A NEUTRAL MODEL FOR INVESTIGATION OF $\pi\pi$ SCATTERING

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An equation describing the scattering of low energy neutral pseudoscalar mesons is investigated. A general solution of the equation is derived, similar to that obtained by Castillejo, Dalitz, and Dyson [12] for the Low equation. Two different types of asymptotic behavior of the solutions at high energies are possible. If the amplitude decreases as $(\ln E)^{-1}$ at high energies, the solution corresponds to the renormalizable perturbation theory. In the second case, when the amplitude decreases as E^{-4}, the solution does not correspond to perturbation theory. In a certain sense it can be connected with the nonrenormalizable Lagrangian $[(\partial\varphi/\partial x_n)(\partial\varphi/\partial x_n)]^2$. This second solution possesses some interesting properties. In particular, it becomes degenerate when the interaction is switched off.

1. INTRODUCTION

ATTEMPTS have been made in recent years to construct a theory of strong interactions in the low-energy region by starting from the analytic properties of the scattering amplitudes in the form proposed by Mandelstam [1] and from the unitarity conditions.

The arguments were based on the premise that the phenomena in the low-energy region admit of description in closed form. Mathematically this hypothesis reduces to the assumption that the behavior of an analytic function is determined in a small region by the near-lying singularities.[2]

The restriction to the low-energy region enables us to write down an approximate unitarity condition in which only two-particle intermediate states are taken into account. This approximate unitarity condition, together with the spectral representation, makes it possible to write down [1,3] a closed system of nonlinear integral equations for the scattering amplitude as a function of two variables.

In view of the complexity of such equations, still another approximation is made by representing the scattering amplitude in the form of a small number of first terms of its expansion and Legendre polynomials. Chew and Mandelstam [4] have obtained in this manner a closed system of nonlinear integral equations for the lower partial waves of the pion-pion scattering. Subsequently analogous equations were obtained for several other processes (see [5,6] and others). In the analysis of the anti-Hermitian part of the amplitude in the cross integral, these authors use analytic continuations in the Legendre polynomials into the region $|\cos\theta| > 1$. But the use of only the first terms of the Legendre series leads to large errors,[7-9] which are particularly large at high-energy crossing processes. The integrals of the higher partial waves are found to be divergent, and the solutions of the equations are unstable against small perturbations in the region of high energies. Analytic continuation in Legendre polynomials, in particular, led to contradictions [10] when attempts were made to determine the parameters of the resonance of the p-phase of $\pi\pi$ scattering from the πN scattering and from the nucleon structure, and also to the impossibility of obtaining a stable solution of $\pi\pi$ scattering equations with large p-wave.[11]

Thus, the use of analytic continuation in Legendre polynomials leads in final analysis to a contradiction of the original assumption that the low-energy region is closed. The foregoing difficulties can raise doubts concerning the possibility of constructing a closed theory of strong interactions at lower energies. In our opinion, however, there are still not enough grounds for such pessimism. It is quite possible that the foregoing difficulties can be overcome by a somewhat different approach, proposed in [8,9], to the derivation of equations for the partial waves. In this deduction no use is made of analytic continuations in Legendre polynomials, and the equations for the partial waves differ from the Chew-Mandelstam equations in the structure of the crossing integrals. These integrals have, in particular, better convergence at high energies.

The aforementioned program may make it possible to describe the phenomena at low energies without internal contradictions. For this purpose, naturally, it is necessary to solve numerically the equations for partial waves of different processes, and above all for the pion-pion scattering. The latter were obtained by Hsien Ting-ch'an, Ho Tso-hsin, and Zoellner, and in simpler form under the assumption that the d and f waves are negligibly small compared with the s and p waves, as was done in [8]. In the derivation of these equations, only rigorously proved dispersion relations for forward scattering were used.

To solve these equations numerically it is necessary to have an idea of at least some general properties of the solutions of equations of this type. However, an analytic investigation of the equations derived in [8] is a complicated matter. We therefore consider first the neutral analogue of the system of equations obtained in [8]. This analysis will lead to several important consequences, which must be taken into consideration in the analytic investigation and numerical solution of such equations.

