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ON THE MAXIMUM VALUE OF THE COUPLING CONSTANT IN FIELD THEORY

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It is assumed that the interactions between elementary particles are characterized by an effective range that depends only on the mass of the particles transmitting the interaction. It is shown that if all masses (and consequently interaction ranges) are fixed, then unitarity and analyticity limit the possible values of the πN interaction coupling constant, as well as the absolute value of the $\pi\pi$ scattering amplitude at zero energy (in the case when the latter is negative). The proof is carried out by means of dispersion relations for the inverse of the forward scattering amplitude.

1. INTRODUCTION

IN modern field theory particle masses and coupling constants between fields describing these particles appear as independent quantities, whose values must be given by experiment. This is true both of the Hamiltonian form of the theory, where renormalization is performed in such a way as to make the renormalized charge and mass the same as the observed ones, and of the dispersion relations approach, in which the location of the singularities of the scattering amplitude and the residues of the pole terms are identified with the experimental values of the particles' masses and coupling constants. One might, however, ask the following question: could not the values of the particle masses impose some restrictions on the coupling constants? The example of nonrelativistic theory with point interaction shows that such a situation is possible; in that example the magnitude of the renormalized coupling constant cannot be in excess of a certain critical value, determined by the masses of the particles.[1,2]

In the relativistic theory no such precise limitation can be proved. This is related to the circumstance that the interaction between particles is effectively smeared out by the existence of virtual processes. On the other hand, it has been assumed in field theory, beginning with Yukawa, that if an interaction is due to the exchange of a particle of mass μ then the effective range of the interaction is of the order of $\hbar/\mu c$, regardless of the interaction strength. If one accepts this point of view, i.e., if it is assumed that the range of the interaction is determined by the masses only, then it becomes possible to deduce limitations on the coupling constants.

In this paper we consider scattering of π mesons by nucleons. Analogous considerations are, apparently, valid for other processes (for example scattering of K mesons by nucleons), however the existence of unphysical regions in the dispersion relations for these processes complicates their analysis.

We find it convenient to make use of dispersion relations for the inverse of the forward scattering amplitude. These dispersion relations possess a number of peculiar properties, although mathematically they are a consequence of the direct dispersion relations. In the first place, they are sensitive to the zeros of the scattering amplitude. The number of these zeros is limited; it is shown below that if the high energy behavior of the cross section does not differ much from a constant, then the forward scattering amplitude of charged pions on nucleons can have in the complex plane one, two or three zeros; the $\pi\pi$-scattering amplitude either has no zeros, or has one or two zeros; the Compton effect amplitude has no zeros. In the second place, although the "inverse" dispersion relations are an identity with respect to the coupling constant, in distinction to the usual dispersion relations they do not represent a term by term identity after the integrands have been expanded in a power series in the charge.

In addition to restrictions on the coupling constants, i.e., on the residues of the pole terms in the scattering amplitudes, it also turns out to be possible to obtain restrictions on the scattering amplitude at zero energy (scattering length) when the latter is negative. In conclusion we discuss the following question: might not the observed pion-nucleon interaction coupling constant have the maximum value allowable by the prescribed masses?

It is possible to give certain indirect arguments in favor of such a hypothesis.

2. SCATTERING OF π^0 MESONS ON PROTONS

Let us consider first scattering of π^0 mesons on protons. This case is simpler than the scattering of charged mesons because the imaginary part of the forward scattering amplitude differs in this case on the left and right cuts by its sign only. The dispersion relations for $A^0(\omega)$, including the form of the pole term, are easily written down following, for example, the work of Goldberger et al,[3] taking into account the fact that $A^0(\omega) = [A^+(\omega) + A^-(\omega)]/2$, where $A^\pm$ are the forward scattering amplitudes for $\pi^\pm$ mesons on protons:

$$A^0(\omega) = A^0(\mu) + 2f^2\omega_0\left[\frac{1}{\omega_0^2-\omega^2} - \frac{1}{\omega_0^2-\mu^2}\right] + \frac{1}{\pi}\int_{\mu^2}^{\infty} \operatorname{Im} A^0(\omega')\left[\frac{1}{\omega'^2-\omega^2} - \frac{1}{\omega'^2-\mu^2}\right] d\omega'^2. \tag{1}$$

Here ω is the energy of the π meson in the laboratory system, μ is the mass of the π meson, $f^2 = 0.08$ is the meson-nucleon coupling constant, $\omega_0 = \mu^2/2m$, and m is the nucleon mass. According to the optical theorem we have

$$\operatorname{Im} A^0(\omega) = (k/4\pi)\,\sigma^0(\omega), \tag{2}$$

where $\sigma^0(\omega)$ is the total π^0-meson-proton interaction cross section.

