ENERGY DEPENDENCE OF CROSS SECTIONS NEAR THE "THRESHOLD" FOR UNSTABLE PARTICLE PRODUCTION

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The energy dependence of the cross sections for elastic scattering, X(aa)X, and for the reaction X(ab)Y are determined near the "threshold" for production of an unstable particle Y, which decays immediately after it is produced ($Y \rightarrow c + d$). The results obtained may be useful in nuclear physics.

1. INTRODUCTION

THE form of the energy dependence of an arbitrary two-particle reaction X(ab)Y near its threshold is well known.[1] It is also known that the cross section for elastic scattering X(aa)X exhibits an anomaly (a peak or cusp) at the threshold point.[2,3] In all the papers cited, however, it is assumed that the particles b and Y are stable, so that both at the beginning and the end of the reaction only two particles are present, namely a + X and b + Y respectively.

If one of the final particles, say Y, is actually unstable (width Γ) and decays after some time

$$Y \rightarrow c + d, \quad (1)$$

then formally one is dealing with a three-particle reaction

$$X + a \rightarrow b + c + d. \quad (2)$$
$$\searrow \; Y+b \; \nearrow$$

As a consequence of the existence of a finite width Γ the threshold will not be sharp in this case, but will be "smeared" out in an energy region $\Delta E \sim \Gamma$ near the energy E_r of the particle Y.

Such a situation arises very often in nuclear reactions, in which one of the reaction products is an excited nucleus, in high energy physics in strange particle production, etc. In this connection it becomes of interest to study the energy dependence of cross sections near the "threshold" for production of an unstable particle Y. This turns out to be possible in a quite general manner. In Sec. 2 we obtain an expression for the cross section for reaction (1), and in Sec. 3 for the elastic scattering process X(aa)X. For simplicity we consider throughout the case of spinless neutral particles.

2. REACTION CROSS SECTION NEAR THRESHOLD

Let us consider reaction (2) in more detail. For energies E of the colliding particles X + a near E_1 (see Fig. 1), the cross section for the reaction $X + a \rightarrow b + c + d$, just like the total cross section for any three-particle reaction, is proportional to $\sigma \sim (E - E_1)^2$ and is consequently very small if E is sufficiently close to E_1.

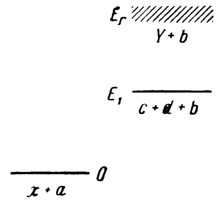

FIG. 1

The distribution of the final particles (b + c + d) in energy (spectrum) near threshold should be statistical (provided there is no zero energy resonance among the particles b, c, and d) as is illustrated by the curve labeled 1 in Fig. 2; the figure shows schematically the differential cross section $\sigma(\epsilon_{cd}, E)$ as a function of the energy of relative motion of particles c + d; $\epsilon_{cd} = E - E_1 - \epsilon_b$, where ϵ_b is the energy carried off by particle b.

The statistical form of the spectrum corresponds to

$$\sigma(\varepsilon_{cd}, E) \approx \text{const} \cdot \sqrt{\varepsilon_{cd}(E - E_1 - \varepsilon_{cd})}. \quad (3)$$

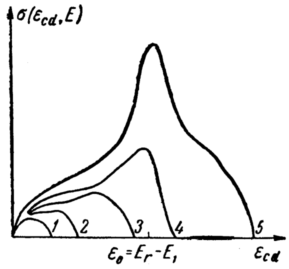

FIG. 2

Let us now increase the energy E, gradually approaching E_r. Simultaneously the form of the spectrum will change in some manner (curve 2 in Fig. 2): the nearer we approach E_r the more pronounced becomes the maximum at the upper end of the spectrum (curve 3). When E exceeds E_r the maximum stabilizes at $\epsilon_{cd} = E_r - E_1 \equiv \epsilon_0$ (curve 4) and as E is increased further remains at the same place (curve 5).

