ERRORS DUE TO THE DEAD TIME OF COUNTERS OPERATING IN CONJUNCTION WITH PULSED SOURCES

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Expressions are derived for the mean counting rate, mean counting rate loss, and dispersions of the recorded and suppressed counts for different relations between the dead time and the pulse duration and repetition rate. Errors due to the dead time are found to depend greatly on the relations between these quantities. The derived formulas can be used to compute the experimental errors due to the dead time.

INTRODUCTION

THE statistics of counting losses associated with counter dead time in the case of pulsed sources is of considerable practical interest, because many investigations in nuclear and particle physics are performed with various types of pulsed accelerators. Earlier literature[1,2] on this subject had been confined to calculating the mathematical expectation of the counting loss for counters with "unprolonged" dead time[3] not exceeding the interval between pulses.

This last condition is not always satisfied in practice. For example, in some cyclic accelerators[4] the beam is bunched with a repetition period equal to the period of orbital revolution (about 10^{-8} sec). The dead time of real detectors usually exceeds this period, and in some instances is considerably longer than the repetition period of bunches. For example, in traveling-wave linear accelerators[5] oscillators in the 10-cm range are used, so that the bunch repetition period is about 3×10^{-10} sec.

It is of interest to study the statistics for arbitrary relations between the dead time and pulse spacing, and to calculate, in addition to the mean values, the dispersions of counts and counting losses (suppressed counts).

The analysis will be based on a sequence of identical pulses with constant repetition frequency f for both "prolonged" and "unprolonged" dead times.[3] The dead time τ is assumed to be constant and unfluctuating. A Poisson distribution $\eta(t)$ will be assumed for the particles impinging on a counter (hits) during any time interval. Since the pulses are identical, the intensity (the mean number of hits per second), averaged over the time of the experimental run, is

$$n = f \int_0^{t_c} \eta(t)\,dt. \qquad (1)$$

Time is measured here from the start of the pulse.

Since the mean number of particles entering the counter in a time T is nT, which is related simply to the mean count $\overline{M}$ and the mean counting loss $\overline{L}$ in the same time by

$$nT = \overline{M} + \overline{L}, \qquad (2)$$

only the expressions for $\overline{M}$ will be given below. Expressions for the dispersion will be given both in the case of the recorded counts (D_M) and of the suppressed counts (D_L), since these quantities, because of the statistical relation between M and L, do not satisfy any equation analogous to (2). In our derivations it will be assumed that the reciprocal pulse duty factor satisfies the realistic condition $Q > 2$.

1. RELATIONS FOR DEAD TIME SHORTER THAN PULSE SEPARATION

In these cases both the count and the counting loss during any pulse are independent of their values during other pulses. $\overline{M}$, D_M, and D_L can therefore be obtained by summations over all pulses.

We shall confine ourselves to the two extreme cases, $t_c \gg \tau$ and $t_c < \tau$. In the first case, using the formulas for constant intensity,[3] we easily obtain for rectangular pulses the following expressions for the mean counts of counters with prolonged and unprolonged dead times ($\overline{M}_p$ and $\overline{M}_u$, respectively):

$$\overline{M}_p = nTe^{-nQ\tau}, \qquad \overline{M}_u = nT/(1 + nQ\tau). \qquad (3)$$

For small loads $nQ\tau \ll 1$, and

$$\overline{M}_{\text{p}} = \overline{M}_{\text{u}} = nT(1 - nQ\tau). \quad (4)$$

The dispersions (standard deviations) for small loads are thus given directly by

$$D_M = nT(1 - 3nQ\tau), \qquad D_L = \overline{L} = n^2QT\tau. \quad (5)$$

It follows from (3)—(5) that all statistical characteristics depend on the single parameter of reciprocal pulse duty factor Q. All relations have the same form as in the case of constant intensity, but with dead time $Q\tau$.

In the second case ($t_c < \tau$) no more than one count can occur during each pulse. For arbitrary pulse shapes we therefore easily obtain the following expressions for the mean count and the dispersions (which coincide for both types of dead time):

$$\overline{M} = fT\{1 - e^{-n/f}\}, \qquad D_M = fTe^{-n/f}\{1 - e^{-n/f}\},$$

$$D_L = fT\{n/f + e^{-n/f} - e^{-2n/f} - ne^{-n/f}/f\}. \quad (6)$$

Unlike the preceding case, all parameters here depend only on the pulse repetition frequency.

For sufficiently high frequencies or low intensities ($f \gg n$), Eq. (6) is simplified as follows:

$$\overline{M} = nT(1 - n/2f), \qquad D_M = nT(1 - n/f),$$

$$D_L = \overline{L} = n^2T/2f. \quad (7)$$

It is thus seen that the counting loss will be greater or less than in the case of constant intensity equal to n, depending on whether 1/2f is larger or smaller than τ.