It is clear from Eq. (1) that, as a function of the complex variable ω^2, $A^0(\omega)$ is an R-function, i.e., the sign of its imaginary part is the same as the sign of the imaginary part of ω^2, and consequently it can have zeros only on the real axis.[1] The inverse function $h^0(\omega) = -1/A^0(\omega)$ is also an R-function and has poles only at the zeros of the function $A^0(\omega)$, i.e., on the real axis. The most general dispersion relation that it satisfies is of the form

$$h^0(\omega) = \frac{1}{\pi}\int_{\mu^2}^{\infty} \operatorname{Im} h^0(\omega')\frac{d\omega'^2}{\omega'^2-\omega^2} + \sum_n \frac{R_n}{\omega_n^2-\omega^2} + b + b_1\omega^2. \tag{3}$$

The constants ω_n^2, R_n, b and b_1 are real; furthermore $b_1 \geq 0$ and $R_n \geq 0$. The latter is necessary in order that $h^0(\omega)$ be an R-function in the ω^2-plane. The dispersion relation (3) has been written without subtractions since $\operatorname{Im} h^0(\omega) = \operatorname{Im} A^0(\omega)/|A^0(\omega)|^2$ goes at large ω like $1/\omega$, if the cross section $\sigma^0(\omega)$ is approximately constant. Besides, two more subtractions in the variable ω^2 would not change the discussion that follows. Actually, some of the terms written on the right side of Eq. (3) drop out. Thus, the quantities b and b_1 are equal to zero if the cross section does not decrease at large energies at least as fast as $1/\omega$. And as regards the sum of the pole terms, it will be shown in the following section that it contains only one term with $\omega_1^2 < \omega_0^2$, if $A^0(\mu) < 0$, and one more term with $\omega_0^2 < \omega_2^2 < \mu^2$, if $A^0(\mu) > 0$. All this, however, is irrelevant for what follows.

The residue of the function $A^0(\omega)$ at the pole, proportional to the coupling constant f^2, equals $-(dh/d\omega^2)^{-1}\big|_{\omega^2=\omega_0^2}$. It is therefore easy to obtain the following expression for the coupling constant:

$$\frac{1}{f^2} = 2\omega_0\left[\frac{1}{\pi}\int_{\mu^2}^{\infty} \frac{\operatorname{Im} A^0(\omega')}{|A^0(\omega')|^2}\frac{d\omega'^2}{(\omega'^2-\omega_0^2)^2} + \sum_n \frac{R_n}{(\omega_n^2-\omega_0^2)^2} + b_1\right]. \tag{4}$$

As a result of the positive nature of R_n and b_1 it follows from here that

$$\frac{1}{f^2} > \frac{2\omega_0}{\pi}\int_{\mu^2}^{\infty} \frac{\operatorname{Im} A^0(\omega')}{|A^0(\omega')|^2}\frac{d\omega'^2}{(\omega'^2-\omega_0^2)^2}. \tag{5}$$

The idea of the subsequent discussion consists of the following. At low energies, when it is possible to limit oneself in the scattering amplitude to s-waves only, one has*

$$\operatorname{Im} A^0(\omega')/|A^0(\omega')|^2 = k' \equiv \sqrt{\omega'^2-\mu^2},$$

and the range of the variable k', within which this assertion is valid, is determined by the inequality $k'\rho \ll 1$ (ρ is the range of the π-meson—nucleon interaction). If one accepts the hypothesis mentioned in the introduction, that the quantity ρ is determined by the masses of the particles only and does not depend on the renormalized coupling constant, then the integral on the right side of Eq. (5) will certainly contain a small region ($k'\rho \ll 1$), whose size is independent of f^2, where the integrand is equal to $k'(\omega'^2 - \omega_0^2)^{-2}$. Since the entire integral can only be larger than the result of the integration over this small region, it follows that the quantity f^{-2} is bounded from below by a certain expression which depends only on the masses of the particles.