The appearance of the maximum is due to a resonant interaction between particles c and d ($c + d \rightleftarrows Y$) and is simplest understood in the following manner. If the interaction of particle b with c and d is neglected then the differential cross section for the reaction should be of the following form[4] (by analogy with the well known expression for two-particle reactions):

$$\sigma(\varepsilon_{cd}, E) = (\varepsilon_{cd}\varepsilon_b)^{-1/2} |\psi_{\varepsilon_{cd}} \varphi_{\varepsilon_b}|^2 \zeta, \tag{4}$$

where ψ_ϵ is the value of the wave function describing the relative motion of the pair c + d (with the energy ϵ) at the surface of the region within which the reaction takes place, φ_ϵ is the corresponding value of the wave function describing the motion of particle b relative to the pair c + d, and ζ is some function determined by the details of the nuclear interaction.

It follows from the assumption of the existence of the quasistable particle Y that a resonance appears in the elastic scattering $c + d \rightarrow c + d$ at the energy $\epsilon_{cd} = \epsilon_0$ i.e., that ψ_ϵ has as a function of the energy a pole at $\epsilon = \epsilon_0 - i\Gamma$.[5] This means that near $\epsilon_{cd} \sim \epsilon_0$

$$|\psi_{\varepsilon_{cd}}|^2 \sim [(\varepsilon_{cd} - \varepsilon_0)^2 + \Gamma^2]^{-1} \tag{5}$$

and, correspondingly, the spectrum (4) has a maximum near $\epsilon_{cd} \approx \epsilon_0$

$$\sigma(\varepsilon_{cd}, E) \sim (\varepsilon_{cd}\varepsilon_b)^{-1/2} \frac{|\varphi_{\varepsilon_b}|^2}{(\varepsilon_{cd} - \varepsilon_0)^2 + \Gamma^2}, \tag{6}$$

with the form of the spectrum in this region given by the Breit-Wigner curve.

We are now in a position to find the energy dependence of the total cross section near the threshold for production of Y

$$\sigma(E) = \int_0^{E-E_1} \sigma(\varepsilon, E)\, d\varepsilon. \tag{7}$$

To this end we note that the main contribution to (7) in this energy region comes from the resonance maximum (6) at $\epsilon \approx \epsilon_0$, and the lower part of the spectrum ($\epsilon < \epsilon_0$) leads to the appearance in $\sigma(E)$ of a small term, whose energy dependence is weak. We may therefore replace in Eq. (7) the lower limit by $\epsilon_0 - \Delta$ (Δ is some quantity $\gg \Gamma$) and obtain, after discarding all factors that vary slowly with the energy near $\epsilon = \epsilon_0$, (it is assumed that Y + b are produced in an S state, so that $|\varphi_{\epsilon b}|^2 \sim \epsilon_b = E - E_1 - \epsilon$, and that $\Gamma \ll \epsilon_0$)

$$\sigma(E) \sim \int_{\varepsilon_0 - \Delta}^{E-E_1} d\varepsilon \frac{\sqrt{E - E_1 - \varepsilon}}{(\varepsilon - \varepsilon_0)^2 + \Gamma^2} \sim \int_{1-\Delta/\varepsilon_0}^{1} dx \frac{(1-x)^{1/2}}{[x - 1 - \alpha]^2 + \gamma^2}$$
$$= \int_0^{\Delta/\varepsilon_0} dy \frac{y^{1/2}}{(y-\alpha)^2 + \gamma^2},$$

where $\alpha = (E - E_r)/\epsilon_0$, $\gamma = \Gamma/\epsilon_0$.

In the case of interest to us, when $|\alpha| \ll 1$ and $\gamma \ll 1$, the main contribution to the last integral comes from the region of small values of y so that we may let the upper limit go to infinity; we then obtain immediately

$$\sigma(E) \sim \int_0^\infty dy \frac{y^{1/2}}{(y-\alpha)^2 + \gamma^2} = \frac{\pi}{\gamma} \mathrm{Re}(\alpha + i\gamma)^{1/2} = \frac{\pi}{\Gamma} \varepsilon_0^{1/2} \mathrm{Re}[E - (E_r - i\Gamma)]^{1/2}. \tag{8}$$

Consequently the cross section for reaction (2) equals near the "threshold"

$$\sigma(E) \approx a\, \mathrm{Re}[E - (E_r - i\Gamma)]^{1/2}, \tag{9}$$

where a is some constant. The physical meaning of this result is perfectly obvious.

