THE UPPER CRITICAL FIELD OF SUPERCONDUCTING ALLOYS

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The value of the upper critical field H_{c1} of superconducting alloys is derived for the case when the mean free path is small compared with the parameter $\xi = v/\Delta$ of the Bardeen, Cooper, and Schrieffer theory of superconductivity. The temperature variation of H_{c1} is nearly linear, and the second derivative with respect to temperature is positive.

It is well known that superconducting alloys have a number of anomalous properties. In particular, the transition from the normal to the superconducting state in a magnetic field starts at some "upper" critical field H_{c1} exceeding the thermodynamic critical field H_c, —the transition being a second order phase transition. From the viewpoint of the Bardeen, Cooper, and Schrieffer (BCS) theory of superconductivity[1] this means that, at fields smaller than the upper critical field, the electron interactions through the phonon field lead to the formation of Cooper pairs and the appearance of a gap in the energy spectrum. As $H \to H_{c1}$ from below, the gap tends to zero, and for $H > H_{c1}$ there are no solutions with a gap. Thus, the upper critical field of superconducting alloys is analogous to the critical field of supercooled, impurity-free superconductors (which has been considered by Gor'kov[2]), in the sense that it is also determined as the stability limit of the normal phase with respect to the appearance of superconducting correlations. In the present paper the value of the field H_{c1} for superconducting alloys is obtained for the whole temperature range, when the mean free path is small in comparison with the parameter $\xi_0 = v/\Delta$ of the BCS theory.

To begin with we consider the absolute zero of temperature. We write the equations of the theory of superconductivity in the form used by Gor'kov.[3] When impurities are present and there is a constant magnetic field, they are

$$\left\{\omega + \frac{1}{2m}\left(\frac{\partial}{\partial \mathbf{r}} - i\frac{e}{c}\mathbf{A}(\mathbf{r})\right)^2 + V(\mathbf{r}) + \mu\right\} G_\omega(\mathbf{r}, \mathbf{r}') + i\Delta(\mathbf{r}) F_\omega^+(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \quad (1)$$

$$\left\{-\omega + \frac{1}{2m}\left(\frac{\partial}{\partial \mathbf{r}} + i\frac{e}{c}\mathbf{A}(\mathbf{r})\right)^2 + V(\mathbf{r}) + \mu\right\} F_\omega^+(\mathbf{r}, \mathbf{r}') + i\Delta^*(\mathbf{r}) G_\omega(\mathbf{r}, \mathbf{r}') = 0, \quad (2)$$

$$\Delta^*(\mathbf{r}) = (2\pi)^{-1}|g| \int_{-\infty}^{+\infty} F_\omega^+(\mathbf{r}, \mathbf{r})\, d\omega, \quad (3)$$

$A_x = -Hy$, $A_y = A_z = 0$ (the magnetic field is taken along the z axis); $V(\mathbf{r})$ is the potential energy of interaction with all the impurity atoms

$$V(\mathbf{r}) = \sum_a u(\mathbf{r} - \mathbf{r}_a), \quad (4)$$

where the sum is taken over all the impurity atoms randomly distributed throughout the lattice.

We need to find the field at which a solution of equations (1), (2), (3) with vanishingly small $\Delta(\mathbf{r})$ and $F_\omega^+(\mathbf{r}, \mathbf{r}')$ first appears as the magnetic field decreases. The system is thus greatly simplified, since we can limit ourselves to first order terms in $\Delta(\mathbf{r})$ and $F_\omega^+(\mathbf{r}, \mathbf{r}')$. Then, instead of (2), we have

$$\left\{-\omega + \frac{1}{2m}\left(\frac{\partial}{\partial \mathbf{r}} + i\frac{e}{c}\mathbf{A}(\mathbf{r})\right)^2 + V(\mathbf{r}) + \mu\right\} F_\omega^+(\mathbf{r}, \mathbf{r}') + i\Delta^*(\mathbf{r})\widetilde{G}_\omega(\mathbf{r}, \mathbf{r}') = 0, \quad (5)$$

where $\widetilde{G}_\omega(\mathbf{r}, \mathbf{r}')$ is the Green's function of an electron in the normal metal when impurities are present and there is a magnetic field; this satisfies the equation

$$\left\{\omega + \frac{1}{2m}\left(\frac{\partial}{\partial \mathbf{r}} - i\frac{e}{c}\mathbf{A}(\mathbf{r})\right)^2 + V(\mathbf{r}) + \mu\right\} \widetilde{G}_\omega(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \quad (6)$$

or

$$\left\{\omega + \frac{1}{2m}\left(\frac{\partial}{\partial \mathbf{r}'} + i\frac{e}{c}\mathbf{A}(\mathbf{r}')\right)^2 + V(\mathbf{r}') + \mu\right\} \widetilde{G}_\omega(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \quad (6')$$

