COMMUTATION FUNCTION OF A NONLINEAR MESON FIELD

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A new definition of the commutation function is proposed. Nonlinear meson fields are considered and, in particular, an expression for the commutation function is presented.

THE usual definition of the commutation function of the boson field

$$D(s) = [\overline{\Psi}(x), \Psi(x')]_-, \qquad s^2 = -x_\mu^2 \qquad (1)$$

[where $\Psi(s)$ is the sum of solutions over the wave vector k] cannot be used in the nonlinear theory, in view of the violation of the superposition principle. The right half of (1) now depends on the wave vector. We therefore propose to start at the very outset from the more general definition

$$D(s) = \sum_{k,\omega}\sum_{k'\omega'}\sum{}' [\overline{\Psi}_{k,\omega}(0), \Psi_{k'\omega'}(x-x')]_- \rho(k,\omega)\rho(k',\omega'), \qquad (2)$$

where $\rho(k, \omega)$ is a certain weighting function.

The summation Σ' is over the physically admissible independent wave solutions of the field equation. In the particular case of the linear theory, where $\rho(k, \omega) = 2|\omega|\delta(k_\mu^2)$, we obtain the well-known result (1). In the case

$$\Psi_k(x) = a_k\varphi_k(x), \qquad [\overline{a}_k, a_{k'}]_- = f(k)\delta_{kk'} \qquad (3)$$

we obtain from (2)

$$D(s) = \sum_{k,\omega}\sum{}' \eta(k,\omega)\varphi_k(x-x'),$$

$$\eta(k,\omega) = f(k)\varphi_k(0)\rho(k,\omega), \qquad (4)$$

where $\varphi_k(x)$ is the wave solution.

In the linear theory the function D(s) is, for any value of $\rho(k, \omega)$, a four-dimensional radially-symmetrical solution of the field equation, made up of the wave solutions. In the case of nonlinear theory this is possible only for a very special choice of $\rho(k, \omega)$. In particular, the simplest choice of the function $2|\omega|\delta(k_\mu^2)$ for $\rho(k, \omega)$, as will be shown later, is found to be unsatisfactory.

If we consider a neutral meson field, obeying the equation

$$(\partial^2/\partial x_\mu^2 + \lambda\varphi^2)\varphi = 0, \qquad (5)$$

for which it is natural to take the commutator between the functions $\varphi_k(x)$ and $\varphi_{-k}(x)$ [in nonquantum theory $\varphi_k(x) = \varphi_{-k}(x)$], then its energy can be represented in the form

$$H_k = \frac{1}{4V}\int\left\{\left[\frac{\partial\varphi_k}{\partial x_n}, \frac{\partial\varphi_{-k}}{\partial x_n}\right]_+ - \left[\frac{\partial\varphi_k}{\partial x_4}, \frac{\partial\varphi_k}{\partial x_4}\right]_+ + k_0^2[\varphi_k, \varphi_{-k}]_+ + \frac{\lambda}{2}[\varphi_k^3, \varphi_{-k}]_+\right\}d^3x. \qquad (6)$$

Calculating further the time average of the energy (with $k_0 = 0$) we obtain[1] ($C = 2/3$):

$$H_k = \frac{C}{2}\omega_k\left(1 - \frac{\lambda}{8\omega_k}[a_k, a_{-k}]_+\right)\frac{1}{2}[a_k, a_{-k}]_+,$$

$$\omega_k = \frac{k}{|k|}\left[k^2 + \frac{\lambda}{2}[a_k, a_{-k}]_+\right]^{1/2}. \qquad (7)$$

We introduce the canonical coordinates

$$q_k = a_k\exp(-i\Omega_k t), \qquad \dot{p}_k = -\partial H_k/\partial q_k,$$

$$\Omega_{-k} = -\Omega_k, \qquad q_{-k} = q_k^*. \qquad (8)$$

Under the assumption

$$[q_k, q_{-k}]_- = f(k) \qquad (*)$$

we obtain

$$p_k(t) = \frac{C}{2}\Omega_k q_k^*, \qquad \Omega_k = \frac{k}{|k|}\left[k^2 + \frac{3\lambda}{4}[a_k, a_{-k}]_+\right]^{1/2}. \qquad (9)$$

From this we get

$$[\Omega_k q_k, q_k]_- = [\Omega_k q_k^*, q_k^*]_- = 0,$$

$$[\Omega_k q_k^*, q_k]_- = -[\Omega_k q_k, q_k^*]_- = -2/C. \qquad (10)$$

Under condition (*), which is equivalent to the approximation $[a_k, a_{-k}]_+ \approx 2a_k^2$, i.e., to neglect of the quantum character of this expression in the formulas for ω_k and Ω_k, we obtain

$$f(k) = 2/C\Omega_k \quad a_k = \xi_k\sqrt{2/C|\Omega_k|}, \quad a_{-k} = \xi_{-k}\sqrt{2/C|\Omega_{-k}|},$$

$$[\xi_k, \xi_{-k}]_- = 1. \qquad (11)$$

Let us attempt now to calculate the sought commutation function D(s) by means of formula (2), using (3) and (11), and assuming $\rho(k, \omega)$ to have a form analogous to that of the linear case

$$\rho(k, \Omega_k) = \delta\left(\Omega_k + \sqrt{k^2 + \tfrac{3}{2}\beta_k^2}\right) + \delta\left(\Omega_k - \sqrt{k^2 + \tfrac{3}{2}\beta_k^2}\right),$$

$$\beta_k^2 \equiv \lambda a_k^2. \qquad (12)$$

We then obtain

$$D_{nl}(s) = \sum_{n=0}^{\infty} A_n D_l(\alpha_n s), \qquad \alpha_n = (2n+1)\lambda a_k \,\mathrm{const},$$

$$A_n = A(n), \qquad (13)$$

where $D_l(s)$ is the commutation function of the linear theory.

