"SCALAR" FORM OF THE DIRAC EQUATION AND CALCULATION OF THE MATRIX ELEMENTS FOR REACTIONS WITH POLARIZED DIRAC PARTICLES

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A formalism first proposed by Sommerfeld is developed, in which the wave function in the Dirac equation is a scalar and the corresponding γ matrices behave like a four-vector under Lorentz transformations. The transition to this representation and its various features are investigated. Solutions of the resulting equation are found, and a study is made of the spin operator of the "scalar" Dirac equation, which is a differential operator. A new method is indicated for calculating matrix elements for reactions involving polarized particles. For this purpose a method is also proposed for writing the γ matrices by means of Kronecker δ symbols and certain discontinuous functions.

THE book by Sommerfeld[1] contains a clear statement of the problem as to whether it is possible that there be two points of view regarding the transformation properties of the wave function and the matrices in the Dirac equation. The first, generally accepted, point of view regards the γ matrices as quantities which do not change under any possible linear transformations of the coordinates, while the ψ function turns out to possess the definite transformation properties of a bispinor. In a certain sense the other point of view is the precise opposite of this. Adopting it, we regard the wave function as an invariant under Lorentz transformations, and the γ matrices turn out to have the properties of a four-vector.

Reference [1], however, gives only a statement of the question in principle, and the second point of view has not been developed. The purpose of the present paper is to present an example and some consequences of a formalism entirely based on the second, alternative, statement of the problem.

1. STATEMENT OF THE DIRAC EQUATIONS IN "SCALAR" FORM

Let us define as follows four mutually orthogonal vectors $n^\mu{}_\alpha$ (the index α written in the second place will always refer to the number of the vector, and the first index μ indicates the component):*

$$g_{\mu\nu} n^\mu{}_\alpha n^\nu{}_\beta = g_{\alpha\beta}. \qquad (1)$$

*Greek letters run through the values 0, 1, 2, 3. The metric used in this paper is $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$; $\hbar = c = 1$.

It is not hard to verify that the components of the vectors $n^\mu{}_\alpha$ also satisfy the relations

$$g^{\alpha\beta} n^\mu{}_\alpha n^\nu{}_\beta = g^{\mu\nu}. \qquad (2)$$

We can change to covariant components in the usual way: $n_{\mu\alpha} = g_{\mu\nu} n^\nu{}_\alpha$. The relations (1) and (2) allow us to interpret the matrix composed of the components of the vectors $n^\mu{}_\alpha$ as the matrix of a certain Lorentz transformation.

Let us also define by means of the vectors $n^\mu{}_\alpha$ certain matrices N^μ:

$$N^\mu = n^\mu{}_\alpha \gamma^\alpha, \qquad N_\mu = g_{\mu\nu} N^\nu. \qquad (3)$$

Here the four-rowed matrices γ^α have the well known properties

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \qquad \gamma_\mu = g_{\mu\nu}\gamma^\nu \qquad (4)$$

and with them one can write the Dirac equation in the usual form

$$(-i\gamma^\mu \partial/\partial x^\mu + m)\psi = 0. \qquad (5)$$

It is not hard to verify from Eqs. (1) and (4) that

$$N^\mu N^\nu + N^\nu N^\mu = 2g^{\mu\nu}. \qquad (6)$$

Owing to this it is possible to construct a first-order equation by means of the matrices N^μ

$$(-iN^\mu \partial/\partial x^\mu + m)\omega = 0. \qquad (7)$$

In view of Eq. (6) multiplication of Eq. (7) on the left by the operator $(iN^\mu \partial/\partial x^\mu + m)$ leads to the second-order equation $(\Box + m^2)\omega = 0$.

