# HIGH-FREQUENCY DIELECTRIC CONSTANT OF A PLASMA

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Submitted to JETP editor April 3, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 861-870 (September, 1961)

An expression is found for the complex dielectric constant of a dilute totally ionized plasma; the expression is accurate to quadratic terms in the number of particles per unit volume and applies for frequencies much higher than the electron Langmuir frequency. An isotropic plasma and a plasma in a magnetic field are considered. In the latter case the applied frequency is assumed to be greater than the Langmuir frequency, but can be greater or smaller than the gyromagnetic frequencies of the plasma particles.

**1.** The present communication is concerned with the high-frequency dielectric constant of a plasma. Specifically, we are interested in the frequency region in which the applied frequency $\omega$ is much higher than the electron Langmuir frequency $\omega_{Le} = \sqrt{4\pi e^2 N_e/m}$. On the other hand, we shall assume that the frequency is much lower than the frequency $\omega_{max} = (\kappa T)^{3/2}(2m)^{-1/2}/|e_i e|$. To describe a plasma under these conditions we find it convenient to use the kinetic equation for rapid processes given earlier by the author.[1]

Under the conditions given above it is well-known[2] that the imaginary part of the dielectric constant of an isotropic plasma must be modified; the correction is quadratic in the number of particles per unit volume and exhibits a frequency dependence of the form $\omega^{-3}\ln(\omega/\omega_{max})$. Below we obtain the corresponding real correction, which exhibits a frequency dependence of the form $\omega^{-3} \times \operatorname{sign}\omega$. Finally, we obtain the correction to the dielectric tensor of a plasma in a strong magnetic field when neither the applied frequency nor the electron (ion) gyromagnetic frequency are small compared with the electron Langmuir frequency.

**2.** We find the dielectric constant of the plasma through the use of the kinetic equation for rapid steady-state processes. For a plasma in a spatially uniform alternating electric field **E** and a fixed magnetic field **B** this equation is:[1]

$$\frac{\partial f_\alpha}{\partial t} + e_\alpha\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}_\alpha\mathbf{B}]\right)\frac{\partial f_\alpha}{\partial \mathbf{p}_\alpha} = \sum_\beta N_\beta \frac{\partial}{\partial p^i_\alpha}\int d\mathbf{p}_\beta\, d\mathbf{r}_\beta \times \frac{\partial U_{\alpha\beta}(|\mathbf{r}_\alpha - \mathbf{r}_\beta|)}{\partial r^i_\alpha}\int_{-\infty}^{0} d\tau \left\{\frac{\partial}{\partial r^j_\alpha} U_{\alpha\beta}(|\mathbf{R}_\alpha[t+\tau, t, \mathbf{p}_\alpha, \mathbf{r}_\alpha] - \mathbf{R}_\beta[t+\tau, t, \mathbf{p}_\beta, \mathbf{r}_\beta]|)\right\}\left\{\frac{\partial}{\partial P^j_\alpha[t+\tau, t, \mathbf{p}_\alpha]} - \frac{\partial}{\partial P^j_\beta[t+\tau, t, \mathbf{p}_\beta]}\right\} f_\alpha(\mathbf{P}_\alpha[t+\tau, t, \mathbf{p}_\alpha], \mathbf{R}_\alpha[t+\tau, t, \mathbf{p}_\alpha, \mathbf{r}_\alpha], t+\tau)\, f_\beta(\mathbf{P}_\beta[t+\tau, t, \mathbf{p}_\beta], \mathbf{R}_\beta[t+\tau, t, \mathbf{p}_\beta, \mathbf{r}_\beta], t+\tau). \quad (1)^*$$

*$[\mathbf{v}_\alpha\mathbf{B}] = \mathbf{v}_\alpha \times \mathbf{B}$; $(\mathbf{B}\mathbf{p}_\alpha) = \mathbf{B}\cdot\mathbf{p}_\alpha$.

Here

$$\mathbf{P}_\alpha[t+\tau, t, \mathbf{p}_\alpha] = \mathbf{B}\frac{(\mathbf{B}\mathbf{p}_\alpha)}{B^2} - \sin\Omega_\alpha\tau\frac{[\mathbf{B}\mathbf{p}_\alpha]}{B} - \cos\Omega_\alpha\tau\frac{[\mathbf{B}[\mathbf{B}\mathbf{p}_\alpha]]}{B^2} + e_\alpha\int_t^{t+\tau} dt'\left\{\mathbf{B}\frac{(\mathbf{B}\mathbf{E}(t'))}{B^2} - \frac{[\mathbf{B}\mathbf{E}(t')]}{B}\sin\Omega_\alpha(t+\tau-t') - \frac{[\mathbf{B}[\mathbf{B}\mathbf{E}(t')]]}{B^2}\cos\Omega_\alpha(t+\tau-t')\right\}, \quad (2)$$

$$\mathbf{R}_\alpha[t+\tau, t, \mathbf{p}_\alpha, \mathbf{r}_\alpha] = \mathbf{r}_\alpha + \mathbf{B}\frac{(\mathbf{B}\mathbf{v}_\alpha)}{B^2}\tau - \frac{1-\cos\Omega_\alpha\tau}{\Omega_\alpha}\frac{[\mathbf{B}\mathbf{v}_\alpha]}{B} - \frac{\sin\Omega_\alpha\tau}{\Omega_\alpha}\frac{[\mathbf{B}[\mathbf{B}\mathbf{v}_\alpha]]}{B^2} + \frac{e_\alpha}{m_\alpha}\int_t^{t+\tau} dt'\int_t^{t'} dt''\left\{\mathbf{B}\frac{(\mathbf{B}\mathbf{E}(t''))}{B^2} - \frac{[\mathbf{B}\mathbf{E}(t'')]}{B}\sin\Omega_\alpha(t'-t'') - \frac{[\mathbf{B}[\mathbf{B}\mathbf{E}(t'')]]}{B^2}\cos\Omega_\alpha(t'-t'')\right\}, \quad (3)$$

where $e_\alpha$, $m_\alpha$, $\mathbf{r}_\alpha$, $\mathbf{v}_\alpha$, and $\mathbf{p}_\alpha$ are respectively the charge, mass, coordinate, velocity, and momentum of a particle of type $\alpha$; $\Omega_\alpha = e_\alpha B/m_\alpha c$ is the gyromagnetic frequency; $N_\alpha$ is the number of particles of type $\alpha$ per unit volume and, $U_{\alpha\beta}(r) = e_\alpha e_\beta/r$.

