ON THE THEORY OF ELECTROMAGNETIC FLUCTUATIONS IN A NONEQUILIBRIUM PLASMA

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The elements of the tensor $\varphi_{\alpha\beta}(\omega)$, which describes the spectral intensity of the fluctuations in electrical current in a nonrelativistic magnetoactive plasma in a strong constant or rapidly varying electric field, are calculated. Expressions are also derived for the elements of the effective-temperature tensor $T_{\text{eff}}^{\alpha\beta}$, which is introduced as a formal extension of the well-known fluctuation-dissipation theorem (Nyquist formula) to the case of a nonequilibrium plasma. Several particular cases are considered.

1. The theory of fluctuation spectra in systems in thermodynamic equilibrium derives chiefly from the so-called fluctuation-dissipation theorem, which establishes the connection between the spectral intensity of the fluctuations of an arbitrary physical quantity and the corresponding (to this quantity) conductivity (which determines the dissipation of energy in the system) and the absolute temperature T.[1-3] In particular, the tensor $\Phi_{\alpha\beta}(\mathbf{r}, \mathbf{r}', \omega)$, which describes the spectral intensity of the fluctuations in electric current density $\mathbf{j}(\mathbf{r}, t)$ in a uniform anisotropic absorbing medium, is given by (only classical fluctuations are taken into account, i.e., $\hbar\omega \ll kT$, and spatial dispersion is neglected)

$$\Phi_{\alpha\beta}(\mathbf{r}, \mathbf{r}', \omega) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle j_\alpha(\mathbf{r}, t)\, j_\beta(\mathbf{r}', t+\tau)\rangle e^{i\omega\tau} d\tau = \frac{kT}{\pi}\sigma_{\alpha\beta}(\omega)\,\delta(\mathbf{r}-\mathbf{r}'), \tag{1}$$

where $\sigma_{\alpha\beta}(\omega)$ is the conductivity tensor.* The theorem in (1) makes it possible to formulate and solve the problem of thermal radiation of hot bodies (media) within the framework of macroscopic electrodynamics;[2,4,5] in this case the fluctuation current $\mathbf{j}(\mathbf{r}, t)$ is regarded as a transverse current.

If we now consider media that are not in thermodynamic equilibrium (although they may be stationary in the sense that the conditions do not change in time), obviously there is no universal relation between the tensor $\Phi_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ such as that given by (1) for equilibrium systems. A relation can be obtained between these tensors only by the formal introduction of an additional, (generally, frequency-dependent) Hermitian tensor $T_{\text{eff}}^{\alpha\beta}(\omega)$ (which we call the effective-temperature tensor)*

$$\Phi_{\alpha\beta}(\mathbf{r}, \mathbf{r}', \omega) = (k/\pi)\, T_{\text{eff}}^{\alpha\gamma}(\omega)\,\sigma_{\gamma\beta}(\omega)\,\delta(\mathbf{r}-\mathbf{r}'), \tag{2}$$

where $\sigma_{\alpha\beta}(\omega)$ is the conductivity tensor corresponding to the given nonequilibrium state of the medium.

It is clear that this formal generalization of the fluctuation-dissipation theorem does not yield any advantage as far as determining the tensor $\Phi_{\alpha\beta}$ is concerned; this follows because in general there will be no method of computing the elements of the tensor $T_{\text{eff}}^{\alpha\beta}$ other than independent calculations of the elements of the tensors $\Phi_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ and the expansion of the matrix equation (2) with respect to $T_{\text{eff}}^{\alpha\beta}$. Nevertheless, in certain cases it is convenient to use (2) (cf. below).

One of these cases arises when the tensor $T_{\text{eff}}^{\alpha\beta}(\omega)$ reduces to a frequency-independent scalar T_{eff} (more precisely, when $T_{\text{eff}}^{\alpha\beta} = T_{\text{eff}}^{\alpha\beta}\delta_{\alpha\beta}$). In this case, as in the case of media in thermodynamic equilibrium, $\Phi_{\alpha\beta}$ is determined completely by $\sigma_{\alpha\beta}$ and, as far as electromagnetic fluctuations are concerned, the nonequilibrium medium behaves