Let us now examine the transformation properties of the equation (7). The Lorentz transformation

$$x'^{\mu} = a^{\mu}{}_{\nu} x^{\nu}, \quad n'^{\mu}{}_{\alpha} = a^{\mu}{}_{\nu} n^{\nu}{}_{\alpha}, \quad a^{\nu}{}_{\mu} a_{\lambda}{}^{\mu} = \delta^{\nu}_{\lambda}$$

does not change the form of (7)

$$\left(-iN'^{\mu} \frac{\partial}{\partial x'^{\mu}} + m\right)\omega' = 0, \qquad \omega' = S\omega, \tag{8}$$

if the following relations hold:

$$S\gamma^{\alpha} = \gamma^{\alpha} S \qquad (\alpha = 0, 1, 2, 3). \tag{9}$$

In other words, according to Schur's lemma the matrix S must be the unit matrix, and owing to this the wave function ω is a scalar under Lorentz transformations:

$$S = I, \qquad \omega' = \omega. \tag{10}$$

2. SOLUTIONS OF THE "SCALAR" DIRAC EQUATION

It is easy to find the solutions of our equation (7) by using the usual methods. We note certain features that turn up in this connection.

The condition of solubility of the two systems of equations

$$(N^{\mu} p_{\mu} - m)\,\omega_0^{(1)} = 0, \qquad (N^{\mu} p_{\mu} + m)\,\omega_0^{(2)} = 0, \tag{11}$$

which correspond to positive-frequency and negative-frequency solutions, is the vanishing of the determinants of their coefficients. This condition takes the form

$$(n^{\mu}{}_0 p_{\mu})^2 - (n^{\mu}{}_1 p_{\mu})^2 - (n^{\mu}{}_2 p_{\mu})^2 - (n^{\mu}{}_3 p_{\mu})^2 - m^2 = 0. \tag{12}$$

We can easily convince ourselves of the truth of Eq. (12) if we go from the system of vectors $n^{\mu}{}_{\alpha}$ to another system of mutually orthogonal vectors $n'^{\mu}{}_{\alpha}$ with the components $n'^{\mu}{}_{\alpha} = \delta^{\mu}_{\alpha}$. In this system of coordinates the condition that the determinant be zero takes the usual form: $p'_{\mu} p'^{\mu} - m^2 = 0$. It is obvious that because of the Lorentz invariance of Eq. (12) the equation holds also in any coordinate system.

We now give a table of the solutions of the "scalar" Dirac equation (the solutions are computed to within a normalization factor)

$$\omega_0 \sim \begin{vmatrix} n^{\mu}{}_0 p_{\mu} + m & 0 & -n^{\mu}{}_3 p_{\mu} & -(n^{\mu}{}_1 - i n^{\mu}{}_2)\, p_{\mu} \\ 0 & n^{\mu}{}_0 p_{\mu} + m & -(n^{\mu}{}_1 + i n^{\mu}{}_2)\, p_{\mu} & n^{\mu}{}_3 p_{\mu} \\ -n^{\mu}{}_3 p_{\mu} & -(n^{\mu}{}_1 - i n^{\mu}{}_2)\, p_{\mu} & n^{\mu}{}_0 p_{\mu} + m & 0 \\ -(n^{\mu}{}_1 + i n^{\mu}{}_2)\, p_{\mu} & n^{\mu}{}_3 p_{\mu} & 0 & n^{\mu}{}_0 p_{\mu} + m \end{vmatrix} \tag{13}$$

When one chooses the special coordinate system in which $n'^{\mu}{}_{\alpha} = \delta^{\mu}_{\alpha}$, the solutions (13) take the form of the well known solutions u (first two columns) and v (last two columns) (see, for example, reference 2).

3. THE TRANSFER OF THE TRANSFORMATION PROPERTIES FROM THE ψ FUNCTION TO THE γ MATRICES

Comparing Eqs. (4) and (6), we see that the matrices γ^{μ} and N^{μ} are connected with each other by a similarity transformation (cf., e.g., reference 3)

$$N^{\mu} = R\gamma^{\mu} R^{-1}, \tag{14}$$

where R is a certain nonsingular matrix. On the other hand, remembering the definition of the matrices N^{μ}, we can write the equation

$$R\gamma^{\mu} R^{-1} = n^{\mu}{}_{\alpha} \gamma^{\alpha}. \tag{15}$$

Since the coefficients $n^{\mu}{}_{\alpha}$ are the matrix of a Lorentz transformation, it is easy to identify R^{-1} with the well known transformation matrix that acts on the bispinor in a Lorentz transformation. Thus the change from the usual representation (5) of the Dirac equation to the new "scalar" Dirac equation (7) is made by means of the relations

$$\omega = R\psi, \qquad N^{\mu} = n^{\mu}{}_{\alpha} \gamma^{\alpha} = R\gamma^{\mu} R^{-1}. \tag{16}$$

Thus we have obtained the Dirac equation in a "scalar" form, i.e., in a form in which the wave function does not change in Lorentz transformations and the transformation properties that in the usual theory inhere in the wave function are transferred to the N matrices. We note that the R transformation that accomplishes this transfer preserves eigenvalues, as do the unitary transformations of the bispinor in the usual Dirac equation.