Equation (1) has been obtained under the assumption of a weak particle interaction and does not apply for small impact parameters. In this connection, in integrating over the impact parameters below, we introduce a cutoff at $\rho_{min}$. On the other hand we have not considered shielding of the Coulomb interaction at large distances in Eq. (1). Equation (1) may not be applicable to an analysis of collisions at high impact parameters in sufficiently slow processes, in which case we must introduce a cutoff at $\rho_{max}$.*

**3.** We first consider an isotropic plasma with no fixed magnetic field. We neglect spatial dis-

*If collisions are neglected the following relation holds:

$$f_\alpha(\mathbf{P}_\alpha[t+\tau, t, \mathbf{p}_\alpha], \mathbf{R}_\alpha[t+\tau, t, \mathbf{p}_\alpha, \mathbf{r}_\alpha], t+\tau) = f_\alpha(\mathbf{p}_\alpha, \mathbf{r}_\alpha, t),$$

and allows us to transform the arguments of the functions in the right side of Eq. (1).

persion in the dielectric constant, assuming the particle distribution to be uniform in space. Assuming a small departure from the Maxwellian distribution $f_{\alpha}^{(0)}$ and linearizing the kinetic equation, taking account of the fact that $\delta f_{\alpha}$ is proportional to the electric field, which is assumed to be weak, we have

$$\frac{\partial \delta f_{\alpha}}{\partial t} - \frac{e_{\alpha}}{\varkappa T}\mathbf{E}\mathbf{v}_{\alpha} f_{\alpha}^{(0)} = \sum_{\beta} N_{\beta}\frac{\partial}{\partial p_{\alpha}^{i}}\int d\mathbf{p}_{\beta}\, d\mathbf{r}_{\beta} \times \int_{-\infty}^{0} d\tau \left\{\frac{\partial U_{\alpha\beta}(|\mathbf{r}_{\alpha}-\mathbf{r}_{\beta}+(\mathbf{v}_{\alpha}-\mathbf{v}_{\beta})\tau|)}{\partial r_{\alpha}^{i}} \times \frac{\partial U_{\alpha\beta}(\mathbf{r}_{\alpha}-\mathbf{r}_{\beta})}{\partial r_{\alpha}^{i}}\left[\frac{\partial}{\partial p_{\alpha}^{i}} - \frac{\partial}{\partial p_{\beta}^{i}}\right][f_{\alpha}^{(0)}\delta f_{\beta}(\mathbf{p}_{\beta}, t+\tau) + \delta f_{\alpha}(\mathbf{p}_{\alpha}, t+\tau) f_{\beta}^{(0)}] - \frac{f_{\alpha}^{(0)} f_{\beta}^{(0)}}{(\varkappa T)^2}\frac{\partial U_{\alpha\beta}(\mathbf{r}_{\alpha}-\mathbf{r}_{\beta})}{\partial r_{\alpha}^{i}} U_{\alpha\beta}(|\mathbf{r}_{\alpha} - \mathbf{r}_{\beta} + (\mathbf{v}_{\alpha}-\mathbf{v}_{\beta})\tau|)(e_{\alpha}\mathbf{v}_{\alpha} + e_{\beta}\mathbf{v}_{\beta}, \mathbf{E}(t+\tau))\right\}. \quad (4)$$

In solving Eq. (4) we assume that the collision integral is small. If there is a periodic time dependence ($e^{-i\omega t}$) the inequality $\omega \gg \nu_{\text{eff}}$ must be satisfied, where $\nu_{\text{eff}}$ is defined below. In the first approximation

$$\delta f_{\alpha}^{(1)} = i\frac{e_{\alpha}}{\varkappa T}\frac{\mathbf{E}\mathbf{v}_{\alpha}}{\omega} f_{\alpha}^{(0)}. \quad (5)$$

Substituting Eq. (5) for $\delta f$ in Eq. (4), we have as a second approximation

$$\delta f_{\alpha}^{(2)} = \frac{1}{\omega^2}\sum_{\beta} N_{\beta}\frac{\partial}{\partial p_{\alpha}^{i}}\int d\mathbf{p}_{\beta}\, d\mathbf{r}_{\beta}\frac{\partial U_{\alpha\beta}(|\mathbf{r}_{\alpha}-\mathbf{r}_{\beta}|)}{\partial r_{\alpha}^{i}}\int_{-\infty}^{0} d\tau e^{-i\omega\tau} \times \frac{\partial U_{\alpha\beta}(|\mathbf{r}_{\alpha}-\mathbf{r}_{\beta}+(\mathbf{v}_{\alpha}-\mathbf{v}_{\beta})\tau|)}{\partial r_{\alpha}^{j}}\frac{1}{(\varkappa T)^2} f_{\alpha}^{(0)} f_{\beta}^{(0)}\left(\frac{e_{\beta}}{m_{\beta}} - \frac{e_{\alpha}}{m_{\alpha}}\right)E_j. \quad (6)$$