Below we shall obtain a more precise inequality based on replacing the integrand by a quantity independent of f^2 at all energies. We have for $\operatorname{Im} A^0/|A^0|^2$

$$\frac{\operatorname{Im} A^0(\omega)}{|A^0(\omega)|^2} = \frac{k\sigma^0(\omega)}{4\pi\sigma_e^0(\omega, 0)} \geqslant k\,\frac{\int \sigma_e^0(\omega, \theta)\, d\Omega/4\pi}{\sigma_e^0(\omega, 0)}. \tag{6}$$

Here $\sigma^0(\omega)$ is as before the total π^0-meson-proton interaction cross section, and $\sigma_e^0(\omega, \theta)$ is the dif-

*Strictly speaking, this equality applies only to pure isotopic states, however the refinements connected with this remark are trivial and do not lead to any new results.

ferential elastic scattering cross section through the angle θ. We next carry out a phase shifts expansion of the right side of Eq. (6):*

$$\frac{\operatorname{Im} A^0(\omega)}{|A^0(\omega)|^2} \geqslant k \frac{\Sigma |a_l|^2 (2l+1)}{|\Sigma a_l (2l+1)|^2} \geqslant k \frac{\Sigma |a_l|^2 (2l+1)}{[\Sigma |a_l| (2l+1)]^2}. \quad (7)$$

We shall, further, assume that the partial wave amplitudes a_l decrease rapidly for $l > l_0(k) = k\rho(k)$, where $\rho(k)$ is the interaction range characteristic of the given energy. If the effective dimensions of the system do not increase indefinitely with increasing k, then there exists a certain maximum ρ, such that at all energies the a_l are vanishingly small if $l > l_0 = k\rho$. It should be emphasized that the quantity ρ need not coincide with the quantity $\rho(k)$ as $k \to \infty$. It is easy now to observe that the right side of the inequality (7) reaches a minimum, when all the a_l are equal to each other. Thus

$$\frac{\operatorname{Im} A^0(\omega)}{|A^0(\omega)|^2} \geqslant k \Big/ \sum_{l=0}^{l_0} (2l+1) = \frac{k}{(l_0+1)^2} = \frac{k}{(k\rho+1)^2}. \quad (8)$$

For $k\rho \ll 1$ one obtains the correct value $\operatorname{Im} A/|A|^2 = k$. Substituting Eq. (8) into Eq. (5) and performing the integration we find

$$\frac{1}{f^2} > \frac{4\omega_0}{\pi}\left[\frac{2\rho}{(1+\rho^2)^2} + \frac{2\rho(1-\rho^2)}{(1+\rho^2)^3}\ln\rho + \frac{\pi}{4}\,\frac{1-6\rho^2+\rho^4}{(1+\rho^2)^3}\right]. \quad (9)$$

In Eq. (9) we have set $\mu = 1$ and have neglected the small quantity ω_0 in comparison with unity. The choice of the quantity ρ is fairly arbitrary. Since the π-meson-nucleon interaction proceeds via the exchange of at least two π mesons one might expect that $\rho \sim 1/2$. Then $f^2 < 60$ ($\omega_0 = 0.07$). When $\rho = 1$, $f^2 < 100$; when $\rho = 0$, $f^2 < 15$. Let us also note that even if ρ were to increase with energy an estimate of f^2 would still be possible. One would only need to take into account this k-dependence of ρ when carrying out the integration in Eq. (8). Such a dependence, however, is of little importance since the main contribution to the integral in Eq. (5) comes from $\omega' \sim \mu$.

The limitation here obtained on the magnitude of the residue has the following meaning: as f^2 goes through a certain critical value the scattering amplitude ceases to satisfy the unitarity and analyticity requirements. That this is so can be clearly seen in the example of nonrelativistic theory.[2]

The magnitude of the residue of the pole term in the amplitude for the scattering of π^0 mesons on protons, in contrast to the residue in the amplitudes for the scattering of charged mesons, is proportional to the small quantity ω_0. This numerically worsens the estimate of f^2 and results in a complete disappearance of the inequality in the limit when the nucleon mass becomes infinite. It is therefore of interest to study the amplitude for the scattering of charged mesons on protons. We shall show in what follows that in this case one obtains a much stronger restriction on f^2.

*The inclusion of nucleon spin leads to no new results, naturally.

3. ZEROS OF THE FORWARD SCATTERING AMPLITUDE

In order to obtain restrictions on the residue of the amplitude for the scattering of charged π mesons on nucleons, it is necessary to know the number and the location in the complex ω-plane of the zeros of the amplitude. In this section we shall derive a formula for the number of zeros and will discuss their location.

Let us consider the amplitude $A^+(\omega)$ for the scattering of π^+ mesons on protons and the amplitude $A^-(\omega)$ for the scattering of π^- mesons on protons, which is connected to A^+ by the crossing symmetry condition. $A^+(\omega)$ is a function analytic in the complex ω-plane except for the two cuts from $\omega = \mu$ to $\omega = +\infty$ and from $\omega = -\mu$ to $\omega = -\infty$, and the pole at $\omega = \omega_0 = \mu^2/2m$. On the right cut the imaginary part of $A^+(\omega)$ is positive above the cut [$\operatorname{Im} A^+(\omega) = k\sigma^+(\omega)/4\pi$] and differs by a sign below the cut. On the left cut the situation is reversed: the imaginary part is positive below the cut and equal to $k\sigma^-(\omega)/4\pi$ and negative above the cut. [$\sigma^{\pm}(\omega)$ are the total interaction cross sections for π^+ and π^- mesons with protons.]

