathbf{H}(\mathbf{R}, t)/H(\mathbf{R}, t)$ determine the position of the "current leaf" in $\mathbf{p}$ space.

The period $T_0(t) = eH/m^*c$ is a function of the "instantaneous" variables ϵ, p_H, and $\boldsymbol{\xi}$. The functions $\boldsymbol{\rho}$, $\tilde{p}_H$, and $\tilde{\epsilon}$ are rapidly oscillating increments, and they satisfy the relations $|\boldsymbol{\rho}| \sim R_0$, $\tilde{p}_H a/\hbar \sim \gamma \ll 1$ (where a is the period of the lattice), and $\tilde{\epsilon}/\epsilon \sim \gamma \ll 1$. Henceforth we shall devote our attention to the time variation of the "averaged" variables and derive their equations of motion.

*We shall henceforth define $\bar{X}$ by

$$\frac{1}{T_0(t)} \int_t^{t+T_0(t)} X(t')\,dt' \tag{3a}$$

We first write the exact equations of motion in terms of $\mathbf{r}$, p_H, and ϵ.

From (1) we obtain

$$\dot{p}_H = \mathbf{p}_\perp(\mathbf{v}\nabla)\boldsymbol{\xi} + \mathbf{p}_\perp \partial\boldsymbol{\xi}/\partial t + eE_\xi, \quad \dot{\varepsilon} = e\mathbf{E}\mathbf{v}, \quad \dot{\mathbf{r}} = \mathbf{v}_\perp + v_\xi \boldsymbol{\xi}, \tag{4}$$

where $\mathbf{p}_\perp$ and $\mathbf{v}_\perp$ are the projections of the momentum and velocity, respectively, on a plane perpendicular to the magnetic field.

The variables $\mathbf{R}$, P_H, and $\mathcal{E}$ satisfy the relation $\dot{\bar{x}} = \dot{\bar{x}}(1 + O(\gamma))$. In averaging (4) the integration over the true time t' can be replaced, accurate to terms of order γ, by integration over the angle variable* τ along the section $\epsilon = \mathcal{E}$, $p_H = P_H$, $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{r}, t)$. For this integration $d\tau = c\,dp_l/eHv_\perp$, where dp_l is the element of arc length of the curve $\epsilon = \mathcal{E}$, $p_H = P_H$. As is well known, such averaging gives $\bar{\mathbf{v}}_\perp = 0$, $\overline{p_x v_y} = \overline{p_y v_x} = 0$, and $\overline{p_x v_x} = \overline{p_y v_y} = S/2\pi m^*$, where $S = S(p_H, \epsilon, \boldsymbol{\xi})$ is the area enclosed by the intersection of the surface $\epsilon = \mathcal{E}$ in the plane $p_H = P_H$, and $m^* = (2\pi)^{-1}\partial S/\partial\epsilon$.

Noting that $\mathbf{E}(\mathbf{R} + \boldsymbol{\rho}) = \mathbf{E}(\mathbf{R}) + (\boldsymbol{\rho}\cdot\nabla)\mathbf{E}$, and using the equation $\nabla \times \mathbf{E} = -c^{-1}\partial\mathbf{H}/\partial t$, we proceed to obtain the averages and arrive finally, after several relatively simple operations, to the set of equations

$$\dot{P}_H = \frac{H}{2\pi m^*}(\boldsymbol{\xi}\nabla)\frac{S}{H} + \mathbf{p}_\perp^0 \frac{\partial\boldsymbol{\xi}}{\partial t} + eE_\xi, \quad \dot{\mathcal{E}} = eE_\xi v_\xi^0 + \mathbf{M}\partial\mathbf{H}/\partial t, \quad \dot{\mathbf{R}} = v_\xi^0 \boldsymbol{\xi}, \tag{5}$$

where $\mathbf{M} = (e/2c)\overline{\mathbf{r}\times\mathbf{v}} - (e/2c)\overline{\mathbf{R}\times\mathbf{v}}$ is the mean magnetic moment of the particle about the orbital center $\mathbf{R}$, and