An expression for the current density can be obtained from Eqs. (5) and (6)

$$\mathbf{j} = \sum_{\alpha} e_{\alpha}N_{\alpha}\int d\mathbf{p}_{\alpha}\mathbf{v}_{\alpha}\delta f_{\alpha}, \quad (7)$$

and can then be used to find the complex conductivity tensor $\sigma_{ij}$ ($j_i = \sigma_{ij}E_j$) or the complex dielectric tensor $\epsilon_j^i = \delta_{ij} + 4\pi i\sigma_{ij}/\omega$. For the isotropic plasma being considered here these tensors are diagonal and, from Eqs. (5) — (7), we have

$$\varepsilon(\omega) = 1 - \sum_{\alpha}\frac{4\pi e_{\alpha}^2 N_{\alpha}}{\omega^2 m_{\alpha}} + \frac{4\pi i}{\omega^3}\sum_{\alpha\beta}\frac{e_{\alpha}}{m_{\alpha}}\left(\frac{e_{\alpha}}{m_{\alpha}} - \frac{e_{\beta}}{m_{\beta}}\right)\frac{N_{\alpha}N_{\beta}}{\varkappa T}\int d\mathbf{p}_{\alpha}\, d\mathbf{p}_{\beta} \times f_{\alpha}^{(0)} f_{\beta}^{(0)}\int_{-\infty}^{0} d\tau e^{-i\omega\tau}\int_{k_{min}}^{k_{max}}\frac{d\mathbf{k}}{(2\pi)^3}\frac{(4\pi e_{\alpha}e_{\beta})^2}{3k^2}\exp\{i\mathbf{k}(\mathbf{v}_{\alpha}-\mathbf{v}_{\beta})\tau\}, \quad (8)$$

where $k_{max} = \rho_{min}^{-1} = \kappa T/|e_{\alpha}e_{\beta}|$ while $k_{min} = \rho_{max}^{-1} \approx r_D^{-1}$ ($r_D$ is the Debye radius).

If terms containing positive powers of the electron-ion mass ratio are neglected in the right side of Eq. (8) and if it is assumed that only one type of ion is present, we have

$$\varepsilon(\omega) = 1 - \frac{\omega_{Le}^2}{\omega^2} + i\frac{\omega_{Le}^2}{\omega^3}\frac{4}{3}\frac{\sqrt{2\pi}(ee_i)^2 N_i}{\sqrt{m}(\varkappa T)^{3/2}}F(\omega); \quad (9)$$

$$F(\omega) = \int_{0}^{\infty}\frac{d\tau}{\tau}e^{i\omega\tau}\left[\Phi\left(\tau\sqrt{\frac{\varkappa T}{2m}}k_{max}\right) - \Phi\left(\tau\sqrt{\frac{\varkappa T}{2m}}k_{min}\right)\right],$$

$$\Phi(x) = \frac{2}{\sqrt{\pi}}\int_{0}^{x} e^{-t^2}dt. \quad (10)$$

Keeping in mind that $k_{max} \gg k_{min}$ and also that these quantities are determined to an accuracy of order unity, we can write Eq. (10) in the form

$$F(\omega) = F'(\omega) + iF''(\omega) = \int_{\tau_{min}}^{\tau_{max}}\frac{d\tau}{\tau}\cos\omega\tau + i\int_{\tau_{min}}^{\tau_{max}}\frac{d\tau}{\tau}\sin\omega\tau,$$

$$\tau_{min} = \sqrt{2m/\varkappa T}\, k_{max}^{-1}, \quad \tau_{max} = \sqrt{2m/\varkappa T}\, k_{min}^{-1} \approx \omega_{Le}^{-1}. \quad (11)$$

When $\omega \ll \omega_{Le}$,

$$F'(\omega) = \ln\frac{k_{max}}{k_{min}}, \quad F''(\omega) \approx \frac{\omega}{\omega_{Le}}. \quad (12)$$

The last expression for $F''$ leads to a small correction (proportional to $\omega^{-2}$) in $\epsilon(\omega)$. Hence, $\omega_{Le}^2$ changes by an amount $\Delta\omega_{Le}^2$, where

$$\Delta \approx \frac{4}{3}\frac{\sqrt{2\pi}(ee_i)^2 N_i}{\sqrt{m}(\varkappa T)^{3/2}}\frac{1}{\omega_{Le}}.$$

Equation (12) for $F'(\omega)$ leads to the usual effective collision frequency, which then yields the following expression for the dielectric constant:[2-4]

$$\varepsilon(\omega) = 1 - \frac{\omega_{Le}^2}{\omega^2}(1+\Delta) + i\frac{\omega_{Le}^2}{\omega^3}\nu_{\text{eff}}^{(0)}, \quad (13)$$

where

$$\nu_{\text{eff}}^{(0)} = \frac{4}{3}\frac{\sqrt{2\pi}(ee_i)^2 N_i}{\sqrt{m}(\varkappa T)^{3/2}}\ln\left(\frac{\varkappa T}{|ee_i|}r_D\right). \quad (14)$$

At frequencies much greater than the electron Langmuir frequency ($\omega \gg \omega_{Le}$) we have from Eq. (11)

$$F'(\omega) = \ln\left(\frac{k_{max}}{\gamma|\omega|}\sqrt{\frac{\varkappa T}{2m}}\right), \quad F''(\omega) = \frac{\pi}{2}\operatorname{sign}\omega; \quad (15)$$

$\gamma = 1.781$ is the Euler constant. Substituting (15) in (9) we have

$$\varepsilon(\omega) = \varepsilon' + i\varepsilon'' = 1 - \frac{\omega_{Le}^2}{\omega^2} + i\frac{\omega_{Le}^2}{\omega^3}\nu_{\text{eff}}^{(\omega)} - \frac{\omega_{Le}^2}{\omega^3}\omega_{\text{eff}}\operatorname{sign}\omega, \quad (16)$$

where

$$\omega_{\text{eff}} = \frac{(2\pi)^{3/2}(ee_i)^2 N_i}{3\sqrt{m}(\varkappa T)^{3/2}}, \quad (17)$$

while $\nu_{\text{eff}}(\omega)$ is of the form known from the theory of absorption of radio waves in interstellar gases:[2]

$$\nu_{\text{eff}}^{(\omega)} = \frac{4}{3}\frac{\sqrt{2\pi}\,(ee_i)^2 N_i}{\sqrt{m}\,(\varkappa T)^{3/2}} \ln\left|\frac{(\varkappa T)^{3/2}}{\gamma\omega\sqrt{2m}\,|ee_i|}\right|. \tag{18}$$

The quantity $\nu_{\text{eff}}$ differs from $\omega_{\text{eff}}$ by the large logarithmic term. Since the logarithmic term depends on the choice of $k_{\max}$, there is some question as to the usefulness of keeping the term proportional to $\omega_{\text{eff}}$. However, $\nu_{\text{eff}}$ appears in the imaginary part of the dielectric constant while $\omega_{\text{eff}}$ appears in the real part, so that we are justified in keeping the term proportional to $\omega_{\text{eff}}$.