Consider the integral

$$\frac{1}{2\pi i}\oint \frac{A^{+\prime}(\omega)}{A^+(\omega)}\,d\omega \quad (10)$$

where the contour consists of lines enclosing both cuts on both sides joined at infinity by two large semicircles. According to Cauchy's theorem, the integral is equal to the number of zeros (k) minus the number of poles (p) of the function $A^+(\omega)$ contained inside the contour (in our case $p = 1$). On the other hand the integral is equal to the increment in the phase of the function on traversing the contour, divided by 2π:

$$k - p = \Delta\varphi / 2\pi. \quad (11)$$

As will be seen from what follows the increment $\Delta\varphi$ depends on the signs of the real quantities $A^+(\mu)$ and $A^+(-\mu) = A^-(\mu)$, i.e., on the signs of the scattering lengths. Let us assume for definiteness sake that $A^+(\mu) < 0$ and $A^-(\mu) > 0$ and start calculating $\Delta\varphi$ by beginning to traverse the contour at $\omega = \mu$. Then the initial value of the phase, $\varphi(\mu)$, may be taken equal to π $[A^+(\mu) < 0]$. The value of the phase at the point $\omega = +\infty + i\epsilon$ will lie between zero and π if the total interaction cross section (and, consequently, the imaginary part of the amplitude) does not vanish anywhere. The total cross section is determined by many partial waves and we shall assume that not all scattering phase shifts can be simultaneously equal to multiples of π. This assumption can be proved theoretically, since the phase shifts corresponding to large orbital angular momenta can be calculated from diagrams with lowest in mass intermediate states.[4]

Let us now assume that the asymptotic behavior of the amplitude $A^+(\omega)$ at large ω has the character ω^n, where n is an odd integer. n must be odd in order that the imaginary part of $A^+(\omega)$ be negative above the left cut. On traversing the large semicircle, joining the points $+\infty + i\epsilon$ and $-\infty + i\epsilon$, the phase increment amounts to $n\pi$. As one proceeds along the upper edge of the left cut, the phase varies between $n\pi$ and $(n+1)\pi$ and consequently is equal to $(n+1)\pi$ [an even multiple of π since $A^-(\mu) > 0$] at the point $\omega = -\mu$. Continuing these considerations it is easy to show that the total phase increment upon traversing the contour equals $\Delta\varphi = 2\pi n$. From here, according to Eq. (11), we get

$$k = n + p \qquad (A^+(\mu) < 0, \qquad A^-(\mu) > 0). \qquad (12a)$$

It is easy to see that Eq. (12a) remains valid if $A^+(\mu) > 0$, $A^-(\mu) < 0$, but

$$k = n + p - 1 \qquad (A^+(\mu) < 0, \qquad A^-(\mu) < 0), \qquad (12b)$$

$$k = n + p + 1 \qquad (A^+(\mu) > 0, \qquad A^-(\mu) > 0). \qquad (12c)$$

If the asymptotic form of the amplitude does not have a pure power law character, but rather is multiplied by a slowly varying function (for example, by $\sim \ln^{-2}\omega$, which would insure the decreasing of the total cross section[5]), then Eqs. (12) remain valid. If instead the behavior of the amplitude at infinity is governed not by an integral power of ω, then it is easy to show that the n in Eqs. (12) is equal to the odd integer nearest to the exponent of ω in the asymptotic form of the amplitude. It should be added that in view of the positive nature of the imaginary parts on the cuts the scattering amplitude cannot, apparently, have an asymptotic behavior that depends on the direction in the complex plane along which the point at infinity is approached.

If it is accepted that at large energies the total cross section is approximately constant (n = 1), then it follows from Eq. (12) that the forward scattering amplitude of π^+ mesons on protons has one $[A^\pm(\mu) < 0]$, two $[A^\pm(\mu)$ of opposite signs$]$, or three $[A^\pm(\mu) > 0]$ zeros. Qualitatively the location of these zeros can be easily determined by investigating the amplitude $A^+(\omega)$ for real values of ω in the interval $(-\mu, \mu)$. For ω close to ω_0 $A^+(\omega)$ tends to $+\infty$ if ω lies to the left of ω_0, and to $-\infty$ if $\omega > \omega_0$. It therefore follows that in the case when $A^+(\mu) < 0$ and $A^-(\mu) < 0$ the single zero of the amplitude lies on the real axis to the left of the point ω_0. When $A^+(\mu) > 0$, $A^-(\mu) < 0$, the amplitude has two zeros: one to the left and one to the right of the point ω_0. When $A^+(\mu) < 0$, $A^-(\mu) > 0$ (the experimentally observed situation) one has various possibilities. The two zeros of the amplitude could both lie to the left of ω_0, or both to the right of ω_0, or lie in the complex plane (in which case they must be, of course, complex conjugate). All the cases with three zeros can be obtained from these last ones by the addition of a zero on the real axis to the right of ω_0.