2. EQUATION AND BEHAVIOR OF SOLUTION AT HIGH ENERGIES

In the case considered here, the scattering amplitude A is a scalar function of three ordinary invariant arguments

$$s = 4(\nu + 1), \qquad u = -2\nu(1 + c), \qquad t = -2\nu(1 - c), \tag{2.1}$$

where $\nu = q^2/\mu^2$ and $c = \cos\theta$; q and θ are the momentum and scattering angle in the center-of-mass system. By virtue of the crossing symmetry, A is symmetrical under commutation of any pair of arguments (2.1). Consequently its Legendre series contains only even l. In accordance with our program,[8,9] we identify the forward scattering amplitude with the s-wave

$$A(\nu, c) = A_0(\nu) \equiv A(\nu). \tag{2.2}$$

Using the symmetry of the amplitude with respect to commutation of s and u, and taking into account the fact that

$$A(\nu) = \lim_{\varepsilon\to+0} A(\nu + i\varepsilon),$$

we obtain

$$A(-\nu - 1) = A^*(\nu). \tag{2.3}$$

The unitarity condition for the s-wave can be written in the form

$$\operatorname{Im} A(\nu) = K(\nu)\,|A(\nu)|^2, \qquad \nu > 0;$$

$$K(\nu) = \sqrt{\nu/(\nu + 1)}. \tag{2.4}$$

This formula is accurate only up to the threshold of the first inelastic process at $\nu = 3$. We shall assume, however, that (2.4) is valid for all positive ν, presupposing that this assumption hardly influences the solution at small ν. Formula (2.4) limits the function $A(\nu)$, and consequently in writing the dispersion relation it is sufficient to employ one subtraction, which we perform at the symmetrical point $\nu = -\frac{1}{2}$. We obtain

$$A(\nu) = \lambda + \frac{\nu + 1/2}{\pi}\int_0^\infty \frac{\operatorname{Im} A(\nu')}{\nu' + 1/2}\left\{\frac{1}{\nu' - \nu} + \frac{1}{\nu' + \nu + 1}\right\} d\nu'. \tag{2.5}$$

From (2.5) it is seen that the assumption

$$\lim_{\nu\to\infty} \operatorname{Im} A(\nu) = C > 0$$

leads to a logarithmic growth of the real part, and consequently, to a contradiction. Consequently $A(\infty) = 0$, and we can therefore write the equation for $A(\nu)$ without subtraction:

$$A(\nu) = \frac{1}{\pi}\int_0^\infty \operatorname{Im} A(\nu')\left\{\frac{1}{\nu' - \nu} + \frac{1}{\nu' + \nu + 1}\right\} d\nu'. \tag{2.6}$$

From (2.6) it also follows that

$$\lambda = \frac{2}{\pi}\int_0^\infty \frac{\operatorname{Im} A(\nu')}{\nu' + 1/2} d\nu' > 0. \tag{2.7}$$

Thus, Eq. (2.6) without subtraction is mathematically equivalent to Eq. (2.5) with subtraction. It follows therefore that the arbitrariness in the solution, connected with the parameter λ, is not a consequence of the subtraction.

3. SOLUTION OF THE EQUATION

We first introduce a new variable

$$\omega = (2\nu + 1)^2, \quad A(\nu) = B(\omega). \tag{3.1}$$

Equations (2.4) and (2.6) assume the form

$$\operatorname{Im} B(\omega) = k(\omega)\,|B(\omega)|^2, \qquad \omega > 1;$$

$$k(\omega) = [(\sqrt{\omega} - 1)/(\sqrt{\omega} + 1)]^{1/2} = K(\nu) \tag{3.2}$$

and

$$B(\omega) = \frac{1}{\pi}\int_1^\infty \frac{\operatorname{Im} B(\omega')}{\omega' - \omega} d\omega'. \tag{3.3}$$