Then,

$$F_\omega^+(\mathbf{r}, \mathbf{r}') = -i\int \widetilde{G}_\omega(\mathbf{r}'', \mathbf{r}')\widetilde{G}_{-\omega}(\mathbf{r}'', \mathbf{r})\,\Delta^*(\mathbf{r}'')\, d\mathbf{r}''. \quad (7)$$

Using (3), we arrive at the integral equation

$$\Delta^*(\mathbf{r}) = -i(2\pi)^{-1}|g| \int_{-\infty}^{+\infty} d\omega \int \overline{\widetilde{G}_\omega(\mathbf{r}', \mathbf{r})\widetilde{G}_{-\omega}(\mathbf{r}', \mathbf{r})}\,\Delta^*(\mathbf{r}')\, d\mathbf{r}'. \quad (8)$$

The asterisk signifies averaging over all possible

FIG. 1

positions of the impurity atoms. The maximum field for which there is a non-zero solution of this integral equation is the upper critical field sought.

The averaging of the Green's functions over the impurity atom positions is performed using the diagram technique developed by Abrikosov and Gor'kov[4,5] and by Edwards.[6] Each scattering of an electron at an impurity atom contributes a factor $u(\mathbf{q}) \exp(i\mathbf{q}\cdot\mathbf{r}_a)$ to the expression for the Green's function in the momentum representation, where $\mathbf{q}$ is the momentum transmitted, and $u(\mathbf{q})$ is the Fourier component of the impurity atom potential. On averaging over $\mathbf{r}_a$, this factor becomes zero unless there are terms in the averaged expression corresponding to scattering at the same impurity atom, but with $\mathbf{q}' = -\mathbf{q}$; in this case a multiplier $|u(\mathbf{q})|^2$ occurs on averaging, which is proportional to the scattering probability in the Born approximation with transmitted momentum $\mathbf{q}$. It is thus necessary to pick out in pairs all scattering events at the same atoms.

In the diagram each scattering is indicated by a cross; averaging is indicated by joining two crosses corresponding to scattering at the same atom by a broken line. In Fig. 1 several such diagrams are shown. On summing in diagrams with intersecting broken lines (for example, diagram d), the integration is over a region far from the Fermi surface, and they therefore make an insignificant contribution ($\sim 1/p_0 l$, where p_0 is the Fermi momentum, and $l = v\tau$ is the mean free path), which can be neglected. Summation of the remaining diagrams leads, as usual, to an integral equation.

Applying the technique described to the Green's function of a normal metal in the absence of a magnetic field, we have

$$G_\omega(\mathbf{p}) = (\omega - \xi_\mathbf{p} + i\,\mathrm{sign}\,\omega/2\tau)^{-1}, \tag{9}$$

$$G_\omega(\mathbf{R}) = -\frac{m}{2\pi R}\exp\left\{iR\left(p_0 - \frac{\omega}{v}\right)\mathrm{sign}\,\omega - \frac{R}{2v\tau}\right\}, \tag{10}$$

$$\frac{1}{\tau} = \frac{nmp_0}{(2\pi)^2}\int |u(\mathbf{q})|^2\,do \tag{11}$$

(n is the concentration of impurity atoms). In a homogeneous magnetic field along the z axis (if

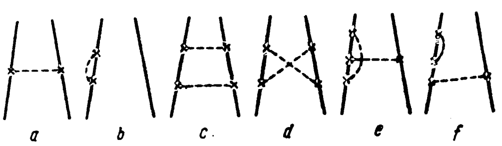

FIG. 2

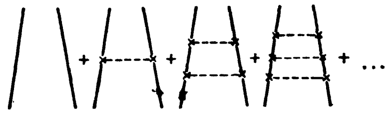

FIG. 3

the curvature of the electron trajectories is small, which is the case in the fields of interest to us), we have

$$\widetilde{G}_\omega(\mathbf{r},\ \mathbf{r}') = \exp\{-i(eH/c)(y+y')(x-x')\}\,G_\omega(\mathbf{r}-\mathbf{r}'). \tag{12}$$