The amplitude for π^0 mesons scattering on protons $\{A^0(\omega) = [A^+(\omega) + A^-(\omega)]/2,\ A^0(\omega) = A^0(-\omega)\}$ has two poles at the points ω_0 and $-\omega_0$, and has consequently either two zeros if $A^0(\mu) < 0$ (which corresponds to the experimental data), or four zeros if $A^0(\mu) > 0$. In the first case these zeros are either on the real axis between $-\omega_0$ and ω_0, placed symmetrically with respect to the origin, or on the imaginary axis. In the second case one must add to them a zero to the right of ω_0 and a zero to the left of $-\omega_0$. In the ω^2-plane this corresponds to what has been said in the previous section.

In an analogous manner it is easy to show that in electrodynamics the amplitude for forward scattering of photons on electrons has no zeros $[A(0) = -e^2/m < 0]$, and the amplitude for scattering of π mesons on π mesons has either no zeros, or one, or two zeros.

4. SCATTERING OF CHARGED π MESONS ON PROTONS AND $\pi\pi$ SCATTERING

Let us return to the estimate of the residue of the pole term in the amplitude for the scattering of π^+ mesons on protons, $A^+(\omega)$. As was shown in the preceding section this function may have one, two or three zeros distributed in various ways. The most interesting, from the point of view of obtaining restrictions on the coupling constant, is the case of two complex zeros and, possibly, one more zero to the right of ω_0. All the other variants can be discussed in a manner analogous to the one described below, and lead to stronger inequalities on the coupling constant — hence may be considered as being included in the estimate obtained below.

Let ω_1 and $\omega_2 = \omega_1^*$ be the complex zeros of $A^+(\omega)$, and ω_3 the additional zero ($\omega_3 > \omega_0$). Consider the functions

$$F^+(\omega) = A^+(\omega)\,(\omega+\mu)/(\omega-\omega_1)(\omega-\omega_2),$$

$$H^+(\omega) = -1/F^+(\omega), \tag{13}$$

which are R-functions (the imaginary parts of these functions due to the pole terms and on the right cut have the same sign as $\operatorname{Im} A^+(\omega)$, and the opposite sign on the left cut). $F^+(\omega)$ has two zeros at $\omega = -\mu$ and $\omega = \omega_3$ and a pole at $\omega = \omega_0'$, $H^+(\omega)$ has poles at $\omega = -\mu$, ω_3 and a zero at $\omega = \omega_0$; both functions behave for large ω approximately like constants $[A^+(\omega) \sim \omega]$. The dispersion relation for $H^+(\omega)$ has the form

$$H^+(\omega) = \frac{1}{\pi}\int_{\mu}^{\infty} d\omega' \frac{\operatorname{Im} A^+(\omega')}{|A^+(\omega')|^2}\frac{|\omega'-\omega_1|^2}{\omega'+\mu}\left[\frac{1}{\omega'-\omega} - \frac{1}{\omega'-\omega_0}\right]$$
$$-\frac{1}{\pi}\int_{\mu}^{\infty} d\omega' \frac{\operatorname{Im} A^-(\omega')}{|A^-(\omega')|^2}\frac{|\omega'+\omega_1|^2}{\omega'-\mu}\left[\frac{1}{\omega'+\omega} - \frac{1}{\omega'+\omega_0}\right]$$
$$-\frac{|\omega_1+\mu|^2}{A^-(\mu)}\left[\frac{1}{\mu+\omega} - \frac{1}{\mu+\omega_0}\right] + R_3\left[\frac{1}{\omega_3-\omega} - \frac{1}{\omega_3-\omega_0}\right]. \tag{14}$$