$$x^0 = \frac{1}{T}\int_0^T x\,d\tau.$$

The right sides of Eqs. (5) contain functions of P_H, $\mathcal{E}$, and $\mathbf{R}$ only. From Eqs. (5) and with the aid of the relations

$$v_\xi^0 = -\frac{1}{2\pi m^*}\frac{\partial S}{\partial p_H}, \qquad \frac{\partial S}{\partial \boldsymbol{\xi}} = \oint \frac{\mathbf{p}_\perp v_\xi}{v_\perp}\,dp_l$$

we can show easily that $J = S(P_H, \mathcal{E}, \boldsymbol{\xi})/H(\mathbf{R}, t)$ is an integral of the motion. Thus S/H is an adiabatic invariant not only for free electrons, but also

*This variable has often been used in studying the motion of quasiparticles in constant magnetic fields.

for charged quasiparticles with arbitrary dispersion law.*

Another important property of motion in a slowly space and time varying electromagnetic field is that the velocity of the orbital center is along the magnetic field.

The existence of the S/H integral of the motion allows one to introduce an important simplification into the discussion of (5). Let us consider some special cases.

1. Inhomogeneous magnetic field constant in time. For this case the orbital center **R** moves along a line of force of the magnetic field. If l is the arc length along such a line of force, the equations of motion can be written

$$\mathscr{E} = \text{const}, \quad S(P_H, \mathscr{E}, \xi(l))/H(l) = \text{const},$$
$$\dot{l} = v_\xi^0(P_H, \mathscr{E}, \xi(l)).$$

2. Electric field parallel to a magnetic field ($\mathbf{E}$ = const, $\mathbf{H}$ = const). The equations of motion become

$$\dot{P}_H = eE, \quad S(P_H, \mathscr{E}) = \text{const}, \quad \dot{l} = v_\xi^0(P_H, \mathscr{E}).$$

For both the above cases the equations can be reduced to quadratures.

3. A varying magnetic field $\mathbf{H} = \mathbf{H}(t)$. Recalling that the electric field induced by the variation of the magnetic field is not energy conserving, we arrive at the following set of equations:

$$\dot{P}_H = \mathbf{p}_\perp^0 \partial\xi/\partial t, \quad J(P_H, \mathscr{E}, t) = \text{const},$$
$$\dot{\mathbf{R}} = v_\xi^0(P_H, \mathscr{E}, t)\,\xi(t).$$

3. OPEN PERIODIC TRAJECTORIES

If the ϵ = const, p_H = const curves are open, the averaged equations of motion are derived quite differently for the two cases of periodic and aperiodic curves. An ϵ = const, p_H = const curve is periodic if the direction in which it is open is parallel to some reciprocal lattice vector **B**; then $\boldsymbol{\xi} \perp \mathbf{B}$. If an open periodic trajectory occurs for at least one direction of $\boldsymbol{\xi} \perp \mathbf{B}$, then it is easily shown that it will occur also for any cross section whose normal lies within some angle φ ($0 < \varphi \le 2\pi$) bounded from above and below.

One must distinguish between two types of surfaces on which open periodic trajectories occur: (a) surfaces for which there exists a one-dimensional set (an angle equal to 2π bounded from above and below) of normal directions leading to open curves; in this case all the open trajectories are periodic; (b) surfaces for which there exists a two-dimensional set (solid angle) of normal directions leading to open trajectories; in this case only rational trajectories (with $\boldsymbol{\xi} \perp \mathbf{B}$) are periodic.