Equation (3.3) is solved by the method of Castillejo, Dalitz, and Dyson.[12] We consider for this purpose the function B(z) in the complex plane $z = \omega + iy$. The function B(z) has the following properties: 1) it is analytic in the plane z with cut $[1, \infty)$, and

$$B^*(z) = B(z^*), \qquad \operatorname{Im} B(\omega + i0) = k(\omega)\,|B(\omega + i0)|^2;$$

2) it is a generalized R-function, i.e.,

$$\operatorname{Im} B(z) = \lambda(z) \operatorname{Im} z,$$

$$\lambda(z) = \frac{1}{\pi}\int_1^\infty k(\omega') \frac{|B(\omega')|^2}{|\omega' - z|^2}\, d\omega' > 0,$$

consequently $B(\omega)$ has zeros nowhere except on the real axis and at an infinitely remote point; 3) when $\omega \leq 1$ we have $B(\omega) > 0$; 4) it can have any number of isolated zeros on the segment of $(1, \infty)$.

Let us consider the function

$$H(z) = 1/B(z). \tag{3.4}$$

$H(z)$ has the following properties: 1) it is analytic and in complex plane with cut $[1, \infty)$, whereupon $H^*(z) = H(z^*)$ and $\operatorname{Im} H(\omega + i0) = -K(\omega)$ on the cut; 2) it is a generalized R-function and consequently has no zeros if $\operatorname{Im} z \neq 0$; 3) it has no poles anywhere except at $(1, \infty)$, where any number of isolated poles of first order is possible (this follows from 2), since higher-order poles do not have the properties of the R-function); 4) it has no zeros on the real axis.

We can therefore write the following general expression for $H(z)$:

$$H(z) = \frac{1}{\lambda} - \frac{z}{\pi}\int_1^\infty \frac{k(\omega')\, d\omega'}{\omega'(\omega' - z)} - cz - zR(z), \tag{3.5}$$

where

$$R(z) = \sum_n \frac{R_n}{\omega_n(\omega_n - z)}, \qquad 1 < \omega_n < \infty. \tag{3.6}$$

Let us verify the property 2). From (3.5) we have

$$\operatorname{Im} H(z) = -\operatorname{Im} z\,\Big(\lambda'(z) + c + \sum_n R_n / |\omega_n - z|^2\Big),$$

with

$$\lambda'(z) = \frac{1}{\pi}\int_1^\infty \frac{k(\omega')\, d\omega'}{|\omega' - z|^2} > 0.$$

It follows therefore that

$$R_n \geqslant 0, \qquad c \geqslant 0. \tag{3.7}$$

Let us verify now property 4). When $\omega < 1$ $H(\omega)$ is a monotonically decreasing function and in order for it not to have any zeros on this interval it is therefore sufficient to have

$$1/\lambda \geqslant I(1)/\pi + c + R(1), \tag{3.8}$$

where

$$I(\omega) = \omega\int_1^\infty \frac{k(\omega')\, d\omega'}{\omega'(\omega' - \omega)} = \pi - 2\sqrt{x}\, Q_0(\sqrt{x}) - \frac{2}{\sqrt{x}}\, Q_0\Big(\frac{1}{\sqrt{x}}\Big),$$

$$x = \frac{\nu}{\nu + 1}, \tag{3.9}$$

and $Q_0(x) = \frac{1}{2}\ln[(x+1)/(x-1)]$ is the Legendre function of the second kind.

As a result we obtain a solution of (3.3) in the form

$$B(\omega) = \lambda/[1 - \lambda I(\omega)/\pi - \lambda c\omega - \lambda\omega R(\omega)]. \tag{3.10}$$

Here $R(\omega)$ is determined from (3.6) while the constants γ, c, and R_n satisfy the conditions (3.7) and (3.8). The limitation (2.7) is a consequence of these conditions.

