SCATTERING EQUATIONS AT LOW ENERGIES

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Submitted to JETP editor February 8, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 762-764 (September, 1961)

Equations are proposed for the partial scattering amplitudes at low energies. The derivation of the equations is based on the double Mandelstam representation and the unitarity condition in the two-particle approximation. The characteristic feature of the equations is that the number of subtractions allowed in them is independent of the number of waves taken into account.

WE consider here the expansions of the scattering amplitude in partial-waves, used in the derivation of equations for the partial amplitudes in the low-energy region by methods based on the Mandelstam representation and unitarity condition in the two-particle approximation.[1-3]

The results of this investigation enable us to obtain for the partial amplitudes at low energies new equations, in which the number of subtractions is independent of the number of waves taken into account.

We base our analysis on the scattering of scalar particles of unit mass. Let s_1, s_2, and s_3 be the Mandelstam variables and let z_1, z_2, and z_3 be the corresponding cosines of the scattering angles. We start from the double Mandelstam representation with one subtraction (see, for example,[4]), which can be written in the form[5]

$$A(s_1, s_2, s_3) = \lambda + \sum_{i=1}^{3} \alpha(s_i, z_i);$$

$$\alpha(s_i, z_i) = (s_i - s_{i0}) \int_4^\infty \frac{a_i(x)\,dx}{(x - s_i)(x - s_{i0})} + (s_i - s_{i0}) \int_4^\infty \frac{dx}{(x - s_i)(x - s_{i0})} \times \int_{16}^\infty dy\, \rho(x, y) \left[\frac{s_j - s_{j0}}{(y - s_j)(y - s_{j0})} + \frac{s_k - s_{k0}}{(y - s_k)(y - s_{k0})}\right], \qquad (1)$$

where $i \neq j \neq k$.

The partial amplitudes $f_l(s)$ are the coefficients of the expansion of the scattering amplitude in Legendre polynomials of the cosines of the angles in reaction I:

$$A(s_1, s_2, s_3) = \sum_l f_l(s_1) P_l(z_1) \qquad (2)$$

We introduce the quasi-partial amplitudes defined by the relation

$$\tilde{f}_l(s) = \int_{-1}^{1} dz P_l(z)\, \alpha(s, z). \qquad (3)$$

Unlike the partial amplitudes, the quasi-partial amplitudes are the coefficients of the expansion of only part of the scattering amplitude in terms of angles of one process.

Let us write down the formal expansion of the amplitude in quasi-partial waves; this expansion is known to be correct in the region of elastic scattering ($s_1, s_2, s_3 < 16$):

$$A(s_1, s_2, s_3) = \lambda + \sum_{i,l} \tilde{f}_l(s_i) P_l(z_i). \qquad (4)$$

It follows from the crossing symmetry that we need sum only over even l.

From (1) and (3) we see that $\tilde{f}_l(s_i)$ has a left-hand cut in s_i when $s_i < -12$ (this cut is connected with the vanishing of the denominators in the integral with respect to y in formula (1), possible only when s_1 or s_3 are greater than 16).

It is easy to show from (3) and (4) that $\tilde{f}_l(s)$ has a zero of order l at the point $s = 4$. Taking into account the unitarity condition in the two-particle approximation, taking into account also the threshold behavior of $\tilde{f}_l(s)$ and neglecting the left-hand cut of $\tilde{f}_l(s)$, we can write for the quasi-partial amplitude a dispersion equation which together with (4) yields the following integral representation for the scattering amplitude:

$$A(s_1, s_2, s_3) = \lambda + \sum_{i,l} \tilde{f}_l(s_i) P_l(z_i);$$

$$\tilde{f}_l(s_i) = (s_i - s_{i0})(s_i - 4)^l \int_4^\infty \frac{|f_l(x)|^2\,dx}{(x - s_i)(x - s_{i0})(x - 4)^l}. \qquad (5)$$

For large values of x, the factor $\left(\frac{s_i - 4}{x - 4}\right)^l$ in (5) tends to unity and consequently does not change the asymptotic behavior of $\tilde{f}_l(s)$. This factor compensates for the pole in $P_l(z_j)$, since $z_i = (s_j - s_k)/(s_i - 4)$.

It is seen from (5) that neglect of the l-th partial wave in the physical region of the corresponding process is equivalent to neglecting the l-th quasi-partial wave in the physical regions of all the processes.

From representation (5) we can obtain equations for the partial amplitudes as proposed in several papers.[1-3]

Investigating the analytic properties of the partial amplitudes with the aid of representation (5), we obtain the Chew and Mandelstam equations[1] (this was done for $K\pi$ scattering in a recently published paper by Gourdin et al[6]). When $z_1 = 1$, the term $i = 2$ has the form

$$P_l\left(1 - \frac{s_1}{2}\right)(-s_{20})\,4^l \int_4^\infty \frac{|f_l(x)|^2\,dx}{x\,(x - s_{20})\,(x - 4)^l}. \tag{6}$$

This term increases at infinity as s_1 raised to the number of waves taken into account, and calls for the same number of subtractions in the equations of Chew and Mandelstam (more accurately, an analogous term in the absorptive part). Essentially the same fact is noted in the paper of Efremov et al[2]. Choosing $s_{20} = 0$, we leave an analogous term in the derivative of the amplitude with respect to z_1 for $z_1 = 1$.