From the above discussion it is clear that for motion in a plane magnetic field such that $\boldsymbol{\xi}(\mathbf{R}, t) \perp \mathbf{B}$, the $\epsilon = \epsilon(t)$, $p_H = p_H(t)$ curves remain periodic. In dealing with motion on periodic trajectories, we introduce the following set of coordinates: $\boldsymbol{\xi}$ is the unit vector in the direction of the magnetic field, $\mathbf{e}_1$ is the unit vector in the direction along which the curve is open, and $\mathbf{e}_2 = \boldsymbol{\xi} \times \mathbf{e}_1$. In a plane field, $\mathbf{e}_1$ remains constant, and $\dot{\boldsymbol{\xi}} \parallel \mathbf{e}_2$. The averaged equations of motion can be derived in the same way as for the closed trajectories. By an averaged quantity $\overline{x}$ (where x may be any function of the coordinates **r**, p_H, and ϵ) we shall now understand an average as defined in Eq. (3a) with $T_0(t)$ equal to the time of flight in passing through an elementary cell of the reciprocal lattice.

The averaged equations of motion are

$$\dot{P}_H = \frac{H}{2\pi m^*}(\xi\nabla)\frac{S}{H} + \mathbf{p}_\perp^0\frac{\partial\xi}{\partial t} + eE_\xi,$$
$$\dot{\mathscr{E}} = e(E_\xi v_\xi^0 + E_2 v_2^0) + \mathbf{M}\partial\mathbf{H}/\partial t,$$
$$\dot{\mathbf{R}} = v_\xi^0\,\xi + v_2^0\mathbf{e}_2;$$
$$S = \int_0^B p_2 dp_1, \quad 2\pi m^* = \frac{\partial S}{\partial \varepsilon}, \quad v_2^0 = \frac{B}{2\pi m^*}, \tag{6}$$

where **M** is defined in the same way as in Eqs. (5).

The main difference between motion on open trajectories and motion on closed ones is that in the present case $v_2^0 \neq 0$, which means that the average velocity is not directed along a line of force of the magnetic field. This means that J = S/H is not conserved. The equation for J now becomes

$$\dot{J} = \frac{B}{H}\left(eE_2 - p_2^0\frac{\dot{H}}{H}\right) + \frac{B}{2\pi m^*}(\mathbf{e}_2\nabla)J. \tag{7}$$

We note that in the case of an electric field $\mathbf{E} \parallel \mathbf{H}$ (with $\mathbf{E}$ = const, $\mathbf{H}$ = const), J is an adiabatic invariant, as is the case for closed trajectories.

4. OPEN APERIODIC TRAJECTORIES

When dealing with open aperiodic trajectories, the average must be taken over a time interval T such that $T_0 \ll T \ll t_0$, where T_0 is a time of the order of the time of flight of a quasiparticle through an elementary cell ($T_0 \sim c\hbar/eHva$). When averaging bounded quantities defined on an ϵ = const, p_H = const curve, the integration may be extended over the entire cross section, since the difference

$$\lim_{T'\to\infty}\frac{1}{2T'}\int_{-T'}^{T'} x\,d\tau - \frac{1}{T}\int_t^{t+T} x\,dt'$$

*Assuming that the trajectory ε = const, p_H = const is closed.

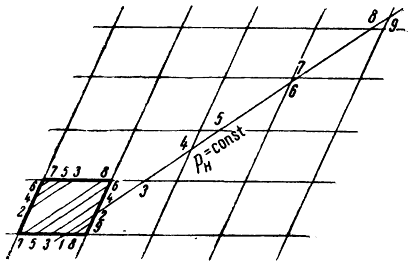

FIG. 1. One of the crystallographic surfaces of the reciprocal lattice, intersected by a p_H = const surface. Equal numbers denote equivalent points.

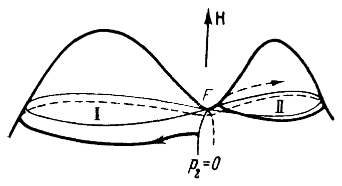

FIG. 2

is a rapidly oscillating expression with period of the order of xT_0/T (the variable τ is defined in the same way as for the case of closed trajectories

