RADIATIVE CORRECTIONS TO BETA DECAY

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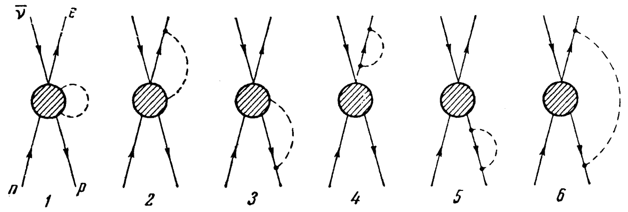
The problem of radiative corrections to β decay is discussed. A method is indicated for introducing different form-factors for the proton and electron (for the interaction with the photon) without coming into conflict with gauge invariance. An estimate is made of the contribution from diagrams that correspond to the emission of a virtual photon directly from the four-fermion vertex, as renormalized by the strong interactions. This estimate shows that the contribution of these diagrams can decidedly change the size of the radiative correction to β decay and eliminate the discrepancy between the predictions of the theory with a conserved vector current and the experimental results.

1. In recent times there has been a number of papers[1-3] on the calculation of the radiative corrections to β decay and μ decay. This interest in the radiative corrections is due to the fact that if the radiative corrections have been calculated correctly and if the experiments on the lifetimes of the O^{14} nucleus and the μ meson are correct, then the hypothesis of the conservation of a vector current in β decay[4] is untrue.[5] The radiative corrections to μ decay have undoubtedly been calculated correctly, since μ mesons do not take part in the strong interactions, and the result is finite in any order of perturbation theory[6] and does not depend on a "cut-off."

For β decay the situation is different. It has been shown[2,3] that the correction to the lifetimes of the neutron and the O^{14} nucleus is logarithmically divergent in the region of large momenta of the virtual γ-ray quantum. To get finite results, which could be compared with experiment, one has made a "cut-off" in the vertex parts for the interaction of the photon with the proton and electron, using for both the same value of the momentum of the virtual photon, $\Lambda \sim M_p$. The electromagnetic form-factor of the proton is due to the strong interactions and actually cuts off at $\Lambda_p \sim M_p$, but the electron does not take part in the strong interactions, and its form-factor is cut off by the weak interactions at values $\Lambda_e \sim 50$—400 Bev.[7]

If, however, one simply introduces different cut-off limits for the electron and proton, the calculation is not gauge invariant. Since the neutron and proton that take part in the β decay can be replaced by an arbitrary number of virtual π mesons, it is possible for a γ-ray quantum to be emitted directly from the complete four-fermion vertex. These processes, which have not been considered in references 2 and 3, are represented by diagrams 1, 2, 3 (see figure). In the present paper we shall take into account the difference between the form-factors of the proton and electron, without destroying the gauge invariance, and shall estimate the contribution from diagrams 1, 2, and 3, which correspond to the emission of a γ-ray quantum from the four-fermion vertex.



2. We shall prove the gauge invariance of the second-order radiative corrections to β decay. This means that if we take the propagation function of the photon with an arbitrary longitudinal part,

$$D_{\mu\nu}(k) = -ik^{-2}(\delta_{\mu\nu} + C(k^2)k^{-2}k_\mu k_\nu), \qquad (1)$$

the result will not depend on the arbitrary function $C(k^2)$. With effects of a possible renormalization on account of strong interactions included, the matrix element for β decay is of the form

$$2^{-1/2} G \sum_i (\bar{\psi}_p F^{(i)}(q) \psi_n)(\bar{\psi}_e O_i (1+\gamma_5)\psi_\nu),$$

where i = 1, . . . , 5 numbers the possible types of weak interaction, O_i are the usual local operators of β decay ($O_i = 1, \gamma_\mu, \sigma_{\mu\nu}, i\gamma_5\gamma_\mu, \gamma_5$ for i = 1, . . . , 5), and $F^{(i)}(q)$ is the most general form of the operator producing the change of a neutron to a proton in the i-th type of β-decay interaction (including effects of any number of virtual π mesons); $F^{(i)}$ depends only on the total momentum q transferred to the leptons. It is obvious that $F^{(i)}(q)$ is represented by the sum of all diagrams with one vertex O_i for the emission of the lepton pair (e, ν) and any number of virtual π-meson and nucleon-antinucleon lines.