The equations of [3] are obtained by neglecting the term with $i = 2$, which in the vicinity $4 < s < 16$ can be readily shown to be greater than the term with $i = 3$, which gives the left-hand cut in s_1. It can be shown that the polynomial (6) cannot be compensated for in the region of elastic scattering by the analytic contribution of the higher approximations. Thus, the analytic continuation in the region $s_1 > 16$ for the forward scattering amplitude distorts the asymptotic properties of the solutions of the equations for the partial amplitudes.

We shall show now that we can obtain equations with asymptotic properties independent of the number of waves taken into consideration. For this purpose we equate the right-hand parts of (5) and (2) and their derivatives with respect to z_1 at $z_1 = 0$ (scattering by 90°). The first equation of the system has the form

$$\sum_l f_l(s)\,P_l(0) = \lambda + \sum_l P_l(0)\,(s - s_{10})\,(s - 4)^l \times \int \frac{|f_l(x)|^2\,dx}{(x - s)\,(x - s_{10})\,(x - 4)^l} + 2\sum_l P_l\left(\frac{3s+4}{s+4}\right)\left(2 - \frac{s}{2} - s_{20}\right)\left(2 - \frac{s}{2} - 4\right)^l \int_4^\infty \frac{|f_l(x)|^2\,dx}{(x - 2 + s/2)\,(x - s_{20})\,(x - 4)^l}, \tag{7}$$

where we put for simplicity $s_{20} = s_{30}$.

The angle 90° is singled out by the fact that in this case we obtain an optimal distance from the regions of non-vanishing spectral functions; this is quite important because the cosines of the scattering angles of reactions II and III in formula (5) are nonphysical.

The procedure proposed is being applied at the present time to specific processes.

The author is grateful to V. Ya. Fainberg for numerous discussions and valuable advice.

[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[2] Efremov, Meshcheryakov, Chou, and Shirokov, Preprint E-539, Joint Inst. Nuc. Res., 1960.

[3] Hsien, Ho, and Zoellner, JETP **39**, 1668 (1960), Soviet Phys. JETP **12**, 1165 (1961).

[4] K. A. Ter-Martirosyan, JETP **39**, 827 (1960), Soviet Phys. JETP **12**, 575 (1961).

[5] M. Cini and S. Fabini, Ann. Physics **9**, 352 (1960).

[6] Gourdin, Noirot, and Salin, Nuovo cimento **18**, 651 (1960).

Translated by J. G. Adashko
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ERRATA

Vol	No	Author	page	col	line	Reads	Should read
13	2	Gofman and Nemets	333	r	Figure	Ordinates of angular distributions for Si, Al, and C should be doubled.	
13	2	Wang et al.	473	r	2nd Eq.	$\sigma_\mu = \frac{e^2f^2}{4\pi^3}\omega^2(\ln\frac{2\omega}{m_\mu} - 0.798)$	$\sigma_\mu = \frac{e^2f^2}{9\pi^3}\omega^2\left(\ln\frac{2\omega}{m_\mu} - \frac{55}{48}\right)$
			473	r	3rd Eq.	$(e^2f^2/4\pi^3)\,\omega^2 \geqslant \ldots$	$(e^2f^2/9\pi^3)\,\omega^2 \geqslant \ldots$
			473	r	17	242 Bev	292 Bev
14	1	Ivanter	178	r	9	1/73	1.58×10^{-6}
14	1	Laperashvili and Matinyan	196	r	4	statistical	static
14	2	Ustinova	418	r	Eq. (10) 4th line	$[-\frac{1}{4}(3\cos^2\theta - 1)\ldots$	$-[\frac{1}{4}(3\cos^2\theta - 1)\ldots$
14	3	Charakhchyan et al.	533	Table II, col. 6 line 1		1.9	0.9
14	3	Malakhov	550			The statement in the first two phrases following Eq. (5) are in error. Equation (5) is meaningful only when s is not too large compared with the threshold for inelastic processes. The last phrase of the abstract is therefore also in error.	
14	3	Kozhushner and Shabalin	677 ff			The right half of Eq. (7) should be multiplied by 2. Consequently, the expressions for the cross sections of processes (1) and (2) should be doubled.	
14	4	Nezlin	725	r		Fig. 6 is upside down, and the description "upward" in its caption should be "downward."	
14	4	Geĭlikman and Kresin	817	r	Eq. (1.5)	$\ldots\left[b^2\sum_{s=1}^{\infty}K_2(bs)\right]^2$	$\ldots\left[b^2\sum_{s=1}^{\infty}(-1)^{s+1}K_2(bs)\right]^2$
			817	r	Eq. (1.6)	$\Phi(T) = \ldots$	$\Phi(T) \approx \ldots$
			818	l	Fig. 6, ordinate axis	$\frac{\varkappa_s(T)}{\varkappa_n(T_c)}$	$\frac{\varkappa_s(T)}{\varkappa_n(T)}$
14	4	Ritus	918	r	4 from bottom	two or three	2.3
14	5	Yurasov and Sirotenko	971	l	Eq. (3)	1 < d/2 < 2	1 < d/r < 2
14	5	Shapiro	1154	l	Table	2306	23.6