There is an important difference in the correction to the real part of $\epsilon(\omega)$ when $\omega \gg \omega_{Le}$ and $\omega \ll \omega_{Le}$. The absolute magnitude of the correction is smaller in the second case than in the first (low frequency); however, a new dependence on frequency arises in the region $\omega \gg \omega_{Le}$ and in principle, makes it possible to observe the corresponding correction. The difficulty of an observation of this kind is due to the necessity for satisfying the condition $\omega \ll (\kappa T)^{3/2} m^{-1/2}/|ee_i|$. For example, in the expression for the refractive index

$$n = \sqrt{\varepsilon'} \approx 1 - \frac{\omega_{Le}^2}{2\omega^2} - \frac{1}{8}\frac{\omega_{Le}^4}{\omega^4} - \frac{1}{2}\frac{\omega_{Le}^2\omega_{\text{eff}}}{\omega^3} - \frac{1}{16}\frac{\omega_{Le}^6}{\omega^6} + \cdots \tag{19}$$

the third term ($\sim \omega^{-4}$) is always greater than the fourth, which is proportional to $\omega^{-3}$. Thus the correction will appear only when the frequency dependence of the refractive index is determined with very high accuracy. We note that the fourth term of the right side of Eq. (19) is smaller than the third, but greater than the fifth, if the following condition holds:

$$N_e^{1/3}(\varkappa T)^{1/2} m^{-1/2} \ll \omega \ll (\varkappa T)^{3/2} m^{-1/2}/|ee_i|.$$

This frequency range is rather wide since it is given by the relation $|ee_i|N_e^{1/3} \ll \kappa T$.

**4.** We now consider a plasma in a fixed magnetic field. In this case the linearized kinetic equation for weak departures from the Maxwellian distribution can be written in the form

$$\begin{aligned}\frac{\partial \delta f_\alpha}{\partial t} &+ \frac{e_\alpha}{c}[\mathbf{v}_\alpha \mathbf{B}]\frac{\partial \delta f_\alpha}{\partial \mathbf{p}_\alpha} - \frac{e_\alpha}{\varkappa T}\mathbf{E}\mathbf{v}_\alpha f_\alpha^{(0)} = \sum_\beta N_\beta \frac{\partial}{\partial p_\alpha^i}\int d\mathbf{p}_\beta\, d\mathbf{r}_\beta \\ &\times \frac{\partial U_{\alpha\beta}(|\mathbf{r}_\alpha - \mathbf{r}_\beta|)}{\partial r_\alpha^i}\int_{-\infty}^{0} d\tau\left\{\left[\frac{\partial}{\partial r_\alpha^j} U_{\alpha\beta}(|\mathbf{R}_\alpha^0(\tau, \mathbf{p}_\alpha, \mathbf{r}_\alpha)\right.\right. \\ &- \mathbf{R}_\beta^0(\tau, \mathbf{p}_\beta, \mathbf{r}_\beta)|)\Big]\left[\frac{\partial}{\partial P_\alpha^{0j}(\tau, \mathbf{p}_\alpha)} - \frac{\partial}{\partial P_\beta^{0j}(\tau, \mathbf{p}_\beta)}\right][f_\alpha^{(0)}\delta f_\beta \\ &\times (\mathbf{P}_\beta^0(\tau, \mathbf{p}_\beta), t+\tau) + f_\beta^{(0)}\delta f_\alpha(\mathbf{P}_\alpha^{(0)}(\tau, \mathbf{p}_\alpha), t+\tau)] \\ &- U_{\alpha\beta}(|\mathbf{R}_\alpha^0(\tau, \mathbf{p}_\alpha, \mathbf{r}_\alpha) - \mathbf{R}_\beta^0(\tau, \mathbf{p}_\beta, \mathbf{r}_\beta)|) \\ &\times \frac{1}{(\varkappa T)^2} f_\alpha^{(0)} f_\beta^{(0)}\left(\mathbf{E}(t+\tau), \frac{e_\alpha}{m_\alpha}\mathbf{P}_\alpha^0(\tau, \mathbf{p}_\alpha) + \frac{e_\beta}{m_\beta}\mathbf{P}_\beta^0(\tau, \mathbf{p}_\beta)\right)\Big\}.\end{aligned} \tag{20}$$

Here, $\mathbf{P}^0$ and $\mathbf{R}^0$ are given by Eqs. (2) and (3) if the electric field is set equal to zero.