The product of two Green's functions is averaged similarly. The lowest order diagrams arising from averaging in pairs are shown in Fig. 2. The diagrams with intersecting broken lines (for example, d and e) are unimportant; the broken lines joining crosses sited on one electron line (for example, diagram f) give complete Green's functions for this line. Thus, we must sum the "ladder" diagrams shown in Fig. 3. We denote the sum of these diagrams by

$$\Gamma_\omega(\mathbf{r}_1,\ \mathbf{r}_1';\ \mathbf{r}_2,\ \mathbf{r}_2') = \overline{\widetilde{G}_\omega(\mathbf{r}_1,\ \mathbf{r}_1')\,\widetilde{G}_{-\omega}(\mathbf{r}_2,\ \mathbf{r}_2')}. \tag{13}$$

It is not difficult to write down the integral equation for $\Gamma_\omega(\mathbf{r}_1, \mathbf{r}_1'; \mathbf{r}_2, \mathbf{r}_2')$. From this equation (which we do not write down because it is very cumbersome) it is possible to extract an exponential factor caused by the superposition of the magnetic field, and to transform the equation from coordinate to momentum space, i.e., to express $\Gamma_\omega(\mathbf{r}_1, \mathbf{r}_1'; \mathbf{r}_2, \mathbf{r}_2')$ as:

$$\Gamma_\omega(\mathbf{r}_1,\ \mathbf{r}_1';\ \mathbf{r}_2,\ \mathbf{r}_2') = \exp\{-i(eH/2c)(y_1+y_1')(x_1-x_1') + (y_2+y_2')(x_2-x_2')\}\,\Gamma_\omega(\mathbf{r}_1-\mathbf{r}_1';\ \mathbf{r}_2-\mathbf{r}_2'). \tag{14}$$

We introduce the Fourier components

$$\Gamma_\omega(\mathbf{r},\ \mathbf{r}') = (2\pi)^{-6}\iint e^{i\mathbf{p}\mathbf{r}+i\mathbf{p}'\mathbf{r}'}\,\Gamma_\omega(\mathbf{p},\ \mathbf{p}')\,d\mathbf{p}\,d\mathbf{p}'. \tag{15}$$

They satisfy the equation

$$\begin{aligned}\Gamma_\omega(\mathbf{p}_1,\ \mathbf{p}_2) &= G_\omega(\mathbf{p}_1)\,G_{-\omega}(\mathbf{p}_2) + \frac{n}{(2\pi)^3}\int |u(\mathbf{q})|^2\,G_\omega(\mathbf{p}_1-\mathbf{m}_1+2\mathbf{l}_2)\\ &\times G_{-\omega}(\mathbf{p}_2-\mathbf{m}_2+2\mathbf{l}_2)\,\Gamma_\omega(\mathbf{p}_1+\mathbf{q}-\mathbf{m}_1 + \mathbf{l}_1,\ \mathbf{p}_2-\mathbf{q}-\mathbf{m}_2+\mathbf{l}_2)\\ &\times \delta_H(\mathbf{m}_1,\ \mathbf{l}_1)\,\delta_H(\mathbf{m}_2,\ \mathbf{l}_2)\,d\mathbf{m}_1\,d\mathbf{m}_2\,d\mathbf{l}_1\,d\mathbf{l}_2\,d\mathbf{q},\end{aligned} \tag{16}$$

where

$$\delta_H(\mathbf{m},\ \mathbf{l}) = (2c/eH)^4\exp\{i(2c/eH)(m_x l_y - m_y l_x)\}\,\delta(m_z)\,\delta(l_z),$$

$$\delta_H(\mathbf{m},\ \mathbf{l}) \to \delta(\mathbf{m})\,\delta(\mathbf{l})\ \text{ as }\ H\to 0. \tag{16'}$$