Having in mind Eq. (16), the definition

$$\bar{\omega} = \omega^{+} \gamma^0, \tag{17}$$

and also the relation characteristic of Lorentz transformations, $\gamma^0 R^{+} \gamma^0 = R^{-1}$, we can easily show that all quantities bilinear in ω have the same physical natures and transformation properties as in the usual formalism. We shall demonstrate this with the example of the current $\bar{\omega} N^{\mu} \omega$:

$$\bar{\omega} N^{\mu} \omega = \bar{\psi} \gamma^0 R^{+} \gamma^0 N^{\mu} R\psi = \bar{\psi} R^{-1} N^{\mu} R\psi = \bar{\psi} \gamma^{\mu} \varphi.$$

It can also be shown simply in the case of the other bilinear physical quantities, including the spin tensor

$$\tfrac{1}{4}\,\bar{\omega}\,(N^{\mu} N^{\nu} - N^{\nu} N^{\mu})\,\omega = \tfrac{1}{4}\,\bar{\psi}\,(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})\,\psi. \tag{18}$$

The formal proof of Eq. (18), however, is not enough to establish the fact that the quantity $\frac{1}{4}(N^\mu N^\nu - N^\nu N^\mu)$ actually is the spin operator for the particle described by (7). This question requires some additional investigation.

4. THE SPIN OPERATOR OF THE "SCALAR" EQUATION

In the framework of the usual theory, in the case of the Dirac equation the infinitesimal Lorentz transformation

$$x'^\mu = a^\mu{}_\nu x^\nu \qquad (x' = ax), \tag{19}$$

$$a^\mu{}_\nu = \delta^\mu_\nu + \varepsilon^\mu{}_\nu = \delta^\mu_\nu + \varepsilon^{\mu\lambda} g_{\lambda\nu} \qquad (\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu}) \tag{20}$$

generates the well known transformation of the wave function

$$\psi_\alpha(x) \to \psi'_\alpha(x') = S'_{\alpha\beta}\psi_\beta(a^{-1}x') = U_{x'x} S'_{\alpha\beta}\psi_\beta(x). \tag{21}$$

If a^μ_ν is of the form (20), then

$$S'_{\alpha\beta} = \delta_{\alpha\beta} + \tfrac{1}{2}\varepsilon_{\mu\nu}\Sigma^{\mu\nu}_{\alpha\beta}, \quad \Sigma^{\mu\nu} = \tfrac{1}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu), \tag{22}$$

and the operator $U_{x'x}$ (an integration over d^4x is understood with respect to the repeated "index" x) can be written in the form

$$U_{x'x} = \delta(x'-x) + \tfrac{1}{2}\varepsilon^{\mu\nu}M_{\mu\nu}\delta(x'-x),$$
$$M_{\mu\nu} = x_\nu\partial/\partial x^\mu - x_\mu\partial/\partial x^\nu. \tag{23}$$

The infinitesimal operators $\Sigma^{\mu\nu}$ and $M_{\mu\nu}$ are closely associated with the spin and orbital angular momentum of the Dirac particle.

It may seem at first glance that the spin of the particle described by Eq. (7) is zero [since in Eq. (8) $S_{\alpha\beta} = \delta_{\alpha\beta}$]. In our case, however, we cannot use the usual formalism expressed in Eqs. (19) – (22). Therefore our problem is to find a mathematical apparatus suitable for the description of the changes of the wave function in (8) which are generated by Lorentz transformations.

