TWISTED SPACE AND NONLINEAR FIELD EQUATIONS

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It is shown that in a twisted space the nonlinear spinor equations considered in the field theory of elementary particles follow from a variational principle in which the Dirac Lagrangian is supplemented by the scalar torsion of the space.

1. One of the simplest generalizations of Euclidean space is a twisted Riemann space, in which the metric is Galilean and the geodesics are straight lines. Such a space (see Appendix) is characterized by a completely antisymmetric torsion tensor $\Phi_{\alpha\beta\epsilon}$. We note that unlike the graviational field components $\Gamma_{\alpha\beta,\epsilon}$, which, like the field $\Phi_{\alpha\beta\epsilon}$, have a definite geometrical meaning, the latter quantities are a tensor field, and therefore cannot be locally reduced to zero by any change of coordinates. The field $\Phi_{\alpha\beta\epsilon}$ can be given a simple physical interpretation; it can be regarded as the proper field of a particle, since its presence does not affect the motion of the particle (the geodesic line is still straight).

In a twisted space covariant differentiation can be defined (see Appendix) for tensor and spinor quantities, though since the metric is Galilean there is no difference between covariant and contravariant components. Then

$$^{*}(A_\mu)_{,\sigma} = \partial A_\mu/\partial x^\sigma + \Phi_{\sigma\lambda\mu}\mathbf{A}_\lambda, \quad (1)$$

$$D_\sigma\Psi = \partial\Psi/\partial x^\sigma - \left(\tfrac{1}{4}\Phi_{\sigma\alpha\beta}\gamma_\alpha\gamma_\beta + iI\varphi_\sigma\right)\Psi, \quad (2)$$

where φ_σ is interpreted as the vector potential of the electromagnetic field.

We get the field equations by starting with the Einstein variational principle

$$S = \int (L - b^{*}R)\, d^4x, \quad (3)$$

where b is a dimensional constant and *R is the scalar curvature of the space, which in our case is

$$^{*}R = \Phi_{\alpha\beta\epsilon}\Phi_{\alpha\epsilon\beta}. \quad (4)$$

Let us consider in particular the equations of a spinor field for which the invariant Lagrangian L is written in the following way:

$$L = \frac{1}{2i}\{\Psi^{+}\gamma_\alpha (D_\alpha\Psi) - (D_\alpha\Psi)^{+}\gamma_\alpha\Psi\}$$
$$= \frac{1}{2i}\{\Psi^{+}\gamma_\alpha\Psi_{,\alpha} - \Psi^{+}_{,\alpha}\gamma_\alpha\Psi\} - \frac{1}{4i}\Phi_{\alpha\beta\epsilon}(\Psi^{+}\gamma_\alpha\gamma_\beta\gamma_\epsilon\Psi)$$
$$- (\Psi^{+}\gamma_\alpha\Psi)\varphi_\alpha, \qquad \Psi_{,\alpha} = \partial\Psi/\partial x^\alpha. \quad (5)$$

The summation is taken with $\alpha \neq \beta \neq \epsilon$.

Let us examine the case in which $\varphi_\alpha = 0$. Since the derivatives of the field $\Phi_{\alpha\beta\epsilon}$ do not occur in the action integral (3), variation with respect to this field at once gives

$$\Phi_{\alpha\beta\epsilon} = (1/8ib)(\Psi^{+}\gamma_\alpha\gamma_\beta\gamma_\epsilon\Psi). \quad (6)$$

Varying the expression (5) with respect to Ψ^{+} and using Eq. (6), we get

$$\gamma_\alpha\Psi_{,\alpha} + (i/32b)(\Psi^{+}\gamma_\alpha\gamma_\beta\gamma_\epsilon\Psi)\gamma_\alpha\gamma_\beta\gamma_\epsilon\Psi = 0. \quad (7)$$

Equation (6) shows that the field $\Phi_{\alpha\beta\epsilon}$ is produced by the spin, and therefore it is quite natural to interpret it as the proper field of the particle.