A subtraction has been performed at the point $\omega = \omega_0$ in Eq. (14) and the fact that $H(\omega_0) = 0$ has been taken into account, so that for ω close to ω_0 $H(\omega)$ is of the form $H(\omega) \approx H'(\omega_0)(\omega - \omega_0)$. As can be seen from the relation between $H^+(\omega)$ and $A^+(\omega)$, the magnitude $-2f^2$ of the residue of the function $A^+(\omega)$ is proportional to $-1/H'(\omega_0)$. Consequently

$$\frac{1}{2f^2} = \frac{\omega_0+\mu}{|\omega_0-\omega_1|^2}\left\{\frac{1}{\pi}\int_{\mu}^{\infty}\frac{\operatorname{Im} A^+}{|A^+|^2}\frac{|\omega'-\omega_1|^2\,d\omega'}{(\omega'-\omega_0)^2(\omega'+\mu)}\right.$$
$$+\frac{1}{\pi}\int_{\mu}^{\infty}\frac{\operatorname{Im} A^-}{|A^-|^2}\frac{|\omega'+\omega_1|^2\,d\omega'}{(\omega'+\omega_0)^2(\omega'-\mu)}$$
$$\left.+\frac{|\omega_1+\mu|^2}{A^-(\mu)}\frac{1}{(\mu+\omega_0)^2} + \frac{R_3}{(\omega_3-\omega_0)^2}\right\}. \tag{15}$$

We can now proceed in the same manner as in estimating the residue of the function $A^0(\omega)$, i.e., estimate $\operatorname{Im} A^\pm/|A^\pm|^2$ by Eq. (8) and then strengthen the inequality by throwing away the last terms in Eq. (15). If the small quantity ω_0 is neglected everywhere, we arrive at the following inequality:

$$\frac{1}{f^2} > \frac{4I_1}{\pi}\left[1 + \frac{1}{\omega_1} + \frac{1}{\omega_1^*} + \frac{I_2}{I_1|\omega_1|^2}\right];$$
$$I_1 = \int_1^{\infty}\frac{d\omega'}{k'\omega'(k'\rho+1)^2}, \qquad I_2 = \int_1^{\infty}\frac{\omega'\,d\omega'}{k'(k'\rho+1)^2}, \qquad \mu = 1. \tag{16}$$

Let us set $1/\omega_1 = x + iy$; then the polynomial in the square brackets $[(I_2/I_1)(x^2+y^2) + 2x + 1]$ has a minimum at $y = 0$ and $x = -I_1/I_2$. For the quantity f^2 we obtain the estimate

$$\frac{1}{f^2} > \frac{4I_1}{\pi I_2}(I_2 - I_1) = \frac{4}{\pi\rho}\varphi(\rho)[1-\varphi(\rho)];$$
$$\varphi(\rho) = \frac{\rho^2}{(1+\rho^2)} + \frac{\pi}{2}\frac{\rho(1-\rho^2)}{(1+\rho^2)^2} + \frac{2\rho^2\ln\rho}{(1+\rho^2)^2}. \tag{17}$$

For $\rho = 1/2$ $f^2 < 1.7$; for $\rho = 1$ $f^2 < \pi$, for $\rho = 0$ $f^2 < 0.5$. These estimates are somewhat better than those obtained in Sec. 2. It is interesting that corresponding to the maximum value of f^2 the position of the zero turns out to be on the left cut, i.e., at a place where the true scattering amplitude cannot vanish. Indeed,

$$\omega_1 = -I_2/I_1 = -\varphi(\rho) < -1. \tag{18}$$

This circumstance is related to the fact that we have not formulated quantitatively the condition that the total cross section must not vanish anywhere. It is obvious that the restriction obtained on f^2 has been greatly overestimated as compared with the true one. From Eq. (18) follow numerical values for ω_1 for various choices of ρ. For $\rho = 1/2$ $\omega_1 = -2.8$; for $\rho = 1$ $\omega_1 = -2$; for $\rho = 0$ $\omega_1 \to -\infty$.

In what follows we consider the question whether the pion-nucleon interaction is the maximal possible given the masses of the particles. Had we been able to give a good estimate for the critical constant, beyond which unitarity and analyticity of the theory are violated, then this question could be answered by comparing this quantity with the observed value $f^2 = 0.08$. The value of the critical coupling, obtained from Eq. (17) with $\rho = 1/2$, $f^2_{cr} = 1.7$ is 20 times larger than the observed value. This, of course, means nothing since our restriction has been so greatly overestimated.

One can compare the location of the zeros of the amplitude, as obtained from Eq. (18), with the location of the true zeros which can be determined from experimental data, and see to what extent they agree. At that one should remark that since experimentally $A^+(\mu) < 0$, $A^-(\mu) > 0$, it follows that $A^+(\omega)$ has two zeros. It is shown in the Appendix that it follows from experiment that $\omega_{1,2} = -0.9 \pm 0.5i$. These numbers are in qualitative agreement with the zeros obtained from Eq. (18) by requiring that f^2 be maximal, if it is taken into account that this requirement must be supplemented by the condition that $\sigma^{\pm}(\omega)$ must be larger than a certain minimum value, beyond which $\omega_{1,2}$ develops an imaginary part. It may be that this comparison may serve as an indirect indication that the meson-nucleon interaction is maximal.