In calculating the mean value of such a quantity, the integration along an open trajectory passing through an infinite set of elementary cells can be replaced by a sum of integrals over equivalent segments within a single cell (Fig. 1). In the case of an aperiodic trajectory, these segments are dense and uniformly distributed in the cell. It follows then that $\bar{x}$ is independent of p_H. The p_H-dependence of averages can also be neglected for periodic trajectories with large period ($B \gg \hbar/a$). This independence of p_H makes it possible to drop this variable from our considerations, and to reduce the number of equations of motion; then these become

$$\dot{R} = v_\xi^0(\mathscr{E}, \xi)\,\xi(R, t) + v_2^0(\mathscr{E}, \xi)\,e_2(R, t),$$
$$\dot{\mathscr{E}} = e\,(E_\xi v_\xi^0 + E_2 v_2^0) + M\partial H/\partial t, \tag{8}$$

where $\mathbf{e}_2$ and $\mathbf{e}_1$ are defined as in the periodic case, and $M_i = \epsilon_{ikl}T_{kl}/2c$.

The asymmetric tensor T_{kl} is given in terms of integrals along the curve $\epsilon = \mathscr{E}$, $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{R}, t)$ by the equations

$$T_{12} = -(c/eH)\,\overline{p_2\tilde{v}_2},\quad T_{13} = -(c/eH)\,\overline{p_2\tilde{v}_\xi};$$

$$T_{23} = -\lim_{T'\to\infty}\frac{1}{2T'}\int_{-T'}^{T'} d\tau'\tilde{v}_2\int^{\tau'}\tilde{v}_\xi\,d\tau'',\quad \tilde{x} = x - \bar{x}.$$

For constant electric and magnetic fields, the integration reduces to quadratures:

$$\frac{1}{e}\int_{\mathscr{E}_0}^{\mathscr{E}}\frac{d\mathscr{E}'}{E_\xi v_\xi^0(\mathscr{E}') + E_2 v_2^0(\mathscr{E}')} = t - t_0.$$

If the function $\mathscr{E}(t)$ is known, the first of Eqs. (8) can be integrated.

During its motion a particle may enter onto a periodic trajectory with period $B \sim \hbar/a$. In this case the p_H-dependence of $\bar{x}$ cannot be neglected. In the general case of a nonplane field [with $\boldsymbol{\xi}(\mathbf{R}, t)$ not perpendicular to $\mathbf{B}$], a particle is in the neighborhood of such a trajectory for a time of the order of T_0t_0/T. This means that one can use Eqs. (8) for times $\gtrsim t_0$.

5. SCATTERING ON A SINGULAR POINT

Scattering will occur in a slowly varying field if the motion undergoes transition from one type to another. Regions in **p** space with a different type of motion are separated from each other by segments of self-intersecting trajectories formed by the intersection of an ϵ = const surface and a plane tangent to an isoenergetic surface at a hyperbolic point. Classically such points are stationary points for motion in a homogeneous and constant magnetic field. The period of motion diverges logarithmically as $p_H \to f_H$ (where $f_H = \boldsymbol{\xi}\cdot\mathbf{f}$, and $\mathbf{f}$ is the **p**-space position vector of the singular point).

It is easiest to understand the essentials of such scattering in terms of the example of a weakly inhomogeneous time-constant magnetic field which has at least the one straight line of force $\mathbf{r} = \mathbf{r}_0 + l\boldsymbol{\xi}_0$. Let F be a saddle point on the surface $\epsilon = \epsilon_0$, such that the normal to the surface at this point is parallel to $\boldsymbol{\xi}_0$. The intersection of $\epsilon = \epsilon_0$ with $p_H = f_H$ is a figure-eight whose intersection point is at F. If, under the motion in **p** space, the "current leaf" $\epsilon = \epsilon_0$, $P_H = P_H(t)$, $\boldsymbol{\xi} = \boldsymbol{\xi}_0$ contacts the surface at the saddle point, it gets broken up into two "current leaves" in the two regions I and II which are separated by the singular point (Fig. 2). The types of motion are quite different in these two regions. Whether the particle enters region I or region II depends on the exact ("microscopic") initial conditions. The "microscopic" initial conditions are distributed so that each macroscopic energy-surface element determined by the averaged coordinates contains points from which the particle can enter region I as well as region II. From this point of view we can treat the entrance into either of these regions as a random process, and thus speak of the "scattering" of particles in the region of a singular point; then the probabilities w_1 and w_2 for scattering into each of the two regions have well defined values.