Going over from the tensor $\Psi^{+}\gamma_\alpha\gamma_\beta\gamma_\epsilon\Psi$ to the corresponding dual axial vector $\Psi^{+}\gamma_\alpha\gamma_5\Psi$, we get

$$(\Psi^{+}\gamma_\alpha\gamma_\beta\gamma_\epsilon\Psi)^2 = 6(\Psi^{+}\gamma_\alpha\gamma_5\Psi)^2. \quad (8)$$

We can then write Eq. (7) in the form

$$\gamma_\alpha\Psi_{,\alpha} + i\lambda_0^2(\Psi^{+}\gamma_\alpha\gamma_5\Psi)\gamma_\alpha\gamma_5\Psi = 0, \quad (9)$$

where we have introduced the notation $\lambda_0^2 = (^3/_{16})\,b$. Thus we have obtained the nonlinear spinor equation with axial-vector nonlinear term.[4]

2. The interpretation of the Lagrangian (5) and the action integral (3) considered in the preceding section is not the only one possible. We shall consider another aspect of the theory, in which the equation (9) appears as an approximation.

If in the Lagrangian (5) we go over from the tensors $\Phi_{\alpha\beta\epsilon}$ and $\Psi^{+}\gamma_\alpha\gamma_\beta\gamma_\epsilon\Psi$ to the corresponding dual axial vectors, Eq. (5) takes the form

$$L = \frac{1}{2i}\{\Psi^+\gamma_\lambda\Psi_{,\lambda} - \Psi^+_{,\lambda}\gamma_\lambda\Psi\} - (\Psi^+\gamma_\lambda\Psi)\varphi_\lambda - (\Psi^+\gamma_\lambda\gamma_5\Psi)\tilde{\varphi}_\lambda, \quad (10)$$

where φ_λ and $\tilde{\varphi}_\lambda$ are vector (electromagnetic) and axial-vector (quasi-electromagnetic) potentials produced by the corresponding vector and axial-vector currents. Introducing a new constant, we can write the second term of the action integral (3) in the form

$$-b^*R = b\Phi_{\alpha\beta\varepsilon}\Phi_{\alpha\beta\varepsilon} = (k_0^2/8\pi\alpha_1)\tilde{\varphi}_\lambda\tilde{\varphi}_\lambda. \quad (11)$$

In addition to the electromagnetic field tensor $H_{\mu\nu}$ we must now introduce the pseudotensor $\tilde{H}_{\mu\nu}$ of the quasi-electromagnetic field; the total Lagrangian can then be written in the form

$$L_{\text{gen}} = L + \frac{1}{16\pi\alpha}H^2_{\lambda\sigma} + \frac{1}{16\pi\alpha_1}\{\tilde{H}^2_{\lambda\sigma} + 2k_0^2\tilde{\varphi}^2_\lambda\}. \quad (12)$$

Varying this with respect to Ψ^+, φ_λ, $\tilde{\varphi}_\lambda$ leads to the following field equations:

$$\gamma_\alpha\left\{\frac{\partial}{\partial x_\alpha} - i\varphi_\alpha - i\gamma_5\tilde{\varphi}_\alpha\right\}\Psi = 0,$$

$$\frac{\partial^2}{\partial x_\lambda^2}\varphi_\mu = -4\pi\alpha(\Psi^+\gamma_\mu\Psi),$$

$$\frac{\partial^2}{\partial x_\lambda^2}\tilde{\varphi}_\mu - k_0^2\tilde{\varphi}_\mu = -4\pi\alpha_1(\Psi^+\gamma_\mu\gamma_5\Psi). \quad (13)$$

For the fields $H_{\mu\nu}$ and $\tilde{H}_{\mu\nu}$ we obviously get the Maxwell equations and the Proca-Yukawa equations, respectively. In cases in which second derivatives can be neglected in the latter equations, we find

$$\tilde{\varphi}_\mu \approx 4\pi\alpha_1 k_0^{-2}(\Psi^+\gamma_\mu\gamma_5\Psi). \quad (14)$$

Substituting this in the first equation of the system (13) and neglecting φ_α, we get Eq. (9) as an approximate equation. Thus inclusion of a torsion of space leads to a new axial-vector field, to which, as is well known, there correspond axial vector mesons when second quantization is used.