Restrictions on the residues of the pole terms can be easily obtained also in the presence of bound states in the theory. In contrast to the nonrelativistic theory[2] the upper bound on the coupling constant may depend here on the energy differences of the bound states and increase as this difference decreases.

Let us show now that also the scattering amplitudes at zero energy, i.e., the scattering lengths, are restricted in absolute magnitude, provided that they are negative. Qualitatively this can be understood as follows. At low energies ($k\rho \ll 1$, where k is the momentum and ρ is the interaction range) the scattering amplitude is of the form $a/(1-ika)$, where a is the scattering length. For $\omega < \mu$, $k = +i\sqrt{\mu^2-\omega^2}$, so that the amplitude is written here in the form $a/(1+a\sqrt{\mu^2-\omega^2})$. This expression has a pole at $|k| = \sqrt{\mu^2-\omega^2} = 1/a$. If $a < 0$ and $|a|$ is very large then this pole falls into the region of applicability of our formula ($k\rho \ll 1$). Therefore, if it is known that in the given theory there are no bound states with small binding energies, the quantity $|a|$ cannot be too large. For $a > 0$ the pole passes into the second sheet of the complex plane and the restriction disappears. In that case we are dealing with a situation analogous to singlet np scattering.

Let us consider π-meson–π-meson scattering, restricting ourselves for the sake of simplicity to the case when the crossed reaction is the same as the direct reaction. If the scattering length is negative then, according to the results obtained above, the scattering amplitude has no zeros and the dispersion relation for the inverse function may be written in a form analogous to Eq. (3)*

$$h(\omega) = \frac{1}{\pi}\int_{\mu^2}^{\infty} \frac{d\omega'^2}{\omega'^2-\omega^2} \frac{\operatorname{Im} A(\omega')}{|A(\omega')|^2}. \tag{19}$$

The scattering length is $a = -1/h(\mu)$. Hence

$$-\frac{1}{a} = \frac{1}{\pi}\int_{\mu^2}^{\infty} \frac{d\omega'^2}{\omega'^2-\mu^2} \frac{\operatorname{Im} A(\omega')}{|A(\omega')|^2}$$

$$> \frac{1}{\pi}\int_{\mu^2}^{\infty} \frac{d\omega'}{\omega'^2-\mu^2} \frac{k'}{(k'\rho+1)^2} = \frac{2}{\pi\rho}, \tag{20}$$

$$|a| < \pi\rho/2. \tag{21}$$

*For positive a, the right side of Eq. (19) would also contain the pole term corresponding to the zero of $A(\omega)$. The presence of this negative term would make it impossible to obtain an estimate.

If the $\pi\pi$ interaction is characterized by a range $\rho \sim 1/2$ then $|a| < \pi/4$. Since this result most certainly represents a great overestimate it is, apparently, to be expected that if the $\pi\pi$ scattering lengths are found experimentally to be negative, they will turn out to be of the order of $0.1\,\hbar/\mu c$, and not $\hbar/\mu c$ as is frequently asserted.

Analogous inequalities may be also obtained for the πp scattering lengths. For example, for the π^0-meson–proton scattering amplitude one obtains

$$|a| < \pi\rho/\varphi(\rho), \tag{22}$$

where $\varphi(\rho)$ is given by Eq. (17). For $\rho = 1/2$, $|a| < 4.3$. Experimentally the value of this length is ~ -0.02.

5. CONCLUSIONS

Despite the hypothetical nature of the assumption that a range independent of the interaction strength exists, we are convinced that unitarity and analyticity do in fact impose restrictions on the possible values of the coupling constant. This raises the question whether the π-meson–nucleon interaction, as well as other strong interactions of various particles, is the maximal possible given the values of the masses of the particles. Formulating the question in this fashion presupposes that the magnitude of the renormalized coupling constant may vary to some extent independently of the masses. Such an assumption seems reasonable at the present time since in field theory masses and coupling constants appear as independent quantities as a result of the infinite renormalizations. As regards a "future" theory, in which definite values of coupling constants will be correlated with strictly determined particle masses, we remark that in the first place in such a theory the coupling constants themselves will have definite numerical values, and in the second place no such theory exists as yet.

An example of maximal coupling in the nonrelativistic case is provided by the deuteron formula for nucleon-nucleon scattering. In this case the connection between the location of the pole and the size of the residue of the scattering amplitude corresponds to the strongest interaction possible.[2]

The idea that the strong interactions that are present in nature are in a certain sense as strong as possible seems rather attractive, although at this time it cannot be formulated theoretically in a precise manner or verified experimentally.