2. RELATIONS FOR DEAD TIME LONGER THAN PULSE SEPARATION. EXPRESSIONS FOR THE MEAN COUNT

We first obtain the mean count in the case of a prolonged dead time. It is most convenient to use Schiff's formula[6] for an arbitrary time dependence of the intensity, which in the given case is

$$\overline{M} = fT\int_0^{t_c} \eta(t)\exp\left\{-\int_{t-\tau}^{t} \eta(t')\,dt'\right\}dt.$$

The expression for $\overline{M}$ differs depending on the relation between 1/f and τ (Fig. 1). In one case (Fig. 1a), $\overline{M}$ can be calculated by dividing the integral from 0 to t_c into two integrals, from 0 to $t_c - t_1$ and from $t_c - t_1$ to t_c. In the first of these integrals, the integral in the exponent can obviously be put into the form

$$\int_{t-\tau}^{t} \eta(t')\,dt' = \int_{t_1+t}^{t_c} \eta(t')\,dt' + \lambda\int_0^{t_c} \eta(t')\,dt' + \int_0^{t} \eta(t')\,dt'.$$

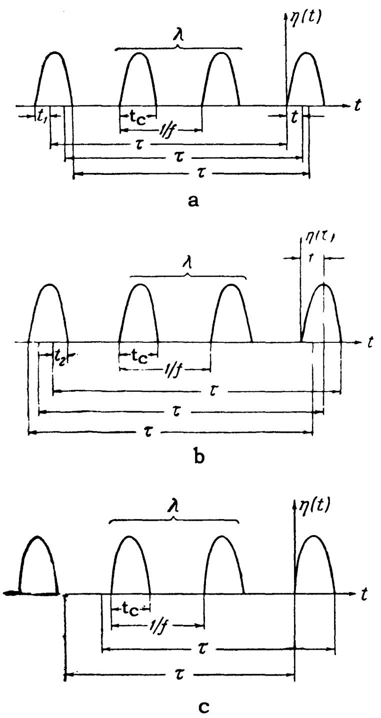

FIG. 1. Three possible relations between τ, f, and t_c. a – the dead time, plotted backward from the start of a given pulse (on the extreme right), includes λ complete pulses and a fraction of an additional pulse; b – the dead time, plotted backward from the end of a given pulse, includes λ complete pulses and a fraction of an additional pulse; c – the dead time, plotted backward from either the start or end of a given pulse, includes only λ complete pulses.

After some elementary transformations involving (1), we have

$$\int_0^{t_c-t_1} \eta(t)\exp\left\{-\int_{t-\tau}^{t} \eta(t')\,dt'\right\}dt$$
$$= e^{-(\lambda+1)n/f}\int_0^{t_c-t_1} \eta(t)\exp\left\{-\int_t^{t+t_1} \eta(t')\,dt'\right\}dt. \quad (10)$$

In the second integral the expression in the exponent is divided into only two terms, the second and third terms in (9). Therefore

$$\int_{t_c-t_1}^{t_c} \eta(t)\exp\left\{-\int_{t-\tau}^{t} \eta(t')\,dt'\right\}dt$$
$$= e^{-\lambda n/f}\int_{t_c-t_1}^{t_c} \eta(t)\exp\left\{-\int_0^{t} \eta(t')\,dt'\right\}dt. \quad (11)$$

Transforming by means of the identity

$$\int_0^x \eta(t)\exp\left\{-\int_0^t \eta(t')\,dt'\right\}dt = 1 - \exp\left\{-\int_0^x \eta(t)\,dt\right\} \quad (12)$$

and using (8) and (10), we obtain

$$\overline{M} = fTe^{-\lambda n/f}\Big\{e^{-n/f}\int_0^{t_u-t_1}\eta(t)\exp\Big[\int_t^{t+t_1}\eta(t')\,dt'\Big]dt - e^{-n/f} + \exp\Big[-\int_0^{t_u-t_1}\eta(t)\,dt\Big]\Big\}. \quad (13a)$$

In the second and third cases (Fig. 1b and c) similar calculations lead to

$$\overline{M} = fTe^{-(\lambda+1)n/f}\Big\{1 - \exp\Big[-\int_0^{t_2}\eta(t)\,dt\Big] + \int_{t_2}^{t_u}\eta(t)\exp\Big[-\int_{t-t_2}^{t}\eta(t')\,dt'\Big]dt\Big\}, \quad (13b)$$

$$\overline{M} = fTe^{-\lambda n/f}\{1 - e^{-n/f}\}. \quad (13c)$$

In the special case $\lambda = 0$, Eq. (13c) reduces to the expression for $\overline{M}$ in (6).