If we write

$$\Gamma'(\mathbf{p},\ \mathbf{k},\ \omega) = \frac{n}{(2\pi)^3}\int |u(\mathbf{q})|^2\,\Gamma_\omega\left(\mathbf{p}+\frac{\mathbf{k}}{2}+\mathbf{q},\ -\mathbf{p}+\frac{\mathbf{k}}{2}-\mathbf{q}\right)d\mathbf{q}, \tag{17}$$

then

$$\Gamma_\omega(\mathbf{p}_1, \mathbf{p}_2) = G_\omega(\mathbf{p}_1)\, G_{-\omega}(\mathbf{p}_2) + \int G_\omega(\mathbf{p}_1 - \mathbf{m}_1 + 2\mathbf{l}_1)\, G_{-\omega}(\mathbf{p}_2 - \mathbf{m}_2 + 2\mathbf{l}_2)\, \Gamma'\left(\tfrac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{m}_2 - \mathbf{m}_1 + \mathbf{l}_1 - \mathbf{l}_2),\ \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{m}_1 - \mathbf{m}_2 + \mathbf{l}_1 + \mathbf{l}_2,\ \omega\right) \times \delta_H(\mathbf{m}_1, \mathbf{l}_1)\, \delta_H(\mathbf{m}_2, \mathbf{l}_2)\, d\mathbf{m}_1\, d\mathbf{m}_2\, d\mathbf{l}_1\, d\mathbf{l}_2, \quad (18)$$

and $\Gamma'(\mathbf{p}, \mathbf{k}, \omega)$ satisfies the equation

$$\Gamma'(\mathbf{p}, \mathbf{k}, \omega) = \frac{n}{(2\pi)^3} \int |u(\mathbf{p} - \mathbf{p}_1)|^2 G_\omega\left(\mathbf{p}_1 + \frac{\mathbf{k}}{2}\right) G_{-\omega}\left(\mathbf{p}_1 - \frac{\mathbf{k}}{2}\right) d\mathbf{p}_1 + \frac{n}{(2\pi)^3} \int |u(\mathbf{p} - \mathbf{p}_1)|^2 G_\omega\left(\mathbf{p}_1 + \frac{\mathbf{k}}{2} - \mathbf{m}_1 + 2\mathbf{l}_1\right) G_{-\omega}\left(\mathbf{p}_1 - \frac{\mathbf{k}}{2} + \mathbf{m}_2 - 2\mathbf{l}_2\right) \times \Gamma'\left(\mathbf{p}_1 - \tfrac{1}{2}(\mathbf{m}_1 - \mathbf{m}_2 - \mathbf{l}_1 + \mathbf{l}_2),\ \mathbf{k} - \mathbf{m}_1 - \mathbf{m}_2 + \mathbf{l}_1 + \mathbf{l}_2,\ \omega\right) \delta_H(\mathbf{m}_1, \mathbf{l}_1)\, \delta_H(\mathbf{m}_2, \mathbf{l}_2)\, d\mathbf{p}_1\, d\mathbf{m}_1\, d\mathbf{m}_2\, d\mathbf{l}_1\, d\mathbf{l}_2. \quad (19)$$

In this equation the product of two Green's functions decreases rapidly on departing from the Fermi surface. On the other hand, $u(\mathbf{p} - \mathbf{p}_1)$ depends weakly on $\mathbf{p} - \mathbf{p}_1$, and changes significantly only when the argument is changed by a quantity of the order of the Fermi momentum. Therefore $\Gamma'(\mathbf{p}, \mathbf{k}, \omega)$ also depends weakly on $|\mathbf{p}|$. By virtue of this we can integrate with respect to $d|\mathbf{p}|$:

$$\Gamma'(\mathbf{p}, \mathbf{k}, \omega) = \frac{inmp_0}{(2\pi)^2} \int |u(\mathbf{p} - \mathbf{p}_1)|^2 \left(2\omega - \frac{\mathbf{pk}}{m} + \frac{i}{\tau}\right)^{-1} do_1 + \frac{inmp_0}{(2\pi)^2} \int |u(\mathbf{p} - \mathbf{p}_1)|^2 \left(2\omega - \frac{\mathbf{p}}{m}(\mathbf{k} - \boldsymbol{\eta}_1 + 2\boldsymbol{\eta}_2) + \frac{i}{\tau}\right)^{-1} \times \Gamma'(\mathbf{p}_1 - \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2, \mathbf{k} - \boldsymbol{\eta}_1 + \boldsymbol{\eta}_2, \omega)\, \delta_H(\boldsymbol{\eta}_2, \boldsymbol{\xi}_1)\, \delta_H(\boldsymbol{\xi}_2, \boldsymbol{\eta}_1) \times d\boldsymbol{\eta}_1\, d\boldsymbol{\eta}_2\, d\boldsymbol{\xi}_1\, d\boldsymbol{\xi}_2\, do_1 \quad (20)$$