*A complex dielectric tensor $\varepsilon'_{\alpha\beta}$ (to take account of absorption) of form $\varepsilon'_{\alpha\beta} = \varepsilon_{\alpha\beta} - i4\pi\sigma_{\alpha\beta}/\omega$ can be defined uniquely, as is well known, if we require that $\varepsilon_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ must be Hermitian tensors. We also note that the tensor $\Phi_{\alpha\beta}(\mathbf{r}, \mathbf{r}', \omega)$ is related to the tensor $\langle j_{\alpha\omega}(\mathbf{r})\, j^*_{\beta\omega'}(\mathbf{r}')\rangle$, where $\mathbf{j}_\omega(\mathbf{r})$ is the Fourier component of the current $\mathbf{j}(\mathbf{r}, t)$, by the relation

$$\langle j_{\alpha\omega}(\mathbf{r})\, j^*_{\beta\omega'}(\mathbf{r})\rangle = \Phi_{\alpha\beta}(\mathbf{r}, \mathbf{r}', \omega)\,\delta(\omega-\omega').$$

*The general approach to the notion of an effective temperature for arbitrary nonequilibrium systems has been given earlier by the author.[6]

like an equilibrium medium at temperature T_{eff}. This case is the situation, for example, in a highly (in particular, in a fully) ionized plasma (cf. below).

In the present paper kinetic theory is used to calculate the elements of the tensor $\Phi_{\alpha\beta}(\mathbf{r}, \mathbf{r}', \omega)$ for a nonrelativistic plasma in a strong, uniform, constant or rapidly varying electric field $\mathbf{E}$ and a fixed uniform magnetic field $\mathbf{H}_0$. By rapidly varying here we mean a field $\mathbf{E}$ characterized by a frequency that satisfies the relation $\Omega \gg \tau_\epsilon^{-1} \sim \delta_{eff}\nu_{eff}$, where τ_ϵ is the relaxation time for the electron energy $\epsilon = mv^2/2$. The quantity δ_{eff} is the mean relative fraction of the energy lost by an electron in a single collision with a heavy particle ($\delta_{eff} \ll 1$); ν_{eff} is the effective frequency of collisions between the electron and heavy particles (more precise definitions of these terms are given, for example, in [7]). If the above condition is satisfied, just as in the case in which $\mathbf{E}$ is constant,* the plasma reaches a stationary state, that is to say, the symmetric part $f_0(v)$ of the electron distribution function

$$f(\mathbf{v}) = f_0(v) + \mathbf{v}\mathbf{f}_1(v)/v$$

reaches some average time independent level (the variable part of the function $f_0(v)$ is of order $(\Omega\tau_\epsilon)^{-1}$ and can be neglected). The function $f_0(v)$ will differ appreciably from a Maxwellian distribution $f_{00}(v)$ corresponding to the temperature of the heavy particles T (i.e., thermal equilibrium no longer holds) when the field E becomes sufficiently high:[8]

$$E \gtrsim E_p = [3kTm\delta_{\text{eff}}(\Omega^2 + \nu_{\text{eff}}^{0\,2})/e^2]^{1/2},$$

where ν_{eff}^0 is the effective collision frequency in the absence of the field E. In further discussion of the nonequilibrium plasma we will keep in mind the fact that the plasma is in a nonequilibrium state due precisely to the presence of such strong fields E.

2. We start by calculating the tensor for the current fluctuations. If spatial dispersion is neglected, in which case the spatial correlation is local (δ-correlation),[9,10] this tensor can be written in the form

$$\langle j_\alpha(\mathbf{r}, t)\, j_\beta(\mathbf{r} + \rho, t + \tau)\rangle = Ne^2\psi_{\alpha\beta}(\tau)\,\delta(\rho), \qquad (3)$$

where N is the electron density, e is the charge of the electron and $\psi_{\alpha\beta}(\tau) = \langle v_\alpha(t) v_\beta(t+\tau)\rangle$ is the correlation tensor for velocity fluctuations of a given electron in the plasma: $\langle v_\alpha\rangle = 0$.