Equation (13c) depends only on the mean intensity and pulse repetition frequency, while (13a) and (13b) depend also on the pulse shape and on the exact relation between f and τ. The simplest forms of these equations are obtained when the dead time contains an integral number μ of periods:

$$\overline{M} = nTe^{-n\mu/f} = nTe^{-n\tau}, \quad (14)$$

i.e., the count will be the same as in the case of a continuous source with constant intensity n.

These results enable us to estimate the mean count when the exact relation between f and τ or the pulse shape is unknown. It follows from (8) that $\overline{M}$ is a monotonic function of τ. Therefore, when $f\tau$ lies between the integers μ and κ, we have, according to (14),

$$nTe^{-\kappa n/f} \leqslant \overline{M} \leqslant nTe^{-n\mu/f}. \quad (15)$$

At high repetition frequencies or low intensities ($f \gg n$) Eqs. (13a)–(13c) are simplified. Series expansions of the expressions in the braces give, to second order terms,

$$\overline{M} = nTe^{-\lambda n/f}\Big\{1 + n\Big(a_1 - \tfrac{1}{2} - b_1 + b_1^2/2\Big)/f\Big\}, \quad (16a)$$

$$\overline{M} = nTe^{-(\lambda+1)n/f}\{1 - n(a_2 + b_2^2/2)/f\}, \quad (16b)$$

$$\overline{M} = nTe^{-\lambda n/f}\{1 - n/2f\}, \quad (16c)$$

$$a_1 = \Big(\frac{f}{n}\Big)^2\int_0^{t_u-t_1}\eta(t)\Big[\int_t^{t+t_1}\eta(t')\,dt'\Big]dt, \quad b_1 = \frac{f}{n}\int_0^{t_u-t_1}\eta(t)\,dt,$$

$$a_2 = \Big(\frac{f}{n}\Big)^2\int_{t_2}^{t_u}\eta(t)\Big[\int_{t-t_2}^{t}\eta(t')\,dt'\Big]dt, \quad b_2 = \frac{f}{n}\int_0^{t_2}\eta(t)\,dt. \quad (17)$$

The coefficients a_1, b_1, a_2, and b_2 do not exceed unity.

At very high frequencies, when $\lambda \gg 1$, we have $\tau \approx \lambda/f$ and Eqs. (16a)–(16c) practically coincide, to within a factor of the order $1 - n/f$, with $\overline{M}$ for the case of constant intensity.

We shall now consider the case of unprolonged dead time. The simplest expression for the mean count is obtained either when the dead time includes an integral number of periods or when the dead time considerably exceeds the repetition period. In these instances the mean count can be calculated by the procedure used in references 3 and 7 for constant intensity. Let us assume that M counts have been obtained during a sufficiently prolonged experiment. Then, if τ contains an integral number of periods λ, the total number of periods during which hits could not be registered is $M\lambda$, without considering in which portions of the pulses the counts occurred. It follows that the mean number of inoperative periods is

$$\bar{\nu} = \lambda\overline{M}. \quad (18)$$

The counting loss will be the number of hits during these periods; the mean loss will thus be

$$\overline{L} = n\bar{\nu}/f = n\lambda\overline{M}/f = n\tau\overline{M}. \quad (19)$$

From (2) we finally obtain

$$\overline{M} = nT/(1 + n\tau). \quad (20)$$

It is easily seen that the same result follows in the case $\tau f \gg 1$, independently of the exact relation between these quantities. The dead time following each count then includes an integral number λ of complete pulses and fractions of two pulses at the beginning and end of the dead time. Since $\lambda \gg 1$ the mean number of hits during these fractions can be neglected compared with the mean number during the large number of complete pulses. All considerations leading to (20) therefore remain valid. Equation (20) differs in no way from the corresponding expression for the constant intensity case, i.e., here also the pulsed character of the source does not affect the results.

For an arbitrary relation between f and τ we shall calculate only the upper and lower limits of $\overline{M}$; this can be done easily in the following manner. If λ is the number of complete periods included in the dead time (neglecting fractional periods), in the case of M counts we obviously have

$$\lambda M \leqslant \nu \leqslant (\lambda + 1)M. \quad (21)$$

Hence, by analogy with the foregoing calculations, we have

$$\frac{nT}{1 + \lambda n/f} \geqslant \overline{M} \geqslant \frac{nT}{1 + (\lambda + 1)n/f}. \quad (22)$$

These narrow limits permit a highly accurate determination of $\overline{M}$ in the cases of large λ and small loads ($\lambda n \ll f$).