(the variables have been changed: $\mathbf{m}_1 + \mathbf{m}_2 = \eta_1$, $\mathbf{m}_1 - \mathbf{m}_2 = 2\xi_1$ etc). Integrating now with respect to $d\xi_1$ and $d\xi_2$, we obtain a δ-type function of η_1 and η_2, the width being, in order of magnitude, $eH/cp_0 = R_H^{-1}$ (R_H is the radius of curvature of the electron trajectories in the magnetic field). If $2\omega - \mathbf{pk}/m + i/\tau$ and $\Gamma'(\mathbf{p}, \mathbf{k}, \omega)$ change little on changing k by the order of R_H^{-1}, we can integrate with respect to $d\eta_1$ and $d\eta_2$:

$$\Gamma'(\mathbf{p}, \mathbf{k}, \omega) = \frac{inmp_0}{(2\pi)^2} \int |u(\mathbf{p} - \mathbf{p}_1)|^2 \frac{1 + \Gamma'(\mathbf{p}_1, \mathbf{k}, \omega)}{2\omega - \mathbf{p}_1\mathbf{k}/m + i/\tau}\, do_1. \quad (21)$$

The applicability of this equation is limited by two conditions:

$$\partial\Gamma'(\mathbf{p}, k, \omega)/\partial k \ll \Gamma'(\mathbf{p}, k, \omega) R_H, \quad (22)$$

$$v\tau \lll R_H; \quad (23)$$

the second is satisfied for the fields considered; a study of Eq. (21) shows that the first condition is violated when $k \lesssim R_H^{-1}$. Such values of k correspond to distances $\gtrsim R_H$ at which the quasi-classical approximation for the Green's function in a magnetic field becomes invalid. In the problem under consideration, as we will see below, smaller distances are important, in fact, distances $\sim \sqrt{c/eH} = \sqrt{R_H/p_0}$.

By a similar line of reasoning, it can be shown that, when conditions (22) and (23) are satisfied, we have, instead of (18):

$$\Gamma_\omega(\mathbf{p}_1, \mathbf{p}_2) = G_\omega(\mathbf{p}_1)\, G_{-\omega}(\mathbf{p}_2)\left[1 + \Gamma'\left(\tfrac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2),\ \mathbf{p}_1 + \mathbf{p}_2,\ \omega\right)\right]. \quad (24)$$

We return to the initial equation of the problem [Eq. (18)], rewritten as:

$$\Delta^*(\mathbf{r}) = \int \exp\left\{-i\frac{eH}{2c}(y + y')(x - x')\right\} K(\mathbf{r} - \mathbf{r}')\, \Delta^*(\mathbf{r}')\, d\mathbf{r}'. \quad (25)$$

From (13) — (15) and (24) we obtain, using the above-mentioned properties of $\Gamma'(\mathbf{p}, \mathbf{k}, \omega)$ and of the product of two G functions,

$$K(\mathbf{r} - \mathbf{r}') = -i|g|\pi^{-1}\int_0^\infty \Gamma_\omega(\mathbf{r} - \mathbf{r}', \mathbf{r} - \mathbf{r}')\, d\omega = \frac{-i|g|}{(2\pi)^6\pi}\int_0^\infty d\omega \int e^{i\mathbf{k}(\mathbf{r} - \mathbf{r}')} G_\omega\left(\mathbf{p} + \frac{\mathbf{k}}{2}\right) G_{-\omega}\left(-\mathbf{p} + \frac{\mathbf{k}}{2}\right) \times [1 + \Gamma'(\mathbf{p}, \mathbf{k}, \omega)]\, d\mathbf{p}\, d\mathbf{k} \quad (26)$$

or

$$K(R) = \frac{-i|g|}{(2\pi)^4}\frac{mp_0}{R}\int_0^\infty d\omega \int e^{ikR} k\, dk \int \frac{1 + \Gamma'(\mathbf{p}, \mathbf{k}, \omega)}{2\omega - \mathbf{pk}/m + i/\tau}\, do, \quad (26')$$