If the fluctuations $v_\alpha(t)$ are to be stationary, we require $\psi_{\alpha\beta}(\tau) = \psi_{\beta\alpha}(-\tau)$. Using this property of $\psi_{\alpha\beta}(\tau)$, we obtain from Eqs. (1) and (3) the following expression for the "frequency" part $\varphi_{\alpha\beta}(\omega)$ of the tensor $\Phi_{\alpha\beta}(\mathbf{r}, \mathbf{r}', \omega) = \varphi_{\alpha\beta}(\omega) \times \delta(\mathbf{r} - \mathbf{r}')$:

$$\varphi_{\alpha\beta}(\omega) = \frac{Ne^2}{2\pi}\left\{\int_0^\infty [\psi_{\alpha\beta}(\tau) + \psi_{\beta\alpha}(\tau)]\cos\omega\tau\, d\tau \right.$$
$$\left. + i\int_0^\infty [\psi_{\alpha\beta}(\tau) - \psi_{\beta\alpha}(\tau)]\sin\omega\tau\, d\tau\right\}. \qquad (4)$$

This expression allows us to compute the tensor $\varphi_{\alpha\beta}(\omega)$ using the velocity tensor $\psi_{\alpha\beta}(\tau)$, which is given only for $\tau > 0$. However, for $\tau > 0$ the elements of $\psi_{\alpha\beta}$ can be determined from the following simple considerations. Let the z axis be along the magnetic field $\mathbf{H}_0$. In the interval between two collisions an electron behaves as though free and moves in a helical path along the magnetic field $\mathbf{H}_0$ with a rotation frequency $\omega_H = |e|H_0/mc$. Hence, for given values of the velocity components v_x, v_y and v_z the nonvanishing second moments at $t = 0$ are

$$\langle v_z(0)\, v_z(\tau)\rangle = v_z^2, \qquad \langle v_x(0)\, v_x(\tau)\rangle = v_x^2\cos\omega_H\tau,$$
$$\langle v_y(0)\, v_y(\tau)\rangle = v_y^2\cos\omega_H\tau,$$
$$\langle v_x(0)\, v_y(\tau)\rangle = -\langle v_y(0)\, v_x(\tau)\rangle = v_x^2\sin\omega_H\tau,$$

if $\tau < \tau_0(v)$. where $\tau_0(v)$ is the time required to travel one mean free path, and zero when $\tau > \tau_0(v)$.

Introducing the step function $p(x)$, which is equal to unity for $x > 0$ and zero for $x < 0$, we can write

$$\psi_{zz}(\tau) = \langle v_z^2 p(s/v - \tau)\rangle,$$
$$\psi_{xx}(\tau) = \langle v_x^2\cos\omega_H\tau\cdot p(s/v - \tau)\rangle,$$
$$\psi_{yy}(\tau) = \langle v_y^2\cos\omega_H\tau\cdot p(s/v - \tau)\rangle,$$
$$\psi_{xy}(\tau) = \psi_{yx}(-\tau) = \langle v_x^2\sin\omega_H\tau\cdot p(s/v - \tau)\rangle, \qquad (5)$$

where the averages are first taken over s, the length of a free path for a given velocity v, by means of the distribution function

$$w(s, v) = \exp[-s/l(v)], \qquad l(v) = v/\nu(v)$$

[$\nu(v)$ is the electron collision frequency], and then with respect to velocity, by means of the electron distribution function $f_0(v)$.*

Substituting (5) in (4) and first carrying out the

*In a highly ionized plasma in a very strong constant field $E > E_c \sim \sqrt{kT_e m}\,\nu_{eff}(T_e)/e$ (T_e is the electron temperature) the phenomenon of electron "runaway" occurs,[7,8] making it impossible to establish a stationary electron distribution $f(\mathbf{v})$. We assume below that the constant field satisfies the condition $E \ll E_c$.

*The function $f(v)$ is normalized to unity (not to the density N).

integration over τ and then over s, we obtain the following expressions for the nonvanishing elements $\varphi_{\alpha\beta}(\omega)$:

$$\varphi_{zz}(\omega) = \langle \chi_0(v, \omega)\rangle,$$

$$\varphi_{xx}(\omega) = \varphi_{yy}(\omega) = \tfrac{1}{2}\{\langle\chi_{(-)}(v, \omega)\rangle + \langle\chi_{(+)}(v, \omega)\rangle\},$$

$$\varphi_{xy}(\omega) = \varphi^*_{yx}(\omega) = \tfrac{1}{2}\{\langle\chi_{(-)}(v, \omega)\rangle - \langle\chi_{(+)}(v, \omega)\rangle\}, \qquad (6)$$