APPENDIX

1. As is well known, in the general case the change of the components of a vector on parallel displacement can be written

$$dA^\mu = -{}^*\Gamma^\mu_{\sigma\lambda}A^\lambda dx^\sigma, \quad (15)$$

where ${}^*\Gamma^\mu_{\sigma\lambda}$ are the affine connection coefficients, which are asymmetrical in the lower indices and can be written as the sum of a symmetric part and an antisymmetric part:

$${}^*\Gamma^\mu_{\sigma\lambda} = \tilde{\Gamma}^\mu_{\sigma\lambda} + C^\mu_{\sigma\lambda}. \quad (16)$$

The quantities $C^\mu_{\sigma\lambda}$, called the components of the torsion, form a tensor and violate the parallelogram law.[1]

On introducing a metric and requiring conservation of the length of a vector on parallel displacement, we can divide ${}^*\Gamma^\mu_{\sigma\lambda}$ into metric and nonmetric parts:

$${}^*\Gamma_{\mu\nu,\sigma} = \Gamma_{\mu\nu,\sigma} + \Pi_{\mu,\nu\sigma}; \quad (17)$$

here $\Gamma_{\mu\nu,\sigma}$ are the ordinary Christoffel symbols, and

$$\Pi_{\mu,\nu\sigma} = -\Pi_{\mu,\sigma\nu} = C_{\sigma\mu,\nu} + C_{\sigma\nu,\mu} + C_{\mu\nu,\sigma}. \quad (18)$$

The equation of the geodesic lines in such a space can be written in the following way:

$$\frac{d^2x^\mu}{d\tau^2} + \tilde{\Gamma}^\mu_{\sigma\lambda}\frac{dx^\sigma}{d\tau}\frac{dx^\lambda}{d\tau} = 0, \quad (19)$$

where by Eqs. (16) – (18) the $\tilde{\Gamma}^\mu_{\sigma\lambda}$ are given by

$$\tilde{\Gamma}^\mu_{\sigma\lambda} = \Gamma^\mu_{\sigma\lambda} + C^{\mu}_{\cdot\sigma,\lambda} + C^{\mu}_{\cdot\lambda,\sigma}. \quad (20)$$

In our special case of a space with Galilean metric, in which the geodesic lines are straight, we have

$$\Gamma^\mu_{\sigma\lambda} = 0, \qquad \tilde{\Gamma}^\mu_{\sigma\lambda} = 0. \quad (21)$$

It then follows from Eq. (20) that the torsion must be a completely antisymmetric tensor. In this case let us introduce the notation

$$C_{\sigma\mu,\nu} = \Phi_{\sigma\mu\nu}. \quad (22)$$

The scalar curvature of the space, which in this case depends only on the torsion components, can be written

$${}^*R = \Phi_{\alpha\beta\varepsilon}\Phi_{\alpha\varepsilon\beta}. \quad (23)$$

2. In order to write the equations of an unquantized spinor field in an affinely connected space, we can use a method developed by Rumer.[2] Let $\Omega_\mu(\alpha)$, $\Omega^\mu(\alpha)$ denote the components of the metric tensor of Lamé. Here μ is the index of the ordinary covariant or contravariant component, and the index in parentheses (α) indicates the number of the invariant orthogonal component, so that

$$g_{\mu\nu} = \Omega_\mu(\alpha)\,\Omega_\nu(\alpha), \qquad g^{\mu\nu} = \Omega^\mu(\alpha)\,\Omega^\nu(\alpha), \ldots$$