When the field is periodic in time ($e^{-i\omega t}$) Eq. (20) can be solved by perturbation methods by taking the right side to be small, so long as the condition $|\omega^2 \pm \Omega_\alpha^2| \gg \nu_{\text{eff}}^2$ is satisfied. We assume below that this condition is satisfied. Then, assuming that the collision integral can be neglected, we have

$$\delta f_\alpha^{(1)}(\mathbf{p}_\alpha, t) = \frac{e_\alpha f_\alpha^{(0)}}{m_\alpha \varkappa T} A_{sr}(\omega, \Omega_\alpha)\, p_\alpha^r E_s e^{-i\omega t}, \tag{21}$$

$$A_{sr} = \left\{\frac{B_s B_r}{B^2}\frac{i}{\omega} - \frac{\Omega_\alpha}{\omega^2 - \Omega_\alpha^2} e_{slr}\frac{B_l}{B} - \frac{i\omega}{\omega^2 - \Omega_\alpha^2}\frac{B_s B_r - B^2\delta_{sr}}{B^2}\right\}. \tag{22}$$

Here, $e_{slr}$ is a completely antisymmetric tensor. Using Eq. (21) we obtain the following equation for the second approximation correction to the nonequilibrium part of the distribution function:

$$\begin{aligned}\frac{\partial \delta f_\alpha^{(2)}}{\partial t} &+ \frac{e_\alpha}{c}[\mathbf{v}_\alpha \mathbf{B}]\frac{\partial \delta f_\alpha^{(2)}}{\partial \mathbf{p}_\alpha} \\ &= \sum_\beta N_\beta \frac{\partial}{\partial p_\alpha^i}\int d\mathbf{p}_\beta\, d\mathbf{r}_\beta \frac{f_\beta^{(0)} f_\alpha^{(0)}}{\varkappa T}\frac{\partial U_{\alpha\beta}(|\mathbf{r}_\alpha - \mathbf{r}_\beta|)}{\partial r_\alpha^i} \\ &\times \int_{-\infty}^{0} d\tau\, e^{-i\omega\tau}\left\{\frac{\partial}{\partial r_\alpha^j} U_{\alpha\beta}(|\mathbf{R}_\alpha^0(\tau, \mathbf{p}_\alpha, \mathbf{r}_\alpha) - \mathbf{R}_\beta^0(\tau, \mathbf{p}_\beta, \mathbf{r}_\beta)|)\right\} \\ &\times \left[\frac{e_\alpha}{m_\alpha} A_{sj}(\omega, \Omega_\alpha) - \frac{e_\beta}{m_\beta} A_{sj}(\omega, \Omega_\beta)\right] e^{-i\omega t} E_s.\end{aligned} \tag{23}$$

In the case being considered (steady-state periodic process) the solution of this equation can be written in the form

$$\begin{aligned}\delta f_\alpha^{(2)}(\mathbf{p}_\alpha, t) &= \int_{-\infty}^{t} dt' \sum_\beta N_\beta \frac{\partial}{\partial P_\alpha^{0i}(t'-t, \mathbf{p}_\alpha)}\int d\mathbf{p}_\beta\, d\mathbf{r}_\beta \frac{f_\beta^{(0)} f_\alpha^{(0)}}{\varkappa T} \\ &\times \frac{\partial U_{\alpha\beta}(|\mathbf{r}_\alpha - \mathbf{r}_\beta|)}{\partial r_\alpha^i}\int_{-\infty}^{0} d\tau\, e^{-i\omega\tau}\left\{\frac{\partial}{\partial r_\alpha^j} U_{\alpha\beta}(|\mathbf{R}_\alpha^0(\tau, \mathbf{P}_\alpha^0[t'\right. \\ &- t, \mathbf{p}_\alpha], \mathbf{r}_\alpha) - \mathbf{R}_\beta^0(\tau, \mathbf{p}_\beta, \mathbf{r}_\beta)|)\Big\}\left[\frac{e_\alpha}{m_\alpha} A_{sj}(\omega, \Omega_\alpha)\right. \\ &- \left.\frac{e_\beta}{m_\beta} A_{sj}(\omega, \Omega_\beta)\right] e^{-i\omega t'} E_s.\end{aligned} \tag{24}$$

To find the complex dielectric tensor we substitute (21) and (24) in (7) and integrate over momentum and time ($t'$). In particular, we assume that

$$\int_0^\infty dt\, e^{i\omega t}\frac{\partial}{\partial p_\alpha^i} P_\alpha^r(t, \mathbf{p}_\alpha) = A_{rl}(\omega, -\Omega_\alpha). \tag{25}$$

Thus

$$\begin{aligned}\varepsilon_{ij}(\omega) &= \delta_{ij} + \sum \frac{4\pi e_\alpha^2 N_\alpha}{\omega m_\alpha} i A_{ji}(\omega, \Omega_\alpha) - \frac{4\pi i}{\omega}\sum_{\alpha\beta}\frac{e_\alpha}{m_\alpha} A_{ir}(\omega, -\Omega_\alpha) \\ &\times \left[\frac{e_\alpha}{m_\alpha} A_{js}(\omega, \Omega_\alpha) - \frac{e_\beta}{m_\beta} A_{js}(\omega, \Omega_\beta)\right]\frac{N_\alpha N_\beta}{\varkappa T}(4\pi e_\alpha e_\beta)^2 \\ &\times \int_{-\infty}^{0} d\tau\, e^{-i\omega\tau}\int_{k_{min}}^{k_{max}}\frac{d\mathbf{k}}{(2\pi)^3}\frac{k_r k_s}{k^4}\exp\left\{-\frac{\varkappa T}{2}\left[\frac{1}{m_\alpha} + \frac{1}{m_\beta}\right]\left(\frac{\mathbf{k}\mathbf{B}}{B}\right)^2\tau^2\right. \\ &- \left. 2\varkappa T\frac{B^2k^2 - (\mathbf{B}\mathbf{k})^2}{B^2}\left[\frac{\sin^2(\Omega_\alpha\tau/2)}{m_\alpha\Omega_\alpha^2} + \frac{\sin^2(\Omega_\beta\tau/2)}{m_\beta\Omega_\beta^2}\right]\right\}.\end{aligned} \tag{26}$$