$$K(R) = -\frac{4|g|}{(2\pi)^4}\frac{mp_0\tau}{R}\int_0^\infty d\omega \int_{-\infty}^{+\infty} \Gamma_0(k, \omega)\, e^{ikR} k\, dk; \quad (26'')$$

$$\Gamma_0(k, \omega) = \frac{1}{4\pi}\int \Gamma'(\mathbf{p}, \mathbf{k}, \omega)\, do = \frac{i}{4\pi}\int \frac{1 + \Gamma'(\mathbf{p}, \mathbf{k}, \omega)}{2\omega - \mathbf{pk}/m + i/\tau}\, do. \quad (27)$$

The last equation can be obtained by integrating both parts of (21) with respect to do (i.e., with respect to the directions $\mathbf{p}$), using (11).

Equation (21) can be solved in the limiting case $k \ll l^{-1}$. Then

$$\Gamma_0(k, \omega) = i\left(2\omega\tau + \tfrac{1}{3} i k^2 v^2 \tau\tau_{tr}\right)^{-1}, \quad (28)$$

$$\frac{1}{\tau_{tr}} = \frac{nmp_0}{(2\pi)^2}\int |u(\mathbf{p} - \mathbf{p}_1)|^2 (1 - \cos\theta)\, do. \quad (29)$$

Whence, for $R \gg l$, we have

$$K(R) = \frac{mp_0|g|}{(2\pi)^3}\frac{2}{R^3}. \quad (30)$$

It is of interest to compare this quantity with the behavior of $K(R)$ at distances small in comparison with the mean free path. In this limiting case values of $k \gg l^{-1}$ are important. For such values of k the behavior of $\Gamma'(\mathbf{p}, \mathbf{k}, \omega)$ is like $(kl)^{-1}\ln(kl)$, and it can be neglected in comparison with unity. Then

$$\Gamma_\omega(\mathbf{p}_1, \mathbf{p}_2) = G_\omega(\mathbf{p}_1)\, G_{-\omega}(\mathbf{p}_2),$$

and for K(R) we obtain the same equation as for an impurity-free metal:

$$K(R)=\frac{mp_0|g|}{(2\pi)^3}\frac{1}{R^3}. \quad (31)$$

We consider the case of a heavily alloyed superconductor, when the mean free path is much smaller than the parameter ξ_0 of the BCS theory. As we shall see below, there exists in this limiting case a solution of (8) or (25), which changes significantly in distances of the order $\sqrt{l\xi_0}\gg l$. The eigenfunctions of this equation, which change rapidly, correspond to small values of the magnetic field. Since we are interested in the maximum field for which a solution exists, $\Delta^*(\mathbf{r})$ can be considered constant at distances of the order of the mean free path. Then, taking into account the behavior of K(R) at large distances, we rewrite (25) as

$$\Delta^*(\mathbf{r})\left\{1-\int\left(K(\mathbf{r}-\mathbf{r}')-\frac{2mp_0|g|}{(2\pi)^3|\mathbf{r}-\mathbf{r}'|^3}\right)d\mathbf{r}'\right\}$$
$$=\frac{mp_0|g|}{4\pi^3}\int\frac{\exp\{ieH(y+y')(x-x')/c\}}{|\mathbf{r}-\mathbf{r}'|^3}\Delta^*(\mathbf{r}')\,d\mathbf{r}'. \quad (32)$$

The right- and left-hand sides of this equation diverge differently for small values of R. This is due to the fact that we have not taken into account the cutoff of the interaction between the electrons in Bardeen's model Hamiltonian at Debye energies. In our problem this implies that the integration with respect to dp in (26) is performed not between infinite limits, but from $p_0-2\tilde{\omega}/v$ to $p_0+2\tilde{\omega}/v$. In (26′) there appears then in the expression under the integral a multiplier

$$\frac{1}{\pi i}\ln\frac{\tilde{\omega}-\omega+\mathbf{pk}/2m-i/2\tau}{-\tilde{\omega}-\omega+\mathbf{pk}/2m-i/2\tau},$$

The expression (31) for K(R) at small values of R is valid only for $R\gg v/\tilde{\omega}$, and the volume integral of K(R) will converge for small values of R.