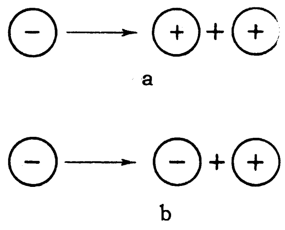

FIG. 3

To find w_1 and w_2 let us consider a classical ensemble of particles whose distribution is given in terms of some directional parameter to be determined later. On its last pass before entering into region I or II, every particle crosses the principal curvature line passing through F (for line $p_2 = 0$ in Fig. 2) for some value of $p_H(0)$. Having gone fully around one of the loops of the figure-eight, the particle finds itself again in the region of the self-intersection point. Then, depending on the sign of $p_H(t) - f_H$, the particle at this instant enters region I or II; the value of $p_H(t) - f_H$ is determined uniquely by $p_H(0)$ at the point where it was last intersected by the trajectory of the particle. It is seen from this that $p_H(0)$ is a convenient impact parameter to use. Let regions I and II correspond to intervals δ_I and δ_{II} of values of $p_H(0)$, and these will then determine the probabilities for entering these regions. The scattering probabilities w_1 and w_2, i.e., the relative number of particles entering regions I and II, respectively, will be proportional to the flux of particles across δ_I and δ_{II}. For a sufficiently smooth distribution function these are proportional, to lowest order in γ, to the intervals themselves.

These intervals can be obtained from the relation

$$p_H(t) = p_H(0) + \int_0^t \dot{p}_H \, dt'.$$

From this we obtain

$$\delta_I = \int_0^{T_1} \dot{p}_H \, dt', \quad \delta_{II} = \int_{T_1}^{T_2} \dot{p}_H \, dt'.$$

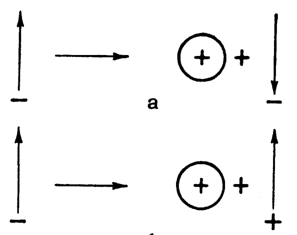

FIG. 4

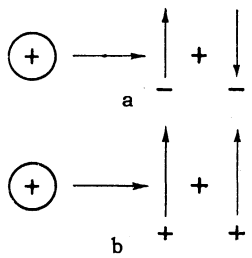

FIG. 5

(Here T_1 is the time it takes to go around the I loop of the figure-eight, and T_2 is the time it takes to go around the entire figure-eight; $T_{1,2} \sim T_0 \ln \gamma$.)

The largest contribution to the variation of p_H during the time it takes to go around the figure-eight is given by those parts of the trajectory which are far from the singular point; in other words, the total change in p_H is of order $\gamma\hbar/a$. This is because at the singular point itself (for a field whose direction is constant) $\dot{p}_H = 0$, and the contribution to the integral from points close to the singular point is of order $\gamma^2 \ln \gamma$. [We bear in mind the fact that the time it takes to go around depends logarithmically on $p_H(0) - f_H$.] It then follows that to first order in γ the integral expressions for δ_I and δ_{II} can be written

$$\delta_I = \oint_I \dot{p}_H \, d\tau, \quad \delta_{II} = \oint_{II} \dot{p}_H \, d\tau,$$

where the integrals are taken, respectively, over the I and II loops of the figure-eight (and τ is the angle variable introduced earlier). Using the first of Eqs. (5) and recalling that the field has a straight line of force, we obtain

$$w_1/w_2 = \delta_I/\delta_{II} = S_1/S_2, \tag{9}$$

where S_1 and S_2 are the areas enclosed by the loops of the figure-eight. From (9) we arrive at

$$w_1 = S_1/(S_1 + S_2), \quad w_2 = S_2/(S_1 + S_2).$$

In general there exist several types of transition from one kind of motion to another. These are indicated schematically in Figs. 3 — 5. The right sides of these figures contain symbols corresponding to the two regions into which the scattering takes place. A circle indicates a closed trajectory, while an arrow indicates an open periodic one, and the direction of the arrow indicates