On being transformed to a different coordinate system the equation

$$\left(-iN^\mu\frac{\partial}{\partial x^\mu} + m\right)\omega(n^\sigma{}_\alpha, x^\nu) = 0 \qquad (\alpha = 0, 1, 2, 3), \tag{24}$$

takes the form

$$\left(-iN'^\mu\frac{\partial}{\partial x'^\mu} + m\right)\omega'(n'^\sigma{}_\alpha, x'^\nu) = 0. \tag{25}$$

It is not hard to find the operator that accomplishes the change from $\omega(n^\sigma{}_\alpha, x^\nu)$ to $\omega'(n'^\sigma{}_\alpha, x'^\nu)$. It can be written in the form

$$\omega'(n'^\sigma{}_\alpha, x'^\nu) = SU_{x'x}T_{n'_\alpha, n_\alpha}\omega(n^\sigma{}_\alpha, x^\nu), \tag{26}$$

where $S_{\alpha\beta} = \delta_{\alpha\beta}$ [by Eq. (10)], the operator $U_{x'x}$ is defined by Eq. (23), and the operator

$$T_{n'_\alpha, n_\alpha} \equiv T_{n'_0 n'_1 n'_2 n'_3, n_0 n_1 n_2 n_3} = \prod_{\alpha=0}^{3}\delta'(n'_\alpha - n_\alpha)$$
$$+ \tfrac{1}{2}\varepsilon^{\mu\nu}\left(n_{\nu\beta}\frac{\partial}{\partial n^\mu{}_\beta} - n_{\mu\beta}\frac{\partial}{\partial n^\nu{}_\beta}\right)\prod_{\alpha=0}^{3}\delta(n'_\alpha - n_\alpha) \tag{27}$$

is written here correct to infinitesimals of the second order in ϵ, just as $U_{x'x}$ is in Eq. (23).

It is easy to verify that by means of the operator $ST_{n'_\alpha, n_\alpha}U_{x'x}$ and its reciprocal Eq. (25) can be written in the form (24):

$$S^{-1}T^{-1}_{n'_\alpha, n_\alpha}U^{-1}_{xx'}\left(-iN'^\mu\frac{\partial}{\partial x'^\mu} + m\right)ST_{n'_\alpha, n_\alpha}U_{x'x}\omega(n^\sigma{}_\alpha, x^\nu)$$
$$= \left(-iN^\lambda\frac{\partial}{\partial x^\lambda} + m\right)\omega(n^\sigma{}_\alpha, x^\nu) = 0,$$

As usual, the infinitesimal operator

$$\eta_{\mu\nu} = n_{\nu\beta}\frac{\partial}{\partial n^\mu{}_\beta} - n_{\mu\beta}\frac{\partial}{\partial n^\nu{}_\beta} \tag{28}$$

must be associated with the spin of the particle described by (7). We note that a peculiarity of this representation is that $\eta_{\mu\nu}$ is written in the form of a differential operator, with a structure somewhat reminiscent of that of the four-dimensional generalization of the orbital angular momentum operator. We shall show that (7) nevertheless describes a Dirac particle with the spin ½. To do so we go over to the usual representation of the Dirac equation

$$\bar\omega\eta_{\mu\nu}\omega = \bar\psi R^{-1}\left(n_{\nu\beta}\frac{\partial R}{\partial n^\mu{}_\beta} - n_{\mu\beta}\frac{\partial R}{\partial n^\nu{}_\beta}\right)\psi \equiv \bar\psi R^{-1}R_{\mu\nu}\psi. \tag{29}$$

To find the result of the action of $\eta_{\mu\nu}$ on R and avoid the long and complicated direct calculations, we resort to the following device. On differentiating the equation $R\gamma^\sigma = n^\sigma{}_\alpha\gamma^\alpha R$ with respect to $n^\mu{}_\beta$ and multiplying the result by $n_{\nu\beta}g_{\sigma\lambda}\gamma^\lambda$ from the right, we get

$$4n_{\nu\beta}\frac{\partial R}{\partial n^\mu{}_\beta} = N_\nu R\gamma_\mu + N^\lambda n_{\nu\beta}\frac{\partial R}{\partial n^\mu{}_\beta}\gamma_\lambda.$$

When we now interchange the indices μ and ν and subtract the resulting expression from the one just written, we get the equation for the required quantity

$$R_{\mu\nu} = \tfrac{1}{4}(N_\nu R\gamma_\mu - N_\mu R\gamma_\nu) + \tfrac{1}{4}N^\lambda R_{\mu\nu}\gamma_\lambda. \tag{30}$$

The solution of (30) can be written in the form

$$R_{\mu\nu} \equiv n_{\nu\beta}\partial R/\partial n^\mu{}_\beta - n_{\mu\beta}\partial R/\partial n^\nu{}_\beta = \tfrac{1}{4}R(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu)$$
$$= \tfrac{1}{4}(N_\nu N_\mu - N_\mu N_\nu)R, \tag{31}$$

which can be verified by direct substitution in (30).