Performing now the volume integration in the left-hand side of (32) to $|\mathbf{r}-\mathbf{r}'|=\epsilon(\epsilon\to 0)$, we have

$$\Delta^*(\mathbf{r})\ln\left(\frac{\tau_{tr}v^2e^2}{3\gamma^2\Delta_0\varepsilon^2}\right)$$
$$=\frac{1}{2\pi}\int\limits_{|\mathbf{r}-\mathbf{r}'|\geq\varepsilon}\frac{\exp\{ieH(y+y')(x-x')/c\}}{|\mathbf{r}-\mathbf{r}'|^3}\Delta^*(\mathbf{r}')\,d\mathbf{r}'. \quad (33)$$

Here, $\gamma=\exp C$, where C is Euler's constant.

An analogous equation was studied by Gor'kov.[2] He showed that the solution of this equation can be obtained from the particular solution for $\Delta^*(\mathbf{r})$, depending only on y. In this case

$$\Delta^*(y)\ln\left(\frac{3\gamma^2\Delta_0}{\tau_{tr}v^2}\varepsilon^2\right)$$
$$=-\int\limits_{|y-y'|\geq\varepsilon}\frac{\exp\{-eH|y^2-y'^2|/c\}}{|y-y'|}\Delta^*(y')\,dy'. \quad (34)$$

Using a variational method [$\exp(-\alpha y^2)$ is taken as the varied function] we obtain the maximum field at zero temperature (which is attained when $\alpha=eH/c$):

$$eH_{c10}/c=3\gamma\Delta_0/2v^2\tau_{tr}. \quad (35)$$

We introduce the constant κ of the phenomenological Ginzburg-Landau theory,[9] defined as

$$\varkappa=\sqrt{2}\,2eH_c\delta^2/\hbar c$$

as $T\to T_c$ (H_c is the thermodynamic critical field, δ is the penetration depth). In the limiting case considered,[5]

$$\varkappa=\frac{3}{2\pi^2}\frac{mc}{e\tau_{tr}}\left(\frac{2\pi m}{p_0^5}\cdot 7\zeta(3)\right)^{1/2}.$$

The theoretical value of the thermodynamic critical field of a pure metal at T = 0 is

$$H_{c0}=\Delta_0\sqrt{2mp_0/\pi}.$$

The ratio of the two critical fields can be expressed in terms of κ:

$$\frac{H_{c10}}{H_{c0}}=\frac{\pi^2\gamma}{2\sqrt{7\zeta(3)}}\varkappa=3.03\varkappa. \quad (36)$$

The upper critical field can be expressed in terms of the critical temperature T_c, the conductivity of the normal metal at low temperatures σ, and the coefficient γ in the linear thermal capacity relationship:

$$H_{c10}=\frac{3}{2\pi}\frac{ec\gamma T_c}{\sigma k},\qquad \frac{H_{c10}}{H_{c0}}=e^C\sqrt{\frac{3}{8\pi^3}}\frac{ec\gamma^{1/2}}{\sigma k}. \quad (37)$$

If at T = 0 the penetration depth $\delta_0\gg l$ (when $\delta_0=(2\pi)^{-1}\times c\sqrt{\hbar/\sigma\Delta_0}$, (cf.[4]), then

$$H_{c10}=\frac{6\pi^2}{e^C}\frac{e\gamma T_c^2\delta_0^2}{c\hbar},\qquad \frac{H_{c10}}{H_{c0}}=\sqrt{6\pi^3}\frac{e\gamma^{1/2}T_c\delta_0^2}{c\hbar}. \quad (37')$$

For non-zero temperatures the corresponding thermodynamic technique must be used.[7,8] Then there will appear in all the formulae sums with respect to $\omega_n=\pi T(2n+1)$ in place of integrals with respect to $d\omega$.