4. COMPARISON WITH PERTURBATION THEORY

Let us establish the correspondence between (3.10) and the results of perturbation theory. Assuming λ to be small and expanding the denominator of (3.10), we obtain

$$A(\nu) = \lambda + \frac{\lambda^2}{\pi} I[(2\nu + 1)^2] + \lambda^2 c\omega + \lambda^2\omega R(\omega) + O(\lambda^3). \tag{4.1}$$

The first two terms of (4.1) correspond to the diagram of first and second orders of perturbation theory, based on the Lagrangian

$$L_{int}^{(0)} = (4\pi/3)\,\lambda\varphi^4. \tag{4.2}$$

The fourth term corresponds to the pole contributions of the diagrams corresponding to the second order of perturbation theory for a Lagrangian of the following form (see in this connection Dyson's paper[13])

$$L_{int}^{(1)} = \sum_n g_n \Phi_n(x)\, \varphi^2(x), \tag{4.3}$$

where the fields Φ_n describe unstable particles with masses $m_n > 2$. The correspondence between g_n, λ, m_n, ω_n, and R_n can be readily established in perturbation theory.

The third term can be set in correspondence with the non-renormalized Lagrangian

$$L_{int}^{(2)} = f\left\{\Big(\frac{\partial\varphi}{\partial x_n}\frac{\partial\varphi}{\partial x_n}\Big)^2 - \frac{\varphi^4}{3}\right\}, \tag{4.4}$$

where $f = 2\pi\lambda^2 c$. Naturally, the correspondence with the Lagrangian (4.4) is arbitrary to a higher degree, since we cannot construct a consistent perturbation theory for such a Lagrangian. However, as shown by one of us (A.E.), such a correspondence can be established in the nonrelativistic theory.

We defer the discussion of this interesting fact and confine ourselves at the present time to an analysis of the terms of (4.1) corresponding to the Lagrangian (4.2).

Calculating the s-wave, we obtain from (4.2)

$$A_{\text{p.t.}}(\nu) = \lambda_0 + \frac{\lambda_0^2}{\pi} A_{\text{p.t.}}^{(2)}(\nu) + \ldots, \tag{4.5}$$

where the amplitude is renormalized at the point x $\nu = 0$, i.e., $\lambda_0 = A_{\text{p.t.}}(0)$ and

$$A_{\text{p.t.}}^{(2)} = 6 - 2\sqrt{x}\,Q_0(\sqrt{x}) - \frac{4}{\sqrt{x}}\,Q_0\left(\frac{1}{\sqrt{x}}\right) - 2\frac{1-x}{x}\,Q_0^2\left(\frac{1}{\sqrt{x}}\right). \tag{4.6}$$

Expressions (4.5) and (4.6) must be compared with the first terms of (4.1), which can be written in the new normalization in the form (4.5), with

$$A_{\text{i.e.}}^{(2)} = 2 - 2\sqrt{x}\,Q_0(\sqrt{x}) - \frac{2}{\sqrt{x}}\,Q_0\left(\frac{1}{\sqrt{x}}\right). \tag{4.7}$$

Let us compare the second-order terms in (4.6) and (4.7). Near the threshold we obtain: a) from perturbation theory

$$A_{\text{p.t.}}^{(2)} \approx -\frac{8}{3}x - \frac{52}{45}x^2 - \frac{248}{315}x^3 + \ldots;$$

b) from the solution of the integral equation

$$A_{\text{i.e.}}^{(2)} \approx -\frac{8}{3}x - \frac{16}{15}x^2 - \frac{24}{35}x^3 + \ldots.$$

At the threshold of the first inelastic process we get for $\nu = 3$

$$A_{\text{p.t.}}^{(2)}(3) = -3.521, \qquad A_{\text{i.e.}}^{(2)}(3) = -3.323.$$

We see therefore that in the region of low energies the solution of the integral equation corresponds with good accuracy to the perturbation-theory results. The error in the second-order term amounts at $\nu = 3$ only to 6 percent. This agreement confirms the hypothesis that the low-energy region can be described in closed form.