Thus, substituting (31) in (29), we find

$$\bar\omega\eta_{\mu\nu}\omega = \tfrac{1}{4}\bar\psi(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu)\psi, \tag{32}$$

i.e., (7) indeed still describes a Dirac particle with spin $1/2$, as also follows from (18).

5. QUANTIZATION AND PHYSICAL INTERPRETATION OF THE WAVE FUNCTION OF THE "SCALAR" EQUATION

By simple arguments it is easy to find the method of quantization of the "scalar" equation, corresponding to the usual quantization:

$$[\bar{\omega}_\rho(x'), \omega_\sigma(x)]_+ = -iS_{\sigma\rho}[N^\mu; x - x']. \quad (33)$$

Here $S_{\sigma\rho}[N^\mu; x-x']$ corresponds to the usual commutator function $S_{\sigma\rho}(x-x')$, with the matrices γ^μ replaced by N^μ. It is obvious that $S_{\sigma\rho}[N^\mu; x-x']$ satisfies the free-particle "scalar" Dirac equation. This method of quantization, and also the replacement of the matrices γ^μ by N^μ in all of the singular functions, establishes the correspondence with the usual formalism and allows us to construct the S-matrix formalism in the standard way.

Let us say a few words about the physical interpretation of the wave function of the "scalar" Dirac equation. Its meaning becomes particularly clear if we transform the "scalar" equation to the coordinate system in which $n'^\mu_\alpha = \delta^\mu_\alpha$, and consequently the axes of the new coordinate system coincide with the four unit vectors of the frame formed by the vectors n_α ($\alpha = 0, 1, 2, 3$). As can be seen from Eqs. (7), (3), and (13), in the new coordinate system the "scalar" equation formally coincides with the usual Dirac equation, and its solutions coincide with the solutions of that equation. But the numerical values of the wave function of the scalar equation remain the same in any coordinate system [cf. Eq. (13)], and therefore they are always numerically equal to the wave function of the usual Dirac equation in the coordinate system in which $n'^\mu_\alpha = \delta^\mu_\alpha$.

These features of the "scalar" Dirac equation reduce to the fact that the requirement of invariance of its form in any transformed coordinate system is satisfies by changes of the matrices N^μ, without affecting the numerical values of the components of the ω function.

6. THE SIMPLEST SYSTEM OF SOLUTIONS

The considerations given above allow us to find the system of coordinates in which the matrix of the solutions, Eq. (13), takes the simplest possible form—namely, is just a four-rowed unit matrix. For brevity we shall call this coordinate system the system K_0. The required transformation R that accomplishes the change to the system K_0 naturally does not affect the exponential factor in the free-particle solution, and changes only the form of the matrix part.

The transition to K_0 corresponds, in the sense of our previous remarks, to the transition to the coordinate system associated with the Dirac particle at rest. Having then obtained the solutions of the simplest form

$$\varphi_0^{(1)} = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}, \quad \varphi_0^{(2)} = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}, \quad \varphi_0^{(3)} = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \end{vmatrix}, \quad \varphi_0^{(4)} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}, \quad (34)$$

we can use them to find the matrix elements of reactions involving polarized Dirac particles by a new method, without using the usual technique of projection operators and the calculation of traces.

In fact, the operator M that stands between the wave functions of the initial and final states, and can be computed, for example, by perturbation theory, is in general a certain four-rowed matrix, independently of how many vertices there are in the process in question. If, however, the initial and final states—solutions of the "scalar" equation—are written in the system K_0 and have the form (34), then the problem of finding the transition probabilities between states of definite polarization reduces in the first stage simply to picking out the corresponding matrix element of M, since $\varphi_0^{(k)} + M\varphi_0^{(l)} = M_{kl}$ ($k, l = 1, 2, 3, 4$). Therefore the solution of the problem breaks up into two steps: a) finding the matrix R that accomplishes the change from the usual Dirac equation to the "scalar" equation with the simplest system of solutions, and b) finding a method for obtaining the matrix element M_{kl}.