Let us consider one of these diagrams and denote it by $f^{(i)}(p_1, p_2, q)$, where p_1, p_2, q are the four-momenta of the neutron and proton and the total four-momentum of the leptons. Let $f_\lambda^{(i)}(p_1, p_2, q, k)$ be the matrix element for the emission of a photon with momentum k and polarization λ from the diagram $f^{(i)}(p_1, p_2, q)$ (the momentum p_1 of the incoming neutron is fixed). The matrix element $f^{(i)}(p_1, p_2, q, k)$ is represented by the sum of the diagrams corresponding to the emission of the photon (k, λ) from each charged line of the original diagram $f^{(i)}(p_1, p_2, q)$.

In electrodynamics the well known generalized Ward's theorem

$$k_\mu \Gamma_\mu (p, p-k, k) = e\{G^{-1}(p-k) - G^{-1}(p)\} \qquad (2)$$

is valid both for fermions and also for bosons.[8] Applying it to the calculation of the quantity $k_\lambda f_\lambda^{(i)}(p_1, p_2, q, k)$, we find the only charged line that contributes is that ending in the proton, and the contributions from the closed charged loops add to zero. The result is that

$$k_\lambda f_\lambda^{(i)}(p_1, p_2, q, k) = e\{f^{(i)}(p_1, p_2, q) - f^{(i)}(p_1, p_2-k, q+k)\}.$$

Summing this equation over all diagrams, we get

$$k_\lambda F_\lambda^{(i)}(p_1, p_2, q, k) = e\{F^{(i)}(p_1, p_2, q) - F^{(i)}(p_1, p_2-k, q+k)\}. \qquad (3)$$

In all there are six diagrams for the radiative corrections (see figure). (Diagrams with emission of the γ-ray quantum by the neutron are always gauge invariant when summed, since the charge of the neutron is zero.) Let us find the contribution to these diagrams from the longitudinal part of $D_{\mu\nu}(k)$ (we denote it by $\delta F^{(i)}$). Let us consider diagram 1. It is obtained from the original diagram $F^{(i)}(p_1, p_2, q)$ by the successive emission of two photons with momenta k and $-k$; according to Eq. (3), we have the following equation for the emission of photons with arbitrary momenta k and k' from $F^{(i)}(p_1, p_2, q)$:

$$\begin{aligned} k_\lambda k'_\mu F_{\lambda\mu}(p_1, p_2-k-k'; q; k, k') &= e\{k'_\mu F_\mu(p_1, p_2-k', q; k') \\ &- k'_\mu F_\mu(p_1, p_2-k-k', q+k; k')\} = e^2\{F(p_1, p_2, q) \\ &+ F(p_1, p_2-k-k', q+k+k') - F(p_1, p_2-k, q+k) \\ &- F(p_1, p_2-k', q+k')\}. \end{aligned}$$

Setting $k' = -k$, we get

$$\begin{aligned} k_\lambda k_\mu F_{\lambda\mu}^{(i)}(p_1, p_2, q; k, -k) &= e^2\{F^{(i)}(p_1, p_2-k, q+k) \\ &+ F^{(i)}(p_1, p_2+k, q-k) - 2F^{(i)}(p_1, p_2, q)\}. \end{aligned} \qquad (4)$$

In our treatment each individual diagram with a virtual photon connecting two lines is counted twice, since we have summed independently over the possible points of emission of the first and second photons; therefore the result must be divided by 2.* We get as the result

$$\begin{aligned} \delta F_1^{(i)} = -\frac{e^2}{(2\pi)^4 i}\int d^4k \frac{C(k^2)}{k^4}[F^{(i)}(p_1, p_2, q) \\ - F^{(i)}(p_1, p_2-k, q+k)] \end{aligned} \qquad (4a)$$