5. RESONANT BRANCH OF SOLUTION

It follows from (3.10) that as $\nu \to \infty$ the solution admits of two different asymptotic approximations:

$$A(\nu) \approx \pi / 2 \ln \nu, \tag{5.1}$$

corresponding to the absence of non-renormalizable interactions, and

$$A(\nu) \approx -1/c\nu^2, \tag{5.2}$$

which corresponds to the non-renormalizable Lagrangian (4.4). These asymptotic expansions do not depend on the part of the R-function corresponding to the unstable particles. We shall henceforth confine ourselves for simplicity to the case when there are no unstable particles or, what is equivalent, to the case when the phase does not vanish when $0 < \nu < \infty$. Then the solution does not admit of resonance in the case (5.1), and is resonant in the case of (5.2) and for small λ at the point

$$\nu_r = \tfrac{1}{2}\{(\lambda c)^{-1/2} - 1\} = \tfrac{1}{2}\{\sqrt{\pi\lambda/2f} - 1\}. \tag{5.3}$$

If λ and f are of the same order of magnitude, the resonance is in the low-energy region.

The resonant solution for small λ can be written in the form

$$A(\nu) = \frac{\lambda}{2} \Big/ \left[1 - \frac{2\nu+1}{2\nu_r+1} - i\frac{\lambda}{2}K(\nu)\,\theta(\nu)\right] + \frac{\lambda}{2} \Big/ \left[1 + \frac{2\nu+1}{2\nu_r+1} + i\frac{\lambda}{2}K(-1-\nu)\,\theta(-1-\nu)\right]. \tag{5.4}$$

In the limit, as $\lambda \to 0$, the imaginary part of $A(\nu)$ is approximated by the δ-functions:

$$\operatorname{Im} A(\nu) \approx \tfrac{1}{2}\pi\lambda\left(\nu_r + \tfrac{1}{2}\right)\{\delta(\nu - \nu_r) - \delta(\nu + \nu_r + 1)\}, \tag{5.5}$$

and the real part is approximated by the pole terms

$$\operatorname{Re} A(\nu) \approx \frac{\lambda}{2}\left(\nu_r + \frac{1}{2}\right)\left\{\frac{1}{\nu_r - \nu} + \frac{1}{\nu_r + \nu + 1}\right\}. \tag{5.6}$$

Thus, for fixed ν_r, the width of the resonance tends to zero in proportion to λ, and at $\lambda = 0$ we obtain a non-zero solution. The scattering phase changes abruptly from 0 to π at the resonance point ν_r, the position of which is arbitrary. Thus, the solution is degenerate when $\lambda = 0$.

We note that in this case the d-wave A_2 will be proportional to the first degree of λ. It is expressed, however, in terms of a crossing integral with large denominator and is consequently small. Thus, for example, when $\nu_r = 3$ and $0 < \nu < 6$, the numerical estimates yield

$$5A_2(\nu)/A_0(\nu) \lesssim 6\%.$$

It is quite probable that the solutions for the charged case also have an arbitrariness of the type (3.10). It can be shown that the solutions of the charged system should decrease at infinity. In addition to the logarithmic branch, corresponding to renormalized perturbation theory, branches can exist in which the decrease at infinity is faster. When the interactions are turned off, these branches should lead to discontinuous phases, similar to what was described above.

Let us make a few remarks on the possibility of obtaining the solutions (3.10) by the "N/D method" of Chew and Mandelstam (see [4]).

The form of the integral representation for the function D depends essentially on the asymptotic form of the phase as $\nu \to \infty$. This representation is determined with accuracy to a polynomial of degree n,[14,15] where

$$n = [\delta(0) - \delta(\infty)]/\pi.$$

The choice of representation in the form (V.12) of [4] corresponds to n = 0. By the same token, Chew and Mandelstam have earlier excluded the possibility of an odd number of resonances in the partial waves. Therefore the equations such as (V.11) and (V.12) of [4] cannot describe solutions of the form (5.4). Resonant solutions such as (5.4) call for the use of a second subtraction in (V.12).