Then in the parallel displacement of a vector the change of the orthogonal components of the vector that corresponds to Eq. (15) is

$$dA(\alpha) = {}^*\Delta_\sigma(\alpha\beta)\,A(\beta)\,dx^\sigma, \quad (24)$$

where ${}^*\Delta_\sigma(\alpha\beta)$ are the generalized Ricci rotation coefficients. Since by hypothesis A^2 remains unchanged in parallel displacement,

$${}^*\Delta_\sigma(\alpha\beta) = -{}^*\Delta_\sigma(\beta\alpha), \qquad {}^*(\Omega_\mu(\alpha))_{,\sigma} = 0, \quad (25)$$

where the brackets ${}^*(\ )_{,\sigma}$ denote the covariant derivative relative to ${}^*\Gamma^\mu_{\sigma\lambda}$ and ${}^*\Delta_\sigma(\alpha\beta)$. The last of the conditions (25) and Eq. (17) give

$$^*\Gamma_{\sigma\mu,\,\nu} = \Gamma_{\sigma\mu,\,\nu} + \Pi_{\sigma,\,\mu\nu} = \Omega_\nu(\alpha)\frac{\partial\Omega_\mu(\alpha)}{\partial x^\sigma} + {}^*\Delta_{\sigma,\,\mu\nu}, \qquad (26)$$

In Riemann space (without torsion) we have

$$\Gamma_{\sigma\mu,\,\nu} = \Omega_\nu(\alpha)\frac{\partial\Omega_\mu(\alpha)}{\partial x^\sigma} + \Delta_{\sigma,\,\mu\nu}, \qquad (27)$$

where $\Delta_{\sigma,\mu\nu}$ are the ordinary Ricci rotation coefficients, which depend on $\Omega_\mu(\alpha)$, $\Omega^\nu(\alpha)$ and their derivatives.

Comparing Eqs. (26) and (27), we find

$$^*\Delta_{\sigma,\,\mu\nu} = \Delta_{\sigma,\,\mu\nu} + \Pi_{\sigma,\,\mu\nu}, \qquad (28)$$

i.e., the same tensor $\Pi_{\sigma,\mu\nu}$ gives the nonmetric parts of both $^*\Gamma_{\sigma\mu,\nu}$ and $^*\Delta_{\sigma,\mu\nu}$.

The change of a spinor on parallel displacement is written

$$d\Psi = {}^*B_\sigma\Psi dx^\sigma, \qquad d\Psi^+ = -\Psi^{+*}B_\sigma dx^\sigma, \qquad (29)$$

where

$$^*B_\sigma = \tfrac{1}{4}\,{}^*\Delta_\sigma(\alpha\beta)\,\gamma(\alpha)\,\gamma(\beta) + iI\varphi_\sigma \qquad (30)$$

are the generalized displacement matrices,[2,3] and $\gamma(\alpha)$ and I are Dirac matrices. Then the covariant derivative of a spinor is written in the following way;

$$D_\sigma\Psi = \partial\Psi/\partial x^\sigma - {}^*B_\sigma\Psi, \qquad (31)$$

and the Lagrangian of the spinor field can be put in the form (the mass term is omitted here)

$$L = (1/2i)\,\{\Psi^+\gamma(\alpha)(D_\sigma\Psi) - (D_\sigma\Psi)^+\gamma(\alpha)\Psi\}\,\Omega^\sigma(\alpha). \qquad (32)$$

In the case of interest to us here, that of a space with the Galilean metric, we obviously have

$$g_{\mu\nu} = \delta_{\mu\nu}, \qquad \Omega_\mu(\alpha) = \delta_{\mu\alpha},$$
$$\Delta_{\sigma,\,\mu\nu} = 0, \qquad \Delta^*_{\sigma,\,\mu\nu} = \Phi_{\sigma\mu\nu}, \qquad (33)$$

and Eqs. (31) and (32) go over into the formulas (2) and (5) used earlier.

[1] P. K. Rashevskii, Риманова геометрия и тензорный анализ (Riemannian Geometry and Tensor Analysis), Gostekhizdat 1953, page 413.

[2] Yu. B. Rumer, Исследования по 5-оптике (Studies in Five-dimensional Optics), Gostekhizdat, 1956, page 110.

[3] V. Fock and D. Ivanenko, Compt. rend. **188**, 1470 (1929).

[4] Nonlinear Quantum Field Theory (Russian Translations), Collection of articles, edited by D. D. Ivanenko, IIL, 1959, pp. 19, 351.

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