(the index 1 indicates that this is the contribution to the matrix element from diagram 1). An analogous treatment can be used for the other diagrams. We give the results:

$$\begin{aligned} \delta F_2^{(i)} = \delta F_3^{(i)} = \frac{e^2}{(2\pi)^4 i}\int d^4k \frac{C(k^2)}{k^4}[F^{(i)}(p_1, p_2, q) \\ - F^{(i)}(p_1, p_2-k, q+k)], \end{aligned} \qquad (4b)$$

$$\delta F_4^{(i)} = \delta F_5^{(i)} = -\frac{e^2}{(2\pi)^4 i}\int d^4k \frac{C(k^2)}{k^4}\frac{1}{2}F^{(i)}(p_1, p_2, q), \qquad (4c)$$

$$\delta F_6^{(i)} = \frac{e^2}{(2\pi)^4 i}\int d^4k \frac{C(k^2)}{k^4}F^{(i)}(p_1, p_2-k, q+k). \qquad (4d)$$

Adding, we get †

$$\sum_k \delta F_k^{(i)} = 0.$$

3. Thus the validity of the generalized Ward's identity (2) at each electromagnetic vertex and of the identity (3) at the four-fermion vertex guarantees that the requirements for gauge invariance of the radiative corrections to β decay are satisfied. This means that the result will not depend on the choice of the arbitrary function $C(k^2)$ in the photon propagator $D_{\mu\nu}(k)$. Equations (2) and (3) impose restrictions only on the longitudinal parts of

*Actually this treatment is a simplified one, since the equation (3) does not relate to the case in which the photon is emitted and absorbed by the same virtual line in the diagram $F^{(i)}$. Furthermore, for the π mesons there are double electromagnetic vertices. A rigorous treatment, however, does not change the result.

†It can be seen from these formulas that the sum of diagrams 4, 5, and 6 is not separately gauge invariant if the form factors $F^{(i)}(q)$ of the weak interaction depend on the momentum q. The μ meson has no strong interactions, and therefore for the decay of the μ meson $F^{(i)}(q)$ reduce to the corresponding local operators $\gamma_\mu(1+\gamma_5)$. The radiative corrections to μ decay are represented by the diagrams 4, 5, and 6 only, and their sum is gauge invariant. With the transverse gauge (1a) each of these diagrams is finite, and consequently the result does not depend on the arbitrary cut-off momenta Λ_μ and Λ_e.

the electromagnetic vertices Γ_μ, $F^{(i)}_\lambda$ and $F^{(i)}_{\lambda\mu}$, and the transverse parts of Γ_μ, $F^{(i)}_\lambda$, $F^{(i)}_{\lambda\mu}$ can be arbitrary.

We shall suppose that the longitudinal parts of Γ_μ, $F^{(i)}_\lambda$, $F^{(i)}_{\lambda\mu}$ are chosen so that Eqs. (2) and (3) are satisfied, and take $C(k^2) = -1$, i.e., we take $D_{\mu\nu}(k)$ in the transverse gauge of Landau and Khalatnikov [9]

$$D_{\mu\nu} = -ik^{-2}(\delta_{\mu\nu} - k^{-2}k_\mu k_\nu). \tag{1a}$$

Then in virtue of the equation $D_{\mu\nu}(k)k_\nu = 0$ the longitudinal parts of Γ_μ, $F^{(i)}_\lambda$, $F^{(i)}_{\lambda\mu}$ do not occur at all in the matrix element, and the result will be gauge invariant for an arbitrary cut-off of the transverse parts of Γ_μ, $F^{(i)}_\lambda$, $F^{(i)}_{\lambda\mu}$.