Were Y a stable particle (with $\Gamma = 0$) then $\sigma(E)$ would be proportional to $(E - E_r)^{1/2}$ above threshold and would be equal to zero for $E < E_r$. Because in our case the threshold is not sharp this picture is smeared out.

The function $\mathrm{Re}[E - (E_r - i\Gamma)]^{1/2}$ is plotted in Fig. 3. This function is equal to

$$\frac{1}{\sqrt{2}} [\sqrt{(E-E_r)^2 + \Gamma^2} + (E - E_r)]^{1/2}$$
$$= \begin{cases} (E - E_r)^{1/2} [1 + \frac{1}{4} \Gamma^2/(E - E_r)^2], & E - E_r > \Gamma \\ \Gamma/2\sqrt{E_r - E}, & E_r - E > \Gamma. \end{cases} \tag{10}$$

By precisely the same method as above one may calculate the cross section for reaction (2) in the case when Y + b are produced in a state with $l \neq 0$. One then finds that, near the threshold,

$$\sigma_l(E) \approx a_l\, \mathrm{Re}[E - (E_r - i\Gamma)]^{l+1/2} + m, \tag{11}$$

where the constant term corresponds to that part of the total cross section that arises from the integration over the lower part of the spectrum ($\epsilon_{cd} < \epsilon_0$). Strictly speaking such a term should also be included in Eq. (9), however for the case $l = 0$ it is, apparently, unimportant since the first term in Eq. (11) is then sufficiently large.

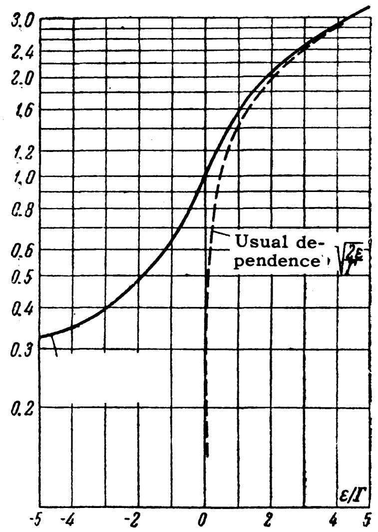

FIG. 3

3. ELASTIC SCATTERING CROSS SECTION

Let us consider now the cross section for elastic scattering X(a, a)X near the "threshold" for the production of Y. If we limit ourselves to S-states, then we have for the elastic and inelastic cross sections the formulas

$$\sigma_{el} = \pi k^{-2} |1 - S|^2, \qquad \sigma_{reac} = \pi k^{-2}(1 - |S|^2), \qquad (12)$$

where k is the wave vector of the colliding particles a + X, and S is usually written as $e^{2i\delta}$ [δ is the scattering phase shift for the reaction X(a, a)X]. From here we obtain right away [see Eq. (9)]

$$|S|^2 = 1 - k^2\sigma_{reac}/\pi = 1 - (k^2a/\pi)\,\mathrm{Re}\,[E - (E_r - i\Gamma)]^{1/2}. \qquad (13)$$

It follows from Eq. (10) that, if the width Γ of the paricle Y is small, the inelastic cross section σ(E) is small in the region near threshold and, correspondingly,

$$|S| = 1 - (k^2a/2\pi)\,\mathrm{Re}\,[E - (E_r - i\Gamma)]^{1/2}. \qquad (14)$$

Noting now the obvious equality

$$|1 + \alpha| = 1 + \mathrm{Re}\,\alpha,$$

valid for an arbitrary complex number α for $|\alpha| \ll 1$, and remembering that S should be an analytic function of the energy, we get

$$S = e^{2i\delta_0}[1 - (k^2a/2\pi)(E - (E_r - i\Gamma))^{1/2}], \qquad (15)$$

where δ_0 is a certain real quantity which may be considered to be constant in the vicinity of the "threshold." In the case of a stable Y (Γ = 0) this formula reduces to the usual expression for S (see Baz'[2]).