Assuming that the plasma contains electrons and only one ion species and neglecting corrections of the order of the electron-ion mass ratio, we have

$$\varepsilon_{ij}(\omega) = \varepsilon_{ij}^{(0,\,h)} + \delta\varepsilon_{ij}^{(a)} + \delta\varepsilon_{ij}^{(h)}, \tag{27}$$

where $\epsilon_{ij}^{(0,h)}$ is the Hermitian part of the dielectric tensor, obtained if the collision integral is neglected completely

$$\varepsilon_{ij}^{(0,\,h)} = \delta_{ij} - \frac{\omega_{Le}^2}{\omega^2}\left\{\frac{B_iB_j}{B^2} - \frac{\omega^2}{\Omega_e}\left[\frac{\Omega_e}{\omega^2-\Omega_e^2} - \frac{\Omega_i}{\omega^2-\Omega_i^2}\right]\frac{B_iB_j-\delta_{ij}B^2}{B^2}\right.$$
$$\left. + \frac{i\omega}{\Omega_e}\left[\frac{\Omega_e^2}{\omega^2-\Omega_e^2} - \frac{\Omega_i^2}{\omega^2-\Omega_i^2}\right]e_{jli}\frac{B_l}{B}\right\}, \tag{28}$$

while $\delta\epsilon_{ij}^{(h)}$ and $\delta\epsilon_{ij}^{(a)}$ are respectively the Hermitian and anti-Hermitian parts of the dielectric tensor, obtained when the collision integral is taken into account:

$$\delta\varepsilon_{ij} = \delta\varepsilon_{ij}^{(a)} + \delta\varepsilon_{ij}^{(h)} = i\frac{\omega_{Le}^2\omega_{\mathrm{eff}}}{\omega^3}\left\{\frac{B_iB_j}{B^2}F_1(\omega)\right.$$
$$\left. + [F_1(\omega) + F_2(\omega)]\,T_{ij}^{\perp}\right\}, \tag{29}$$

where

$$T_{ij}^{\perp} = -\frac{2i\omega^3}{\Omega_e^2}e_{jil}\frac{B_l}{B}\left[\frac{\Omega_e}{\omega^2-\Omega_e^2} - \frac{\Omega_i}{\omega^2-\Omega_i^2}\right]\left[\frac{\Omega_e^2}{\omega^2-\Omega_e^2} - \frac{\Omega_i^2}{\omega^2-\Omega_i^2}\right]$$
$$- \frac{\omega^2}{\Omega_e^2}\frac{B_iB_j-\delta_{ij}B^2}{B^2}\left\{\omega^2\left[\frac{\Omega_e}{\omega^2-\Omega_e^2} - \frac{\Omega_i}{\omega^2-\Omega_i^2}\right]^2\right.$$
$$\left. + \left[\frac{\Omega_e^2}{\omega^2-\Omega_e^2} - \frac{\Omega_i^2}{\omega^2-\Omega_i^2}\right]^2\right\}.$$

The functions $F_1$ and $F_2$ are given by

$$F_1(\omega) = \frac{3}{\pi}\int_0^\infty d\tau\, e^{i\omega\tau}\int_{-1}^{+1}\frac{dx\,x^2}{\sqrt{\varphi(x,\tau)}}\left\{\Phi\left(k_{max}\sqrt{\frac{\varkappa T}{2m}}\sqrt{\varphi(x,\tau)}\right)\right.$$
$$\left. - \Phi\left(k_{min}\sqrt{\frac{\varkappa T}{2m}}\sqrt{\varphi(x,\tau)}\right)\right\}, \tag{30}$$

$$F_2(\omega) = \frac{3}{2\pi}\int_0^\infty d\tau\, e^{i\omega\tau}\int_{-1}^{+1}dx\,(1-3x^2)\frac{1}{\sqrt{\varphi(x,\tau)}}$$
$$\times\left\{\Phi\left(k_{max}\sqrt{\frac{\varkappa T}{2m}}\sqrt{\varphi(x,\tau)}\right)\right.$$
$$\left. - \Phi\left(k_{min}\sqrt{\frac{\varkappa T}{2m}}\sqrt{\varphi(x,\tau)}\right)\right\}, \tag{31}$$

where

$$\varphi(x,\tau) = \left(1+\frac{m}{m_i}\right)x^2\tau^2 + 4(1-x^2)\left[\frac{\sin^2(\Omega_e\tau/2)}{\Omega_e^2}\right.$$
$$\left. + \frac{m}{m_i}\frac{\sin^2(\Omega_i\tau/2)}{\Omega_i^2}\right]. \tag{32}$$

The real parts of the functions $F_1$ and $F_2$ make contributions to the anti-Hermitian part of the dielectric tensor while the imaginary parts appear in the Hermitian part.

At frequencies much higher than the electron gyromagnetic frequency ($\omega \gg \Omega_e$)

$$F_1(\omega) \approx 2\pi^{-1}F(\omega), \qquad F_2(\omega) \approx 0. \tag{33}$$

Thus, in this frequency region and when $\omega \gg \omega_{Le}$, we have

$$\delta\varepsilon_{ij}(\omega) = \frac{\omega_{Le}^2}{\omega^3}\left[i\nu_{\mathrm{eff}}^{(\omega)} - \omega_{\mathrm{eff}}\right]\left\{\frac{B_iB_j}{B^2} + T_{ij}^{\perp}\right\}. \tag{34}$$

If $\omega$ and $\Omega_e \ll \omega_{Le}$, then $\delta\epsilon_{ij}$ is of the same form but $\nu_{\mathrm{eff}}^{(\omega)}$ and $\omega_{\mathrm{eff}}$ must be replaced by $\nu_{\mathrm{eff}}^{(0)}$ and $\Delta\omega_{Le}$ respectively. We assume below that the applied frequency is greater than the electron Langmuir frequency. In particular, this assumption allows us to write $k_{min} = 0$ in Eqs. (30) and (31).