Let us first examine the case in which the form-factor of the weak interaction does not depend on q, i.e., $F^{(i)}(q) = \gamma_\mu(1 + \gamma_5)$. In this case only the diagrams 4, 5, 6 contribute to the radiative corrections (and only these diagrams have been considered in[1,3]). The exact form of the vertices Γ_μ is not known, and therefore we make the simplest choice

$$\Gamma^{(p)}_\mu = a_p(k^2)\gamma_\mu, \quad \Gamma^{(e)}_\mu = a_e(k^2)\gamma_\mu. \tag{5}$$

It is not hard to verify that with the gauge chosen for the photon propagator $D_{\mu\nu}(k)$ the matrix elements 4 and 5 are finite, and in the matrix element 6 there must be the cut-off factor $a_p a_e$. We take the cut-off function $a(k)$ in the form

$$a(k, n) = [\Lambda^2/(k^2 + \Lambda^2)]^n. \tag{6}$$

For agreement with the experimental data of Hofstadter [10] on the behavior of the electromagnetic form-factor of the proton for small k^2 we must take $\Lambda_p = (6n/\overline{r^2})^{1/2}$, where $(\overline{r^2})^{1/2}$ is the root-mean-square radius of the charge distribution in the proton.

In the paper by Behrends et al.[1] the matrix elements 4, 5, and 6 have been calculated on the assumption $n = 1/2$, and the photon propagator was taken in the Feynman gauge: $D_{\mu\nu}(k) = \delta_{\mu\nu}/ik^2$. In our case we must add to the matrix elements 4, 5 from[1] the quantity

$$\Delta M_4 + \Delta M_5 = 2i\pi^2 N\left(-\ln(\Lambda_p/\lambda) + \tfrac{1}{2}\right)M. \tag{7}$$

Here $N \doteq ie^2/4\pi^3$, $e^2 = \alpha = 1/137$, λ is the mass of the photon, and M is the matrix element of β decay without radiative corrections.

The β-decay Hamiltonian of the $V - \Lambda A$ type is usually written in the form

$$H = 2^{-1/2}G(\bar\psi_p\gamma_\mu(1 + \Lambda\gamma_5)\psi_n)(\bar\psi_e\gamma_\mu(1 + \gamma_5)\psi_\nu).$$

Then diagram 6 means the exchange of a virtual photon between the charged particles (p, e) which occur in different brackets. To simplify the calculations it is convenient to go over to a way of writing the Hamiltonian in which the charged particles are in the same bracket; to do this we must perform the operation of charge conjugation and the Fierz transformation[11] (i = 0, 1, 2, 3, 4):

$$H = 2^{-1/2}G\sum_{i=0}^{4} c_i(\bar\psi_p O_i\psi_{\bar e})(\bar\psi_{\bar\nu}(1 - \gamma_5)O_i\psi_n) = \sum_{i=0}^{4} c_i H_i,$$

$$c_i = \left(1 + \Lambda, -\tfrac{1}{2}(1 - \Lambda), 0, \tfrac{1}{2}(1 - \Lambda), -(1 + \Lambda)\right). \tag{8}$$

For the i-th type of β-decay term the change of the matrix element 6 as compared with[1], when the Hamiltonian is written in the form (8), is given by

$$\Delta M_6^{(i)} = NH_i\left\{-\frac{a_i^2}{4}\left[\int\frac{a_p(k,n)a_e(k,n)}{(k^2+\lambda^2)^2}d^4k - \int\frac{a_p^2(k,1/2)}{(k^2+\lambda^2)^2}d^4k\right] + \int\frac{a_p(k,n)a_e(k,n)}{(k^2+\lambda^2)^2}d^4k\right\}.$$