7. THE TRANSITION TO THE SYSTEM K_0

The system of four solutions of (5) can be written in the form of a single four-rowed matrix

$$\psi_0 = [2m(\varepsilon + m)]^{-1/2}\{(\varepsilon + m) I - \gamma^0\gamma^k p_k\} \quad (k = 1, 2, 3). \quad (35)$$

Here I is the four-rowed unit matrix, $\epsilon = |p_0|$, and

$$\gamma^0 = \begin{vmatrix} I' & 0 \\ 0 & -I' \end{vmatrix}, \quad \gamma^r = \begin{vmatrix} 0 & \sigma_r \\ -\sigma_r & 0 \end{vmatrix} \quad (r = 1, 2, 3) \quad (36)$$

where I′ is the two-rowed unit matrix and σ_r are the Pauli matrices.

The transition to the "scalar" equation, written in K_0, is accomplished by means of the relations

$$\varphi_0 = R\psi_0 = I, \tag{37}$$

$$R\gamma^\mu R^{-1} = n^\mu{}_\alpha \gamma^\alpha = N^\mu. \tag{38}$$

Hereafter we shall denote by L(p) the matrix R defined by (37), and shall denote by $l^\mu{}_\alpha$ the vectors $n^\mu{}_\alpha$ defined by (37) and (38). From the form of ψ_0, Eq. (35), it is easy to find the operator L(p):

$$R \equiv L(p) = [2m(\varepsilon + m)]^{-1/2}\{(\varepsilon + m)\, I + \gamma^0\gamma^k p_k\}. \tag{39}$$

In fact, it is easy to verify that (37) is satisfied, and, in addition,

$$L^{-1}(p) = \psi_0 = [2m(\varepsilon + m)]^{-1/2}\{(\varepsilon + m)\, I - \gamma^0\gamma^k p_k\}, \tag{40}$$

$$\gamma^0 L^+(p)\,\gamma^0 = L^{-1}(p). \tag{41}$$

The components of the vectors $l^\mu{}_\alpha$ defined by the relations (38) and (39) are given by the table

	l_0	l_1	l_2	l_3
$l^0{}_\alpha$	$\frac{\varepsilon}{m}$	$\frac{p^1}{m}$	$\frac{p^2}{m}$	$\frac{p^3}{m}$
$l^1{}_\alpha$	$\frac{p^1}{m}$	$1+\frac{(p^1)^2}{m(\varepsilon+m)}$	$\frac{p^1p^2}{m(\varepsilon+m)}$	$\frac{p^1p^3}{m(\varepsilon+m)}$
$l^2{}_\alpha$	$\frac{p^2}{m}$	$\frac{p^2p^1}{m(\varepsilon+m)}$	$1+\frac{(p^2)^2}{m(\varepsilon+m)}$	$\frac{p^2p^3}{m(\varepsilon+m)}$
$l^3{}_\alpha$	$\frac{p^3}{m}$	$\frac{p^3p^1}{m(\varepsilon+m)}$	$\frac{p^3p^2}{m(\varepsilon+m)}$	$1+\frac{(p^3)^2}{m(\varepsilon+m)}$

(42)

The table (42) (in which we have changed to the contravariant components of the vector p) can be represented by the formula

$$l^\mu{}_\alpha = \delta_{\mu\alpha} + \frac{p^\mu p^\alpha}{m(\varepsilon + m)} + \frac{1}{\varepsilon + m}[p^\mu\delta_{\alpha 0} + p^\alpha\delta_{\mu 0}] - \frac{2\varepsilon + m}{\varepsilon + m}\delta_{\mu 0}\delta_{\alpha 0}, \tag{43}$$

if we set $p^0 = \epsilon$. The quantities $l^\mu{}_\alpha$ satisfy Eqs. (1) and (2).