In conclusion the authors would like to express their gratitude to Ya. B. Zel'dovich, who stimulated their interest in this question. We are also grateful to I. Ya. Pomeranchuk and V. A. Anisovich for constructive remarks.

APPENDIX

DETERMINATION OF THE ZEROS OF THE AMPLITUDE FOR THE FORWARD SCATTERING OF π MESONS ON PROTONS

The dispersion relation with one subtraction at $\omega = \omega_0$ for the function $-(\omega - \omega_1)(\omega - \omega_2)/(\omega - \omega_0) A^+(\omega)$ can be written as follows ($\mu = 1$)

$$a(\omega) - 2b(\omega)\xi + c(\omega)\eta = 0, \quad \text{(A.1)}$$

where

$$\xi = (\omega_1 + \omega_2)/2, \quad \eta = \omega_1\omega_2, \quad \text{(A.2)}$$

$$a(\omega) = \frac{\omega_0^2}{2f^2} + \frac{\omega^2}{(\omega-\omega_0)A^+(\omega)} + \frac{\omega-\omega_0}{\pi}\int_1^\infty d\omega'\omega'^2\left[\frac{\operatorname{Im}A^+(\omega')}{|A^+(\omega')|^2}\frac{1}{(\omega'-\omega_0)^2(\omega'-\omega)} + \frac{\operatorname{Im}A^-(\omega')}{|A^-(\omega')|^2}\frac{1}{(\omega'+\omega_0)^2(\omega'+\omega)}\right],$$

$$b(\omega) = \frac{\omega_0}{2f^2} + \frac{\omega}{(\omega-\omega_0)A^+(\omega)} + \frac{\omega-\omega_0}{\pi}\int_1^\infty d\omega'\omega'\left[\frac{\operatorname{Im}A^+(\omega')}{|A^+(\omega')|^2}\frac{1}{(\omega'-\omega_0)^2(\omega'-\omega)} - \frac{\operatorname{Im}A^-(\omega')}{|A^-(\omega')|^2}\frac{1}{(\omega'+\omega_0)^2(\omega'+\omega)}\right],$$

$$c(\omega) = \frac{1}{2f^2} + \frac{1}{(\omega-\omega_0)A^+(\omega)} + \frac{\omega-\omega_0}{\pi}\int_1^\infty d\omega'\left[\frac{\operatorname{Im}A^+(\omega')}{|A^+(\omega')|^2}\frac{1}{(\omega'-\omega_0)^2(\omega'-\omega)} + \frac{\operatorname{Im}A^-(\omega')}{|A^-(\omega')|^2}\frac{1}{(\omega'+\omega_0)^2(\omega'+\omega)}\right]. \quad \text{(A.3)}$$

It follows from Eq. (A.1) that no matter what two values are chosen for ω after evaluating the expressions (A.3) the combinations

$$\xi = (a_1c_2 - a_2c_1)/2(b_1c_2 - b_2c_1),$$

$$\eta = (a_1b_2 - a_2b_1)/(b_1c_2 - b_2c_1) \quad \text{(A.4)}$$

(a_1 and a_2 ... are the values of the functions a ... for those choices of ω) should lead to the same values of ξ and η. The verification of this assertion is by itself equivalent to an additional verification of the dispersion relations. It turns out to be simplest to evaluate (A.3) at the points $\omega = \pm 1$ since, on the one hand, it is then not necessary to evaluate the integral in the principal value sense and, on the other hand, the following quantities are known

$$A^+(1) = a_3\left(1 + \frac{\mu}{m}\right),$$

$$A^+(-1) = \left(\tfrac{2}{3}a_1 + \tfrac{1}{3}a_3\right)\left(1 + \frac{\mu}{m}\right),$$

where a_1 and a_3 are the scattering lengths of Orear.[6] In the numerical integration in Eq. (A.3) we have set $\omega = \pm 1$. The values of the real and imaginary parts of $A^\pm(\omega)$ were taken from[7].

As a result of integration and evaluation of Eq. (A.4) we find $\xi = -0.88$, $\eta = 0.96$. Consequently $\omega_{1,2} = -0.88 \pm 0.44i$. Since the size of the imaginary part of $\omega_{1,2}$ depends on the difference of two rather similar numbers (it is equal to $\sqrt{\eta - \xi^2}$) it is unlikely that our determination is very accurate. For this reason we give in the text the value $\omega_{1,2} = -0.9 \pm 0.5i$.

Note added in proof (July 14, 1961). From a different point of view the question of the strength of the coupling has also been discussed by Chew and Frautschi.[8]

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