The quantities a_i are determined from the condition

$$\sum_{\mu=1}^{4}\gamma_\mu O_i\gamma_\mu = a_i O_i.$$

$$a = (4, -2, 0, 2, -4) \text{ for } i = 0, 1, 2, 3, 4.$$

It can be shown that for $\lambda \to 0$ (λ is the mass of the photon)

$$\int\frac{a_p(k,n)a_e(k,n)}{(k^2+\lambda^2)^2}d^4k = i\pi^2\begin{cases}2\ln(\Lambda_p/\lambda) - 1 - \psi(2n) - \gamma & \text{for } \Lambda_e = \Lambda_p \\ 2\ln(\Lambda_p/\lambda) - 1 - \psi(n) - \gamma & \text{for } \Lambda_e \gg \Lambda_p\end{cases}. \tag{9}$$

Here $\psi(x) = d\ln\Gamma(x)/dx$ is the logarithmic derivative of the Γ function (cf., e. g.,[12]), and $\gamma = 0.5772$ is the Euler constant. From this we find that the total change in the sum of diagrams 4, 5, 6 is

$$\Delta M = i\pi^2 N\sum_{i=0}^{4} c_i\left\{\frac{a_i^2}{4}[\psi(n) - \psi(1)] + \psi(n) - \gamma\right\}H_i. \tag{10}$$

For the $V - A$ interaction the radiative correction to the lifetime of a nucleus against β decay is of the same form as in Eq. (4.5) of[2], in which we have only to replace the cut-off limit Λ_p by Λ_{eff}:

$$\Lambda_{eff} = \Lambda_p\exp\left\{\tfrac{2}{3}[\psi(1) - \psi(n)] + \tfrac{1}{6}(\psi(n) + \gamma)\right\}. \tag{11}$$

Values of Λ_{eff} are shown in the table. As we see, for $n = 1/2$ (cut-off by the Feynman method) choice

n	Λ_p / M	$\Lambda_{eff} / \Lambda_p$	Λ_{eff} / M
0.5	0.47	2	0.94
1.0	0.67	1	0.67
1.5	0.82	0.736	0.60

of Λ_p in accordance with the experiments of Hofstadter and use of the fact that $\Lambda_e \gg \Lambda_p$ gives practically the same result as the assumption $\Lambda_p = \Lambda_e = M$ (M is the mass of the nucleon).

4. In the β-decay theory with a conserved vector current the operator $F^{(v)}(q)$ has the form

$$F^{(v)} = f_1(q^2/M^2)\gamma_\alpha + \tfrac{1}{2}M^{-1}f_2(q^2/M^2)\sigma_{\alpha\beta}q_\beta, \qquad (12)$$

where the form-factors f_1 and f_2 are equal to the isotopic-vector parts of the electric and magnetic form-factors of the nucleon and can be found from the experiments of Hofstadter.[13] In particular,

$$f_1(0) = 1, \qquad f_2(0) = (\mu_p - \mu_n)/e = 3.70$$

(μ_p and μ_n are the anomalous magnetic moments of the proton and neutron).

Since $F^{(v)}$ depends on q^2, it is not only by the proton electromagnetic form-factor a_p that diagram 6 is cut off. For $\Lambda_e \gg \Lambda_p$ the cut-off factor $a_p a_e f_1 \approx a_p^2$, and the result is close to that found in the paper of Behrends and others.[1]

For an estimate of the contribution of diagrams 1, 2, 3 let us find the limiting values of $F_\lambda^{(i)}$ and $F_{\lambda\mu}^{(i)}$ for small k:

$$F_\lambda^{(i)}(k=0) = -e\partial F^{(i)}(q+k)/\partial k_\lambda|_{k=0}$$
$$= -e\partial F^{(i)}(q)/\partial q_\lambda,$$
$$F_{\lambda\mu}^{(i)}(k=0) = e^2\partial^2 F^{(i)}(q+k)/\partial k_\lambda\partial k_\mu|_{k=0}$$
$$= e^2\partial^2 F^{(i)}(q)/\partial q_\lambda \partial q_\mu. \qquad (13)$$