As was shown by the author,[2] formula (15) when substituted into the expression for σ_{el} (12) leads, in the case Γ = 0, to the appearance of characteristic singularities in the elastic cross section at the threshold point (peaks or cusps). The instability of Y (Γ ≠ 0) leads to a smearing out of these singularities (for a more detailed statement see below). Consequently the singularities in the elastic scattering cross section are smoothed out for sufficiently large Γ. With the exception of this circumstance the case of an unstable Y does not differ in any way from the previously discussed[2] case Γ = 0. In particular, all conclusions about the possibility of deducing important information (scattering phase shifts, etc.) from the study of the singularities in the elastic scattering cross section near the threshold point remain valid (provided, of course, that Γ is not so large as to completely "smear out" these singularities).

4. CONCLUDING REMARKS

The above-obtained formulas (9), (11) and (15) may be summarized as follows: the cross section for the reaction X(ab)Y has an energy dependence of the form $\sim (E - E_r)^{l+1/2}$ in the case of a stable Y, and of the form $\sim \mathrm{Re}\,[E - (E_r - i\Gamma)]^{l+1/2}$ in the case of production of an unstable particle; otherwise there is no difference between the two cases. For $E - E_r > \Gamma$ these functional dependences are practically the same (see Fig. 3). They differ from each other only in the region near E_r and below threshold.

If the production cross section of the unstable Y is known experimentally then it is easy to deduce from the form of the energy dependence the width Γ of this particle. To this end one only needs to choose the parameters E_r and Γ in Eq. (9) so as to best approximate the experimental curve.

Equation (9) may turn out to be useful in one further respect: to estimate the magnitude of the cross section for the reaction X + a → b + c + d below "threshold" when the production cross section above "threshold" is known. Indeed, above "threshold" the cross section (9) becomes $\sigma_{reac} \approx a(E - E_r)^{1/2}$ and the magnitude of the constant a may be easily determined since in this region, as a rule, the production cross section is sufficiently large. Knowing a, E_r and Γ, we obtain directly from Eqs. (9) and (10) the magnitude of the three-particle production cross section below threshold

$$\sigma_{reac} \gtrsim a\Gamma / 2(E_r - E)^{1/2}.$$

The "width" of particle Y may also be determined from the behavior of the elastic scattering cross section X(aa)X near threshold by studying the degree to which the threshold singularity is "smeared out." As an illustration of this we calculate the cross section for s-wave scattering. By substituting Eq. (15) into Eq. (12) we get

$$\sigma_{el} = 4\pi k^{-2}\sin^2\delta_0 - 2a\{\sin^2\delta_0 \operatorname{Re}[E-(E_r-i\Gamma)]^{1/2} + \sin\delta_0\cos\delta_0 \operatorname{Im}[E-(E_r-i\Gamma)]^{1/2}\};$$

and the energy dependence of the cross section near threshold, as determined by the second term in this formula, makes it possible to find Γ. As an example we show in Fig. 4 the form of the singularity in σ_{el} for various values of Γ.

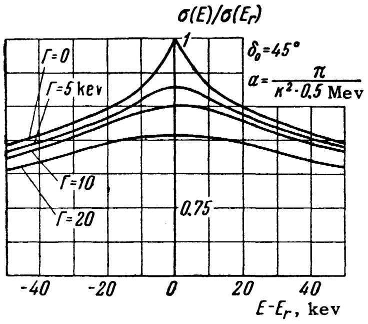

FIG. 4

We have assumed throughout that the unstable particle decays into two stable ones: $Y \to c + d$. Using the same method as above it can be shown that Eqs. (9), (11) and (15) remain valid no matter what the mode of decay of Y (three-particle decay mode, as in the case of the neutron $n \to p + e^- + \nu$, four-particle mode, etc.).

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