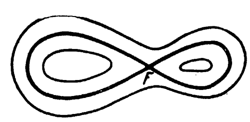

FIG. 6

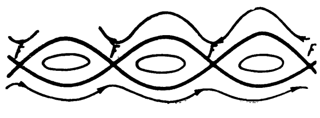

FIG. 7

the direction of motion along this periodic trajectory. The kinds of equal-energy surfaces which correspond to these transitions are shown in Figs. 6 — 8 by means of the p_H = const contour lines. On these figures the self-intersection point is denoted by the letter F. The heavy lines denote self-intersecting curves. The arrows show the direction of motion along the trajectories. Figure 6 corresponds to diagrams 3a and b, Fig. 7 to diagrams 4a and 5a, and Fig. 8 to diagrams 4b and 5b. The signs in Figs. 3 — 5 give the sign of $p_H - f_H$ in the different regions.*

In an arbitrary field satisfying conditions (2), the singular point $\mathbf{f}(\mathscr{E}, \boldsymbol{\xi})$ may move in $\mathbf{p}$ space (unlike the case considered above). Then the criterion for entering regions of different types of motion is the sign of $\Delta(t) = p_H(t) - f_H(t)$ [where $f_H(t) = \mathbf{f}(t) \cdot \boldsymbol{\xi}(t)$] when the particle is in the neighborhood of the singular point. Then the intervals δ_I and δ_{II} of $\Delta(0)$, which determine whether the particle enters regions I and II, are given for cases 3a, 4a, and 5a by

$$\delta_{\mathrm{I}} = \left| \int_0^{T_1} (\dot{p}_H - \dot{f}_H)\, dt' \right|, \quad \delta_{\mathrm{II}} = \left| \int_{T_1}^{T_2} (\dot{p}_H - \dot{f}_H)\, dt' \right|,$$

and for cases 3b, 4b, and 5b by

$$\delta_{\mathrm{I}} = \left| \int_0^{T_2} (\dot{p}_H - \dot{f}_H)\, dt' \right|, \quad \delta_{\mathrm{II}} = \left| \int_{T_1}^{T_2} (\dot{p}_H - \dot{f}_H)\, dt' \right|.$$

Here T_1 and T_2 are the first and second times when the particle is in the neighborhood of the singular points; as before, $T_{1,2} \sim T_0 \ln \gamma$.

Up to terms of order γ the expression for δ_I and δ_{II} can be written

$$\delta_{\mathrm{I}} = \frac{c}{eH} \left| \int_{L_1} (\dot{p}_H - \dot{f}_H) \frac{dp_l}{v_\perp} \right|, \quad \delta_{\mathrm{II}} = \frac{c}{eH} \left| \int_{L_2} (\dot{p}_H - \dot{f}_H) \frac{dp_l}{v_\perp} \right|, \tag{10}$$

where the contours L_1 and L_2 are the segments of the self-intersecting trajectories which bound the regions into which the particle may be scattered. (In the case of an open periodic trajectory, the integral is taken over one period.) Equations (10) hold for all types of transitions. It should be noted that in cases 3b, 4b, and 5b, scattering can occur

*Six other types of transitions, essentially the same as those of Figs. 3–5, are obtained by changing the signs.

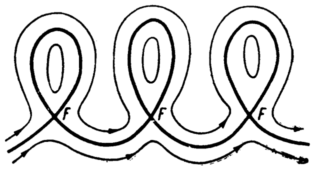

FIG. 8

only if the signs of $\int_{T_1}^{T_2} \dot{\Delta} dt'$ and $\int_0^{T_2} \dot{\Delta} dt'$ are opposite, while in cases 3a, 4a, and 5a, these signs must be the same.