It is impossible to determine from Eqs. (3) and (4) the subsequent terms of the expansions of $F_\lambda^{(i)}$ and $F_{\lambda\mu}^{(i)}$ in powers of k, since there can be transverse parts of $F_\lambda^{(i)}$, $F_{\lambda\mu}^{(i)}$ linear in the photon momentum k. Substituting $F^{(v)}(q)$ in the formulas (13) and omitting in the calculations terms of the order m_e/M, we find the limiting values of the operators $F_\lambda^{(v)}$ and $F_{\lambda\mu}^{(v)}$:

$$F_\lambda^{(v)}(k=0) = -(e/2M)f_2(0)\sigma_{\alpha\lambda},$$
$$F_{\lambda\mu}^{(v)}(k=0) = (2e^2/M^2)f_1'(0)\delta_{\lambda\mu}\gamma_\alpha. \qquad (14)$$

In the calculation of the contribution from diagrams 1, 2, 3 we shall use the expression (14) and cut off the integration over the momentum of the virtual photon at the mass of the nucleon. The use of the values (14) for $F_\lambda^{(v)}$ and $F_{\lambda\mu}^{(v)}$ right up to $k^2 = M^2$ is illegitimate, but we hope that in this way we shall get the correct order of magnitude for the contribution of diagrams 1, 2, 3. With this crude estimate the contribution from these diagrams is almost equal in magnitude to the contribution from diagrams 4, 5, 6, but is opposite in sign. This leads to a great decrease in the size of the radiative correction to the lifetime of a nucleus against β decay, as compared with earlier results,[2,3] and we may suppose that an accurate inclusion of diagrams 1, 2, 3 will remove the discrepancy between the theory of the conserved vector current and experiment.

In all of the foregoing we have used the expression for the electromagnetic vertex of the proton which is valid when the proton is a real particle ($p^2 = M^2$) before and after the emission of a photon. In our case this condition is not satisfied (cf. diagrams 3, 5, 6). Since the "nonreality" of the proton is important only for $k^2 \sim M^2$ we can hope that the "nonreality" of the proton will not change the results very much. The contribution from the anomalous magnetic moments of the neutron and proton has been calculated by Berman,[3] using the data of Hofstadter, and it was found to be small.

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[1] Behrends, Finkelstein, and Sirlin, Phys. Rev. **101**, 866 (1956). T. Kinoshita and A. Sirlin, Phys. Rev. **107**, 593, 638 (1957).

[2] T. Kinoshita and A. Sirlin, Phys. Rev. **113**, 1652 (1959).

[3] S. M. Berman, Phys. Rev. **112**, 267 (1958).

[4] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958). S. S. Gershteĭn and Ya. B. Zel'dovich, JETP **29**, 698 (1955), Soviet Phys. JETP **2**, 576 (1956).

[5] R. P. Feynman, Proc. 1960 Int. Conf. on High-Energy Physics at Rochester, Univ. of Rochester, 1960, page 501.

[6] Ya. A. Smorodinskii and Ho Tso-Hsiu, JETP **38**, 1007 (1960), Soviet Phys. JETP **11**, 724 (1960).

[7] B. L. Ioffe, JETP **38**, 1608 (1960), Soviet Phys. JETP **11**, 1158 (1960).

[8] H. S. Green, Proc. Phys. Soc. **A66**, 873 (1953). Y. Takahashi, Nuovo cimento **6**, 371 (1957).

[9] L. D. Landau and I. M. Khalatnikov, JETP **29**, 89 (1955), Soviet Phys. JETP **2**, 69 (1956).

[10] R. Hofstadter, Revs. Mod. Phys. **28**, 214 (1956).

[11] M. Fierz, Z. Physik **104**, 553 (1957).

[12] I. M. Ryzhik and I. S. Gradshteĭn, Tablitsy integralov, summ, ryadov, i proizvedenii (Tables of Integrals, Sums, Series, and Products), Gostekhizdat, 1951.

[13] Hofstadter, Bumiller, and Croissiaux, Phys. Rev. Letters **5**, 263 (1960).

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