3. RELATIONS FOR DEAD TIME LONGER THAN PULSE SEPARATION. DERIVATION OF THE DISPERSIONS

The expressions for the dispersions will be derived only in the most interesting practical case of small loads ($n\tau \ll 1$). Since the statistical relations for the different types of dead time coincide in this case,[3] we shall consider only the unprolonged dead time, for which the calculations are easier. The dispersion of the counts can now be derived, as in[3], by passing from the statistics of counts to the statistics of pulse intervals. The number r_i of complete periods between two successive counts obviously does not depend on the exact times when preceding counts occurred. On the other hand, in the present case ($\tau > 1/f - t_c$, $n\tau \ll 1$) the mean value of r_i is considerably greater than unity. Therefore in the case of "good" statistics (large M during the experiment) the total number of periods is approximately

$$R = \sum_{i=1}^{M} r_i. \qquad (23)$$

The mean values $\bar{r}$ and the dispersions D_r of all r_i are obviously identical. Therefore, because of the large number of independent r_i, R has a Gaussian distribution with mean value and dispersion given by

$$\overline{R} = M\bar{r}, \qquad D_R = MD_r. \qquad (24)$$

Passing from the statistics of intervals to the statistics of counts by means of Bayes' formula and confining ourselves to "good" statistics, we obtain (as in Chapter 2, Sec. 6 of[3])

$$\overline{M} = R/\bar{r}, \qquad D_M = RD_r/\bar{r}^{\,3}. \qquad (25)$$

D_M will be derived after $\bar{r}$ and D_r are determined. In the given case ($r_i \gg 1$) r_i can be represented, with an error not exceeding unity (Fig. 2), by

$$r_i = \lambda + \rho_i. \qquad (26)$$

It follows from (26) that the probability of r_i complete periods in the time interval between counts equals the probability that no particles arrive during r_i periods but that a particle does hit during the following period:

$$p(r_i) = ne^{-\rho_i n/f}/f. \qquad (27)$$

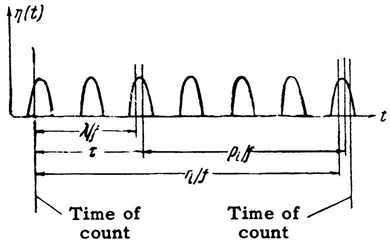

FIG. 2. Relations between r_i, λ, and ρ_i. r_i – number of complete periods between two successive counts; λ – number of complete periods in time τ; ρ_i – number of complete periods between termination of dead time and time of next count.

This formula is used to calculate $\bar{r}$ and D_r (as in Chapter 1, Sec. 6 of[3]):

$$\bar{r} = \lambda + f/n, \qquad D_r = (f/n)^2. \qquad (28)$$

Substituting in (25) and using the formula R = fT, we finally have

$$D_M = nT/(1 + \lambda n/f)^3 = nT(1 - 3\lambda n/f). \qquad (29)$$

This equation is based on the fact that we are considering small loads. When τ contains an integral number of periods or $f\tau \gg 1$, we have $\lambda/f = \tau$, and (29) differs in no way from the corresponding expression for continuous operation with constant intensity n.

The expression for the dispersion of the counting loss can also be derived by a familiar procedure (Chapter 4, Sec. 6 of[3]). Without presenting the calculations, we note only that at small loads and for $f\tau \gg 1$ the result is the same as in the case of constant intensity.

CONCLUSIONS

The foregoing analysis has shown that the counting loss associated with dead time depends essentially on the relation between the dead time and pulse duration and spacing. With increasing pulse repetition frequency and reduced pulse duration for the same mean intensity, the counting loss tends generally to diminish. While for $t_p \gg \tau$ the counting loss increases by a factor Q compared with the case of constant intensity (and Q is often of the order of tens of thousands), when t_p and τ are comparable the counting loss is very close to that obtained with constant intensity (multiplied by no more than a few units). Finally, when $f\tau \gg 1$, there is no difference between these quantities, and all statistical relations are identical for the two cases. The difference also disappears when the condition $f\tau \gg 1$ is unfulfilled in the special case of a dead time containing an integral number of periods.

Our results show particularly that high-frequency bunching of beams in linear accelerators has no effect on the experimental errors associated with the dead time.

[1] C. H. Westcott, Proc. Roy. Soc. (London) **A194**, 508 (1948).

[2] N. Feather, Proc. Cambridge Phil. Soc. **45**, 648 (1949).

[3] Gol'danskii, Kutsenko, and Podgoretskii, Statistika otschetov pri registratsii yadernykh chastits (Counting Statistics in the Registration of Nuclear Particles), Fizmatgiz, 1959.

[4] M. S. Livingston, High-Energy Accelerators, Interscience Publishers, New York, 1954.

[5] E. L. Chu and W. Hansen, J. Appl. Phys. **18**, 996 (1947).

[6] L. I. Schiff, Phys. Rev. **50**, 88 (1936).

[7] B. V. Gnedenko, JETP **11**, 101 (1941).

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