The matrices N^0 and N^s (s = 1, 2, 3) can now be represented in the form

$$N^0 = \frac{\varepsilon}{m}\gamma^0 + \frac{p^n}{m}\gamma^n \qquad (n = 1, 2, 3), \tag{44}$$

$$N^s = \frac{p^s}{m}\gamma^0 + \left[\delta_{sn} + \frac{p^s p^n}{m(\varepsilon + m)}\right]\gamma^n \qquad (s, n = 1,2,3). \tag{45}$$

The transformation L(p) diagonalizes the equations (11), converting them into

$$(m\gamma^0 \mp m)\,\varphi_0 = 0, \tag{46}$$

owing to the relation

$$N^\mu p_\mu = m\gamma^0. \tag{47}$$

8. THE TRANSFORMATION OF THE MATRIX ELEMENT

Let us consider the general form of the matrix element, as calculated, for example, by perturbation theory. For definiteness suppose only one Dirac particle is involved in the process. Then in the framework of the usual theory the general form of the matrix element in the momentum representation, for states with prescribed polarizations k and l, is

$$M_{kl} = \bar\psi_0^{(k)}(p_2)\,\hat A \ldots \hat E\hat F\hat G\psi_0^{(l)}(p_1). \tag{48}$$

Here p_1 and p_2 are the initial and final momenta of the particle, $\bar\psi_0^{(k)} = \psi_0^{+(k)}\gamma^0$ and $\psi_0^{(l)}$ are particular solutions in Eq. (35), and the notations $\hat A$ and so on have their usual meanings: $\hat A = A_\mu\gamma^\mu$.

On the basis of Eqs. (37) and (41) the formula (48) can be rewritten in the form

$$M_{kl} = \bar\varphi_0^{(k)}L(p_2)\,\hat A \ldots \hat E\hat F\hat G L^{-1}(p_1)\,\varphi_0^{(l)}, \tag{49}$$

where $\varphi_0^{(l)}$ (l = 1, 2, 3, 4) are as shown in Eq. (34). If one of the matrix vectors, for example $\hat F$, is equal to $\hat p_1$, then Eq. (49) can be simplified, in view of Eq. (47):

$$M_{kl} = \bar\varphi_0^{(k)}L(p_2)\,\hat A \ldots \hat E L^{-1}(p_1)\,m\gamma^0\hat G_N\varphi_0^{(l)},$$

where the notation $\hat G_N$ has the meaning that one must replace the matrices γ^μ in $\hat G$ by the matrices N^μ, i.e.,

$$\hat G_N = G_\mu N^\mu = G_\alpha^{(N)}\gamma^\alpha, \quad G_\alpha^{(N)} = G_\mu l^\mu{}_\alpha. \tag{50}$$

Remembering the obvious relations

$$\bar\varphi_0^{(k)}L(p_2) = \varphi_0^{+(k)}\hat V, \qquad L^{-1}(p_1)\,\gamma^0\hat G_N\varphi_0^{(l)} = \hat T\hat G_N\varphi_0^{(l)}. \tag{51}$$

$$V_2 = \frac{\varepsilon_2 + m}{\sqrt{2m(\varepsilon_2 + m)}}, \quad V_r = -\frac{p_2^r}{\sqrt{2m(\varepsilon_2 + m)}},$$

$$\varepsilon_2 = |p_2^0| \quad (r = 1, 2, 3),$$

$$T_0 = \frac{\varepsilon_1 + m}{\sqrt{2m(\varepsilon_1 + m)}}, \quad T_r = -\frac{p_1^r}{\sqrt{2m(\varepsilon_1 + m)}}, \quad \varepsilon_1 = |p_1^0| \tag{52}$$

(in Eq. (52) we have for convenience gone over to the contravariant components of the vectors p_1 and p_2), we can finally rewrite M_{kl} in the form

$$M_{kl} = m\varphi_0{}^{+(k)}\hat V\hat A \ldots \hat E\hat T\hat G_N\varphi_0{}^{(l)}. \tag{53}$$

In case of need the factor $L(p_2)$ can be taken to the right across one or more of the matrix vectors, in an analogous way. If the original particle was at rest, then $L^{-1}(p_1) = I$ and Eq. (49) for M_{kl} takes the form

$$M_{kl} = \varphi_0{}^{+(k)}\hat V\hat A \ldots \hat E\hat F\hat G\varphi_0{}^{(l)}. \tag{54}$$

Thus in all cases the operators L and L^{-1} can by simple manipulations be brought into the expression for M_{kl} on an equal footing with the matrix vectors $\hat{A}, \ldots, \hat{E}$, etc.