At large distances the kernel $K_T(R)$ will, instead of (30), be

$$K_T(R)=\frac{mp_0|g|}{(2\pi)^3}\frac{2}{R^3}2\int\limits_0^\infty e^{-\eta}\sin\eta\,\mathrm{th}\left(\frac{v^2\tau_{tr}}{6TR^2}\eta^2\right)\eta\,d\eta. \quad (38)^*$$

At finite temperatures Eq. (34) becomes

*th = tanh.

$$\Delta^*(y)\ln\left(\frac{3\gamma^2\Delta_0}{\tau_{tr}v^2}\varepsilon^2\right) = -\int\limits_{|y-y'|\geqslant\varepsilon} K_T(y, y')\,\Delta^*(y')\,dy',$$

$$K_T(y, y') = \int\limits_0^\infty \mathrm{th}\left(\frac{\tau_{tr}v^2}{6T}\eta\right)$$

$$\times\mathrm{Im}\,\frac{\exp\{-|y-y'|\,[((y+y')eH/c)^2-2i\eta]^{1/2}\}}{[((y+y')eH/c)^2-2i\eta]^{1/2}}\,d\eta. \qquad (39)$$

Again we use a variational method, choosing $\Delta^*(y)$ as $\exp(-\alpha y^2)$. The maximum value of the magnetic field is attained when $\alpha = eH/c$, and is determined by the solution of the equation

$$\ln\left(\frac{3\gamma\Delta_0}{2\tau_{tr}v^2}\,\frac{c}{eH_{c1}}\right) = F\left(\frac{3\pi T}{2\tau_{tr}v^2}\,\frac{c}{eH_{c1}}\right),$$

$$F(x) = 2x\int\limits_0^\infty \frac{d\eta}{\eta+2x}\left(\frac{1}{\eta}-\frac{1}{\mathrm{sh}\,\eta}\right) \qquad (40)^*$$

($F(x) \to 2x\ln 2$ as $x \to 0$ and $F(x) \to \ln x + \pi^2/8x$ as $x \to \infty$).

It is convenient to write this equation introducing from (35) — (37) the critical field H_{c10} at $T = 0$:

$$\ln(H_{c10}/H_{c1}) = F(TH_{c10}/T_cH_{c1}). \qquad (41)$$

For low temperatures ($T \ll T_c$)

$$\frac{H_{c1}}{H_{c10}} = 1-\frac{T}{T_c}2\ln 2, \quad \frac{H_{c1}}{H_c} = \left(1-\frac{T}{T_c}2\ln 2\right)3.03\varkappa. \qquad (42)$$

Close to the critical temperature ($T_c - T \ll T_c$)

$$\frac{H_{c1}}{H_{c10}} = \left(1-\frac{T}{T_c}\right)\frac{8}{\pi^2}, \quad H_c = \frac{4\gamma}{\sqrt{14\zeta(3)}}\left(1-\frac{T}{T_c}\right)H_{c0},$$

$$\frac{H_{c1}}{H_c} = \sqrt{2}\varkappa, \qquad (43)$$

which agrees with the result given by the phenomenological Ginzburg-Landau theory.[9,10] This is natural, since for $T_c - T \ll T_c$ the integral equation (39) can be transformed into the differential equation of the same problem in this theory.

The variation of the upper critical field on temperature is shown in Fig. 4. It is clear from the graph that the variation is almost linear, the curve being convex from below, i.e., $\partial^2 H_{c1}/\partial T^2 > 0$, in distinction from the temperature variation of the thermodynamic critical field, which is almost parabolic and has $\partial^2 H_c/\partial T^2 < 0$.

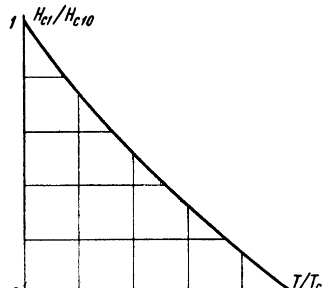

FIG. 4

*sh = sinh.

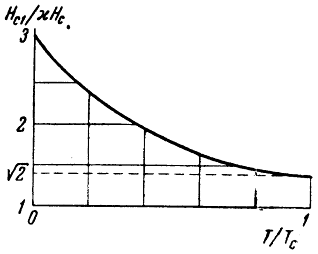

FIG. 5

It should be remembered that the variational principle gives a somewhat reduced value of the upper critical field, and the error increases as the temperature is lowered, so that, in fact, the curvature ought to be somewhat greater than that shown in the figure. Unfortunately, there are no experimental data on the temperature variation of the upper critical field throughout a wide temperature range, with which it would have been possible to compare the results of the present work. Experiments close to T_c give a trivial linear variation. For an accurate comparison of the theory with experiment it is necessary to know the mean free path of the electrons, i.e., the residual resistance of the specimens studied.

Figure 5 shows the temperature variation of the ratio H_{c1}/H_c.

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