where the functions χ_0, $\chi_{(-)}$, and $\chi_{(+)}$ are given by

$$\chi_0 = \frac{Ne^2}{3\pi}\frac{\nu v^2}{\omega^2+\nu^2} = \frac{2Ne^2}{3\pi}\frac{\nu\varepsilon}{\omega^2+\nu^2},$$

$$\chi_{(\mp)} = \frac{Ne^2}{3\pi}\frac{\nu v^2}{(\omega\mp\omega_H)^2+\nu^2} = \frac{2Ne^2}{3\pi}\frac{\nu\varepsilon}{(\omega\mp\omega_H)+\nu^2}, \qquad (7)$$

and the brackets $\langle\ \rangle$ denoted averages over the distribution functions $f_0(v)$.

To proceed further in computing the elements of $\varphi_{\alpha\beta}(\omega)$ we must introduce the concrete properties of the plasma and its environment. More precisely, we must assign the type of collisions i.e., the function $\nu(v)$, the nature of the field E (whether $\Omega = 0$ or $\Omega \gg \tau_\epsilon^{-1}$), and its orientation with respect to the field $\mathbf{H}_0$. Introducing these factors uniquely determines the form of the stationary kinetic equation satisfied by the function $f_0(v)$ (cf. [7]) and thus makes it possible, by means of (6), to obtain expressions for $\varphi_{\alpha\beta}(\omega)$ in terms of such parameters as T —the temperature of the heavy plasma particles, N, E, H_0, Ω, and θ, the angle between $\mathbf{E}$ and $\mathbf{H}_0$.

Certain particular examples will be considered below. First we compute the components of the tensor $T^{\alpha\beta}_{\text{eff}}(\omega)$, which establishes the general relation between the $\varphi_{\alpha\beta}(\omega)$ and $\sigma_{\alpha\beta}(\omega)$.

3. From (2) and the definition of the tensor $\varphi_{\alpha\beta}(\omega)$ we have

$$kT^{\alpha\beta}_{\text{eff}}(\omega) = \pi\varphi_{\alpha\gamma}(\omega)\,\sigma^{-1}_{\gamma\beta}(\omega), \qquad (8)$$

where $\sigma^{-1}_{\alpha\beta}(\omega)$ is the inverse of the matrix $\sigma_{\alpha\beta}(\omega)$, which characterizes the linear conductivity of the plasma in a given equilibrium state with respect to a small (with respect to the field $\mathbf{E}$) harmonic field $\mathbf{e} = \mathbf{e}_0 e^{i\omega t}$. Expressions for the elements of $\sigma_{\alpha\beta}(\omega)$ in terms of the distribution function $f_0(v)$ are known from the kinetic theory of electrical conductivity (cf. [7]) and can be written as follows ($L \equiv d\ln f_0/d\epsilon$):

$$\sigma_{xz} = \sigma_{zx} = \sigma_{yz} = \sigma_{zy} = 0,\quad \sigma_{zz}(\omega) = -\pi\langle\chi_0(v, \omega)L\rangle,$$

$$\sigma_{xx}(\omega) = \sigma_{yy}(\omega) = -\frac{\pi}{2}\{\langle\chi_{(-)}(v, \omega)L\rangle + \langle\chi_{(+)}(v, \omega)L\rangle\},$$

$$\sigma_{xy}(\omega) = \sigma^*_{yx}(\omega) = -\frac{\pi i}{2}\{\langle\chi_{(-)}(v, \omega)L\rangle - \langle\chi_{(+)}(v, \omega)L\rangle\}. \qquad (9)$$

Calculating the inverse matrix $\sigma^{-1}_{\alpha\beta}$ and using (6) we obtain the following expressions for the nonzero elements of the effective-temperature tensor:

$$kT^{zz}_{\text{eff}} = -\langle\chi_0(v, \omega)\rangle/\langle\chi_0(v, \omega)L\rangle, \qquad (10)$$

$$kT^{xx}_{\text{eff}} = kT^{yy}_{\text{eff}} = -\frac{1}{2}\left\{\frac{\langle\chi_{(-)}(v, \omega)\rangle}{\langle\chi_{(-)}(v, \omega)L\rangle} + \frac{\langle\chi_{(+)}(v, \omega)\rangle}{\langle\chi_{(+)}(v, \omega)L\rangle}\right\}, \qquad (10')$$

$$kT^{xy}_{\text{eff}} = -kT^{yx}_{\text{eff}} = -\frac{i}{2}\left\{\frac{\langle\chi_{(-)}(v, \omega)\rangle}{\langle\chi_{(-)}(v, \omega)L\rangle} - \frac{\langle\chi_{(+)}(v, \omega)\rangle}{\langle\chi_{(+)}(v, \omega)L\rangle}\right\}. \qquad (10'')$$