We note that a similar conclusion was reached by Taylor in his latest paper.[16]

We note also that in our opinion the equations of the "N/D method," such as (V.11) and (V.12), cannot ensure crossing symmetry of the real part of the amplitude.

6. DISCUSSION OF RESULTS

Let us make first one formal remark. Solutions (5.1) and (5.2) recall in many respects the model expressions for the Green's functions in the renormalizable and non-renormalizable theories proposed in[17,18].

Solution (5.1) is analogous to the expression for the photon Green's function. This solution satisfies the spectral representation without subtraction, [i.e., Eq. (2.6)]. However, if we attempt the expansion in powers of λ under the sign of the spectral integral, we obtain after integration logarithmically divergent integrals in each order in λ. On the other hand, if we carry out one subtraction in the spectral representation, i.e., if we go over to (2.5), these divergences do not arise.

The solution (5.2) corresponds in this sense to the non-renormalizable theory. If we expand the integrand in (2.6) in powers of λ and f, we obtain integrals whose degree of divergence increases with the power of f. These divergences cannot be removed by any finite number of subtractions. The solution (5.2) has thus no correspondence with perturbation theory. There are no grounds, however, for discarding this solution and for confining ourselves to solutions of type (5.1), which actually are analytic continuations of perturbation theory to the region of not small values of λ. Solutions such as (5.2) are degenerate when the interaction is turned off. As was noted by Bogolyubov,[19] solutions of this type are of great interest in many problems of statistical physics. We see now that such solutions can also turn out to be important in the theory of elementary particles. It is known that the 33-resonance in πN-scattering is sufficiently narrow. A similar conclusion is reached also from preliminary estimates of p-resonance in $\pi\pi$ scattering. However, it is very difficult to obtain a narrow 33-resonance.[20,21] Even greater difficulties arise when attempts are made to obtain a narrow p-resonance in $\pi\pi$ scattering.[22,23] Solutions such as (5.2) lead to narrow resonances in a natural manner.

On the basis of the explicit form of the solution (3.10), we can draw the following important conclusion.

The integral equations obtained from the dispersion relations, from unitarity conditions, and from crossing symmetry do not lead to an ambiguous description of the scattering processes. In order to determine the solution completely, it is necessary to specify an (infinite!) set of parameters.

This fact is not surprising. The dispersion relations reflect only the very general properties of the theory, such as causality and relativistic invariance, and do not give any details on the specific interaction mechanism. In this sense, the situation in relativistic dispersion theory corresponds fully to the situation in the nonrelativistic models (see, for example, [13]).

Thus, in order to obtain a theory from the integral dispersion equations, it is necessary to specify many other properties of the solutions of these equations. For example, in the neutral case under consideration, it is sufficient to specify the value of the amplitude at the threshold of the process, to state the asymptotic behavior at infinity, and to stipulate that the scattering phase not vanish. Similar limitations can be imposed by introducing fixed subtraction constants. The threshold value is specified by the first subtraction. Specification of the second subtraction constant (i.e., the derivative of the amplitude at the threshold) is equivalent, in the absence of phase zeros, to fixing the asymptotic behavior. This method of fixing the solution is the most convenient in numerical solution of the integral equations.

We can now speculate somewhat on the physical meaning of the parameters defining the solutions.

We can, first, establish a correspondence between these parameters and Lagrangians of the type (4.2), (4.3), and (4.4). It may turn out here that an important role is played in pion physics by interactions which are not renormalizable in perturbation theory (see [24] in this connection). In other words, the dispersion approach may decide, through comparison with experiment, the existence of non-renormalizable strong interactions.

Second, it can be assumed that the parameters under consideration take into account the influence of inelastic processes on the elastic processes in the low-energy region. We arrive thereby at the possibility of a phenomenological account of inelastic processes in the two-particle approximation scheme.

In conclusion we note that analysis shows the essential properties of the neutral model, to which this section was devoted, to be possessed also by the scattering of charged pions. The results of an investigation of a real charged case will be reported in future articles.

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