We first analyze the Hermitian part of the correction to the dielectric tensor, for which purpose we must consider the imaginary parts of the functions $F_1$ and $F_2$. Because the integrands of the corresponding imaginary parts do not have singularities at small $\tau$ we can set $k_{max}$ equal to infinity in the appropriate formulas. Taking $\Omega_e < 0$, we have

$$F_1''(\omega) = -\frac{3}{\pi}\int_0^\infty\frac{d\xi}{\xi}\sin\left(\xi\frac{2\omega}{\Omega_e}\right)\left\{\frac{1}{1-\psi(\xi)}\right.$$
$$\left. - \frac{\psi(\xi)}{[1-\psi(\xi)]^{3/2}}\ln\frac{1+\sqrt{1-\psi(\xi)}}{\sqrt{\psi(\xi)}}\right\}, \tag{35}$$

$$F_2''(\omega) = -\frac{3}{\pi}\int_0^\infty\frac{d\xi}{\xi}\frac{\sin(\xi 2\omega/\Omega_e)}{\sqrt{1-\psi(\xi)}}\left\{-\frac{3}{2}\frac{1}{\sqrt{1-\psi(\xi)}}\right.$$
$$\left. + \left[1+\frac{3}{2}\frac{\psi(\xi)}{1-\psi(\xi)}\right]\ln\frac{1+\sqrt{1-\psi(\xi)}}{\sqrt{\psi(\xi)}}\right\}, \tag{36}$$

where

$$\psi(\xi) = \frac{\sin^2\xi + \frac{e^2}{e_i^2}\frac{m_i}{m}\sin^2\left(\frac{e_i}{e}\frac{m}{m_i}\xi\right)}{[1+m/m_i]\,\xi^2}. \tag{37}$$

To obtain relatively simple expressions we consider several limiting cases assuming at all times, however, that the gyromagnetic electron frequency and the frequency $\omega$ are small compared with $\omega_{max} = (\kappa T)^{3/2}m^{-1/2}/|ee_i|$. At the highest frequency $\omega$, the magnetic field has no effect on collisions; and the situation is given by Eqs. (33) and (34). Hence, in the following we assume that $\Omega_e \gg \omega$.

If this inequality is satisfied the basic contribution in the integrals in (35) and (36) comes from the region $\xi \gg 1$ in which $\psi(\xi)$ is small compared with unity. We must distinguish three regions of large values of $\xi$. First, the region $1 \ll \xi \ll \sqrt{m_i/m}$, where

$$\psi(\xi) \approx \xi^{-2}\sin^2\xi. \tag{38}$$

Second, the region defined by the inequality

$$\sqrt{m_i/m} < \xi < |em_i/e_im|,$$

where

$$\psi(\xi) \approx m/m_i. \qquad (39)$$

Finally, the region $\xi > |em_i/e_i m|$, where

$$\psi(\xi) \approx \frac{1}{\xi^2}\frac{e^2}{e_i^2}\frac{m_i}{m}\sin^2\left(\frac{e_i}{e}\frac{m}{m_i}\xi\right), \qquad (40)$$

Since $\psi(\xi)$ is small, we obtain immediately from Eq. (35)

$$F_1''(\omega) = \frac{3}{2}\operatorname{sign}\omega. \qquad (41)$$

The situation is somewhat more complicated for the function $F_2''(\omega)$. Here, corresponding to the three ranges of values of $\xi$ and the corresponding values of the function $\psi(\xi)$, we have

$$F_2''(\omega) \approx \frac{3}{2}\ln\left|\frac{\Omega_e}{\omega}\right|\operatorname{sign}\omega, \qquad \Omega_e\sqrt{\frac{m}{m_i}} \ll \omega \ll \Omega_e, \qquad (42)$$

$$F_2''(\omega) = \frac{3}{4}\left[\ln\frac{4m_i}{m} - 3\right]\operatorname{sign}\omega, \qquad \Omega_i \ll \omega \ll \Omega_e\sqrt{\frac{m}{m_i}}, \qquad (43)$$

$$F_2''(\omega) = \frac{3}{2}\ln\left|\frac{\Omega_i}{\omega}\sqrt{\frac{m_i}{m}}\right|\operatorname{sign}\omega, \qquad \omega \ll \Omega_i. \qquad (44)$$

Equations (29), (41) — (44) allow us to write the Hermitian part of the correction to the complex dielectric tensor in the following form:

$$\delta\varepsilon_{ij}^{(h)}(\omega) = -\frac{3}{2}\frac{\omega_{Le}^2\omega_{\text{eff}}}{\omega^3}\operatorname{sign}\omega\left\{\frac{B_iB_j}{B^2} + \ln\left|\frac{\Omega_e}{\omega}\right|T_{ij}^{\perp}\right\}, \qquad (45)$$

$$\Omega_e\sqrt{\frac{m}{m_i}} \ll \omega \ll \Omega_e;$$

$$\delta\varepsilon_{ij}^{(h)}(\omega) = -\frac{3}{2}\frac{\omega_{Le}^2\omega_{\text{eff}}}{\omega^3}\operatorname{sign}\omega\left\{\frac{B_iB_j}{B^2} + \frac{1}{2}\left[\ln\frac{4m_i}{m} - 1\right]T_{ij}^{\perp}\right\}, \qquad (46)$$

$$\Omega_i \ll \omega \ll \Omega_e\sqrt{\frac{m}{m_i}};$$

$$\delta\varepsilon_{ij}^{(h)}(\omega) = -\frac{3}{2}\frac{\omega_{Le}^2\omega_{\text{eff}}}{\omega^3}\operatorname{sign}\omega\left\{\frac{B_iB_j}{B^2} + \ln\left|\frac{\Omega_i}{\omega}\sqrt{\frac{m_i}{m}}\right|T_{ij}^{\perp}\right\},$$

$$\omega \ll \Omega_i. \qquad (47)$$

We now consider the anti-Hermitian correction. As in the analysis of the imaginary parts of $F_1$ and $F_2$ we can write $k_{max} = \infty$ in the real part of $F_2$. Then, $F_2'(\omega)$ assumes a form similar to (36) with the difference that the cosine appears in place of the sine. When $\omega \ll \Omega_e$, we have

$$F_2'(\omega) = \frac{3}{2\pi}\int_1^\infty \frac{d\xi}{\xi}\cos\left(\xi\frac{2\omega}{\Omega_e}\right)\left\{\ln\frac{4}{\psi(\xi)} - 3\right\}. \qquad (48)$$