As expected, at thermodynamic equilibrium $f_0(v) = C\exp(-mv^2/2kT)$ and these formulas give $T^{\alpha\beta}_{\text{eff}} = T\delta_{\alpha\beta}$.

4. We consider other particular examples. Suppose that a plasma is highly (in particular, fully) ionized so that the electron-electron collision frequency $\nu_e \gg \delta\nu$ where $\nu(v)$ is the frequency of electron collisions with heavy particles. Then (cf. [7]), the function $f_0(v)$ is a Maxwellian distribution corresponding to the electron temperature T_e. In the general case the temperature T_e is a monotonically increasing function of E that also depends on the parameters T, N, H_0, Ω, and θ.* In this case Eqs. (10), (10′), and (10″) yield a unique frequency-independent value of the effective temperature:

$$T^{\alpha\beta}_{\text{eff}} = T_e\delta_{\alpha\beta}.$$

Thus, as in the case of an equilibrium medium, the spectral composition of the electromagnetic radiation from a highly ionized plasma is completely determined by the absorptive capacities (the tensor $\sigma_{\alpha\beta}(\omega)$,[7,12]) while the intensity of this radiation is determined by the electron temperature T_e.

We now consider an isotropic plasma ($H_0 = 0$), In this case

$$\sigma_{\alpha\beta}(\omega) = \sigma(\omega)\,\delta_{\alpha\beta},\qquad T^{\alpha\beta}_{\text{eff}}(\omega) = T_{\text{eff}}(\omega)\,\delta_{\alpha\beta},$$

where $\sigma(\omega)$ and $T_{\text{eff}}(\omega)$ are determined by (9) and (10) respectively. At low frequencies $\omega \ll \nu$ and at high frequencies $\omega \gg \nu$ Eq. (10) yields in accordance with Eq. (7),

$$kT_{\text{eff}}(0) = -\frac{\langle\varepsilon/\nu\rangle}{\langle\varepsilon L/\nu\rangle},\qquad kT_{\text{eff}}(\infty) = -\frac{\langle\varepsilon\nu\rangle}{\langle\varepsilon\nu L\rangle}. \qquad (11)$$

The first of these expressions has been given earlier.[6]

Suppose now that the field E is constant and that the plasma is weakly ionized $\nu_e \ll \delta\nu$. Then, if the only collisions of importance are elastic col-

*Expressions for T_e for actual cases are given in [7] and [11].

lisions with neutral particles—solid spheres of radius a,—the function $f_0(v)$ is the well-known Druyvesteyn function[13]

$$f_0(v) = C \exp\left[-3m^2\delta_{el}v^4/8e^2E^2l^2\right], \tag{12}$$

where $l = v/\nu(v) = (\pi a^2 N_m)^{-1}$, N_m is the density of molecules, $\delta_{el} = 2m/M$, and M is the mass of the molecule. Substitution of (12) in (10) gives the following general expressions for the effective temperature in this case:

$$kT_{eff} = \frac{eEl}{\sqrt{6\delta_{el}}} \int_0^\infty \frac{x^2 e^{-x^2}}{x+\alpha}\,dx \Big/ \int_0^\infty \frac{x^3 e^{-x^2}}{x+\alpha}\,dx,$$

$$\alpha = \sqrt{3\delta_{el}}\, m\omega^2 / [2^{3/2}(\pi a^2 N_m)^2\, eEl]. \tag{13}$$

The dependence of T_{eff} on frequency ω (which is always weak) appears only in the interval $\alpha \lesssim 1$. When $\alpha \gg 1$ we have

$$kT_{eff} = \frac{1}{2}\, eEl \sqrt{\frac{\pi}{6\delta_{el}}}\,. \tag{14}$$

At zero frequency the value of T_{eff} can be only $4/\pi$ times greater at most.

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