To derive the formulas for the scattering probabilities in cases 3a and 3b, we make use of the adiabatic invariance of $J = S(P_H, \mathscr{E}, \boldsymbol{\xi})/H(\mathbf{R}, t)$. Consider $S_{cr}^{1,2}(f_H(\mathscr{E}, \xi), \mathscr{E}, \xi)$ (S_{cr}^1 and S_{cr}^2 are the areas of the loops bounding the regions into which scattering takes place*). Using the fact $J_{1,2}$ = const to terms of order $\gamma \ln \gamma$, we obtain

$$T_{1,2}\, \dot{j}_{\mathrm{cr}}^{1,2} = T^{1,2} (\dot{j}_{\mathrm{cr}}^{1,2} - \dot{j}^{1,2}) = (\partial S_{1,2}/\partial P_H)\, \bar{\dot{\Delta}} T_{1,2}.$$

From this we obtain

$$w_1/w_2 = \delta_{\mathrm{I}}/\delta_{\mathrm{II}} = \dot{j}_{\mathrm{cr}}^{(1)} / \dot{j}_{\mathrm{cr}}^{(2)} |_{p_H = f_H}. \tag{11}$$

In deriving (11) we make use of the fact

$$\lim_{p_H \to f_H} \frac{T_1 \partial S_1/\partial p_H}{T_2 \partial S_2/\partial p_H} = 1.$$

Let us consider this equation for some special cases. In a time-constant weakly inhomogeneous magnetic field,

$$\frac{w_1}{w_2} = \frac{\partial}{\partial l}\left(\frac{S_1}{H}\right) \Big/ \frac{\partial}{\partial l}\left(\frac{S_2}{H}\right), \tag{12}$$

where l is the length of a line of force, and S_1 and S_2 are the areas of each of the loops on the ϵ = const surface defined by the intersection with the plane passing through the singular point and perpendicular to $\boldsymbol{\xi}(l)$. For a straight line of force Eq. (11) goes over into (9).

In parallel electric and magnetic fields ($\mathbf{E}$ = const, $\mathbf{H}$ = const) we obtain

$$\frac{w_1}{w_2} = \frac{d}{dp_H} S_1(p_H, \varepsilon_{\mathrm{cr}}(p_H)) \Big/ \frac{d}{dp_H} S_2(p_H, \varepsilon_{\mathrm{cr}}(p_H)) \Big|_{p_H = f_H} \tag{13}$$

Transitions of types 4 and 5 differ topologically from those of type 3, since the former involve open periodic trajectories. Nevertheless in these cases also the formulas for the transition probabilities are obtained with the aid of the equation for J = S/H (for a periodic trajectory $S = \int_0^B p_2\, dp_1$, and the in-

*In case 3b one of these loops is the entire figure-eight.

tegration is taken over one period). Using Eqs. (7) and proceeding similarly as above, we obtain the following expressions. For transitions of type 4a and b

$$\frac{w_1}{w_2}=\frac{\delta_{\mathrm{I}}}{\delta_{\mathrm{II}}}=j_{\mathrm{cr}}^{(1)}\Big/\left\{j^{(2)}-B\left(eE_2-f_2\frac{\dot{H}}{H}\right)+\frac{\mathbf{f}_{\perp}(\partial\xi/\partial x_2)\,Bv_f}{2\pi H}\right\}; \quad (14)$$

for transitions of type 5a and b

$$\frac{w_1}{w_2}=\frac{\delta_{\mathrm{I}}}{\delta_{\mathrm{II}}}=\frac{j_{\mathrm{cr}}^{(1)}-B\,(eE_2-f_2\dot{H}/H)+\mathbf{f}_{\perp}(\partial\xi/\partial x_2)\,v_fB/2\pi H}{j_{\mathrm{cr}}^{(2)}-B\,(eE_2-f_2\dot{H}/H)+\mathbf{f}_{\perp}(\partial\xi/\partial x_2)\,v_fB/2\pi H}, \quad (15)$$

where v_f is the component of the velocity at the singular point in the direction of the magnetic field.

[1] I. M. Lifshitz and M. I. Kaganov, Usp. Fiz. Nauk **69**, 419 (1959), Soviet Phys.-Uspekhi **2**, 831 (1960).

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