Corresponding to Eqs. (38) — (40), we have from Eq. (48)

$$F_2'(\omega) = \frac{3}{2\pi}\left[\ln\frac{\omega}{\Omega_e}\right]^2, \qquad \Omega_e\sqrt{\frac{m}{m_i}} \ll \omega \ll \Omega_e; \qquad (49)$$

$$F_2'(\omega) = \frac{3}{2\pi}\left\{\ln\left|\frac{\Omega_e}{2\gamma\omega}\right|\left[\ln\frac{4m_i}{m} - 3\right] - \frac{1}{4}\left[\ln\frac{m_i}{m}\right]^2\right\},$$

$$\Omega_i \ll \omega \ll \Omega_e\sqrt{m/m_i}; \qquad (50)$$

$$F_2'(\omega) = \frac{3}{2\pi}\left\{\left(\ln\frac{\Omega_i}{\omega}\right)^2 + \ln\frac{m_i}{m}\ln\left|\frac{\Omega_i}{\omega}\right| + \frac{3}{4}\left[\ln\frac{m_i}{m}\right]^2\right\},$$

$$\omega \ll \Omega_i. \qquad (51)$$

We cannot write $k_{max} = \infty$ in $F_1'(\omega)$. However, since the integrand in Eq. (30) is independent of magnetic field when $\tau \ll 1/\Omega_e$, an expression for $F_1'(\omega)$ can be written that applies for both $\omega \gg \Omega_e$ and $\omega \ll \Omega_e$:

$$F_1'(\omega) = \frac{2}{\pi}\int_{\xi_{min}}^1 \frac{d\xi}{\xi}\cos\left(\frac{2\omega}{\Omega_e}\xi\right) + \frac{3}{\pi}\int_1^\infty \frac{d\xi}{\xi}\cos\left(\frac{2\omega}{\Omega_e}\xi\right),$$

$$\xi_{min} = \sqrt{2m}\,\Omega_e|ee_i|(\varkappa T)^{-3/2}. \qquad (52)$$

In the case of immediate interest we have

$$F_1'(\omega) = \frac{2}{\pi}\ln\frac{(\varkappa T)^{3/2}}{\gamma\sqrt{2m}\,\Omega_e|ee_i|} + \frac{3}{\pi}\ln\left|\frac{\Omega_e}{2\omega}\right|. \qquad (53)$$

Then, using Eqs. (29), (49) — (51) and (53) we can write an expression for the anti-Hermitian part of the complex dielectric tensor

$$\delta\varepsilon_{ij}^{(a)}(\omega) = i\left\{\frac{B_iB_j}{B^2}[\nu_{\text{eff}}^{(\Omega)} + \delta\nu_{\parallel}(\omega)] + T_{ij}^{\perp}[\nu_{\text{eff}}^{(\Omega)} + \delta\nu_{\perp}(\omega)]\right\}\frac{\omega_{Le}^2}{\omega^3}, \qquad (54)$$

where

$$\nu_{\text{eff}}^{(\Omega)} = \frac{4}{3}\frac{\sqrt{2\pi}(ee_i)^2N_i}{\sqrt{m}(\varkappa T)^{3/2}}\ln\frac{(\varkappa T)^{3/2}}{\gamma\Omega_e\sqrt{2m}|ee_i|}, \qquad (55)$$

$$\delta\nu_{\parallel}(\omega) = 2\frac{\sqrt{2\pi}(ee_i)^2N_i}{\sqrt{m}(\varkappa T)^{3/2}}\ln\left|\frac{\Omega_e}{2\omega}\right|, \qquad (56)$$

$$\delta\nu_{\perp}(\omega) = \frac{\sqrt{2\pi}(ee_i)^2N_i}{\sqrt{m}(\varkappa T)^{3/2}}\begin{cases}\left[\ln\frac{\Omega_e}{\omega}\right]^2, & \Omega_e\sqrt{m/m_i} \ll \omega \ll \Omega_e; \quad (57)\\ \left[\ln\frac{4m_i}{m} - 1\right]\ln\left|\frac{\Omega_e}{\omega}\right| - \frac{1}{4}\left[\ln\frac{m_i}{m}\right]^2, & \Omega_i \ll \omega \ll \Omega_e\sqrt{m/m_i}; \quad (58)\\ \left[\ln\frac{\Omega_i}{\omega}\right]^2 + \frac{3}{4}\left[\ln\frac{m_i}{m}\right]^2 + \ln\frac{m_i}{m}\ln\left|\frac{\Omega_i}{\omega}\right|, & \omega \ll \Omega_i \quad (59)\end{cases}$$

A comparison of Eqs. (54) — (59) with Eq. (34) indicates that in strong fields there are two effective collision frequencies or relaxation times.* If $\omega \sim \omega_{Le}$ in Eq. (57) this equation contains a higher order term which, together with Eq. (55), leads to an expression for the transverse relaxation time; the transverse relaxation time gives the coefficient for electron-ion diffusion across the magnetic field.[5]

*Equation (20) of[1] can be regarded as an interpolation formula which gives Eqs. (18) and (15) of the present work as limiting cases. When $\omega \sim \Omega_e$ the approximate kernel of the collision integral, which corresponds to Eq. (17) of,[1] is a poor approximation.

In all formulas in this paper we have assumed that the maximum frequency ($\omega_{max}$) is determined by the limit of applicability of perturbation theory. However, if this restriction does not hold and instead the range of applicability of our formulas in the region of small impact parameters in determined by quantum effects in the kinetic equation (1), which have been neglected, then $\omega_{max}$ is replaced by ($\kappa T/\hbar$).

I am indebted to V. L. Ginzberg for his interest in this work.

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Translated by H. Lashinsky
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