9. THE STRUCTURE OF THE γ MATRICES AND THE CALCULATION OF MATRIX ELEMENTS

In order to proceed further, it is necessary to find a method for calculating the matrix elements of the matrix $\hat{V}\hat{A}\hat{B}\hat{C}\ldots$. This method can be the ordinary law of matrix multiplication, provided it is possible to represent the γ matrices by means of Kronecker δ symbols.

First let us introduce some step functions of discrete variables*

$$\varepsilon_k = \begin{cases} +1, & k = 1, 2 \\ -1, & k = 3, 4, \end{cases}$$
$$\varkappa_k(1, 2) = \begin{cases} +1, & k = 1, 2 \\ 0, & k = 3, 4, \end{cases} \quad (55)$$
$$\varkappa_k(3, 4) = \begin{cases} 0, & k = 1, 2 \\ +1, & k = 3, 4. \end{cases}$$

By means of the step functions (55) and the Kronecker δ symbols one can write expressions for the matrix elements of the γ matrices in Eq. (36). In fact, it can easily be verified directly that

$$\gamma^\mu_{kl} = \varepsilon_k \{\delta_{\mu 0}\delta(k-l) + [\delta_{\mu 1} + (-1)^k \delta_{\mu 2}]\, \delta(k+l-5) - \delta_{\mu 3}(-1)^k [\varkappa_k(1,2)\, \delta(k-l+2) + \varkappa_k(3,4)\, \delta(k-l-2)]\}, \quad (56)$$

where we have used the notation $\delta(m-n) = \delta_{m,n}$. With this way of writing them, the matrix elements of the matrix vector $(\hat{A})_{kl} = (A_\mu \gamma^\mu)_{kl}$ will have the following appearance:

$$A_\mu \gamma^\mu{}_{kl} = \varepsilon_k \{A_0 \delta(k-l) + [A_1 + (-1)^k i A_2]\, \delta(k+l-5) - A_3(-1)^k [\varkappa_k(1,2)\, \delta(k-l+2) + \varkappa_k(3,4)\, \delta(k-l-2)]\}. \quad (57)$$

*We stipulate that hereafter Greek letters shall as before take the values 0, 1, 2, 3, and Latin letters shall take the values 1, 2, 3, 4.

Then, using the law of matrix multiplication, we get, for example:

$$(\hat{A}\hat{B})_{kl} = \sum_{m=1}^{4} A_\mu \gamma^\mu{}_{km} \cdot B_\nu \gamma^\nu{}_{ml}$$
$$= \{A_\mu B^\mu + (-1)^k i (A_1 B_2 - A_2 B_1)\}\, \delta(k-l)$$
$$+ \{A_0 B_1 - A_1 B_0 + (-1)^k i (A_0 B_2 - A_2 B_0)\}$$
$$\times \delta(k+l-5) - (-1)^k (A_0 B_3 - A_3 B_0)\, [\delta(k-l+2)$$
$$+ \delta(k-l-2)] - \{(-1)^k (A_1 B_3 - A_3 B_1)$$
$$+ i (A_2 B_3 - A_3 B_2)\}\, [\delta(k+l-3) + \delta(k+l-7)]. \quad (58)$$

We note that it is enough to indicate the range of variation of the indices k and l, and the step functions $\kappa_k(1, 2)$ and $\kappa_k(3, 4)$ can then be omitted in the final formulas. In intermediate steps they are to be kept, however, to avoid possible mistakes.

The set of six δ symbols appearing in Eq. (58) is enough to exhaust all 16 matrix elements of a four-rowed matrix. Therefore no new δ symbols appear subsequently.

The calculation of matrix elements from the product of a large number of matrix vectors can be continued further. The method of calculation is obvious.

The existence of tables of formulas like Eqs. (57) and (58) decidedly facilitates and quickens the calculation of matrix elements for reactions involving polarized Dirac particles.

In conclusion I regard it as my pleasant duty to express my gratitude to Professor M. A. Markov for his interest in this work and for discussions.

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