The result obtained, in view of the known divergence of the linear function $D_l(s)$, is unsatisfactory (from the point of view of constructing a complete field theory). This does not apply, however, to the expressions (2) or (11). Within the limits of assumption (*) Eq. (11) is valid, and we shall make use of it, as well as of (2). What is unsatisfactory is the choice of the form (12) for $\rho(k, \omega)$.

Inasmuch as we expect no divergences in the linear theory, we must review the definition of the nonlinear commutation function. In this connection we assume that the commutation function is a four-dimensional radially-symmetrical solution of the field equation not only in the linear but also in the nonlinear theory. If we start from such an assumption we obtain for the weighting function $\rho(k, \omega)$, subject to assumption (3), as the main result of the present investigation, a Fredholm integral equation of the first kind.

$$D(s) = (2\pi)^{-4}\int \sum{}' [\Psi_k(0), \Psi_k(x-x')]_- \rho(k, \omega) d^3k d\omega. \qquad (14)$$

Here $D(s)$ is a radially-symmetrical solution of the nonlinear equation of the field.

The summation Σ', as already indicated, is carried out over the physically admissible independent wave solutions $\Psi_k(x)$. Thus, Eq. (14) establishes with the aid of the function $\rho(k, \omega)$ the connection between the wave solutions and radially-symmetrical solutions of the nonlinear field equation.

The condition imposed on (14) is

$$D(0) = 0, \qquad \int \rho(k, \omega) d^3k d\omega = 1. \qquad (15)$$

In this connection, let us analyze in greater detail the radially-symmetrical solution of the initial nonlinear generalized Klein-Gordon equation (5).

Equation (5) has in the real domain two independent wave solutions

$$\varphi = \varphi_0 \,\mathrm{cn}\,(k_\mu^{(1)} x_\mu + c_1; k_1), \qquad \omega_{(1)}^2 = k_{(1)}^2 + \lambda\varphi_0^2, \qquad k_1^2 = 1/2,$$

$$\varphi = \varphi_0 \,\mathrm{sn}\,(k_\mu^{(2)} x_\mu + c_2; i|k_2|) = \frac{\varphi_0}{\sqrt{2}} \frac{\mathrm{sn}\,(k_\mu^{(1)} x_\mu + \sqrt{2}c_2; k_1)}{\mathrm{dn}\,(k_\mu^{(1)} x_\mu + \sqrt{2}c_2; k_1)}, \quad |k_2| = 1,$$

where $k_\mu^{(1)} = \sqrt{2}k_\mu^{(2)}$.

As wave solutions, both solutions are physically admissible. However, Eq. (5) has two independent radially-symmetrical solutions, corresponding to the two independent wave solutions mentioned, namely $\mathrm{cn}(u)$ and $\mathrm{sn}(u)$. The radially symmetric solution corresponding to $\mathrm{sn}(u)$ is in this case complex (when $\lambda > 0$ and $s^2 > 0$). In this connection we neglect both the radially-symmetrical solution corresponding to the wave solution $\mathrm{sn}(u)$ and the wave solution $\mathrm{sn}(u)$ itself. The real radially-symmetrical solution of Eq. (5) is given by the following formulas (see reference 1)

$$\varphi\underset{(\lambda s^2>0)}{(s, k_1)} = \begin{cases} \left(\dfrac{2}{\lambda s_0^2(2-k_2^2)}\right)^{1/2}\left(\dfrac{s_0}{s}\right)\mathrm{dn}\left[\dfrac{\ln(s_0/s)}{\sqrt{2-k_2^2}}\right], & 0 < k_2^2 = k_1^{-2} \leqslant 1, \quad (16a)\\ \left(\dfrac{2k_1^2}{\lambda s_0^2(2k_1^2-1)}\right)^{1/2}\left(\dfrac{s_0}{s}\right)\mathrm{cn}\left[\dfrac{\ln(s_0/s)}{\sqrt{2k_1^2-1}}\right], & \tfrac{1}{2} < k_1^2 \leqslant 1, \quad (16b)\\ 0 & 0 \leqslant k_1^2 \leqslant \tfrac{1}{2}, \quad (16c)\end{cases}$$

where k_1 and k_2 are the moduli of the corresponding elliptic functions.

When $k_2 = 0$ and $k_2^2 = 1$ we have

$$\varphi(s, k_2 = 0) = \sqrt{\frac{1}{\lambda s_0^2}}\left(\frac{s_0}{s}\right),$$

$$\varphi(s, k_2^2 = 1) = \sqrt{\frac{2}{\lambda s_0^2}} \frac{1}{1 + (s/s_0)^2}. \qquad (17)$$

This last particular solution was used by Borgart as a commutation function.[2]

In the region $0 < k_2^2 = k_1^{-2} \le 1$ we have $k_2' = \sqrt{1 - k_2^2} \le \mathrm{dn}(u) \le 1$, and therefore the function tends to zero only as $s \to \infty$. On the other hand, in the region $1/2 < k_1^2 < 1$ the function $\mathrm{cn}(u)$ executes infinite oscillations as $s \to 0$, and we can put

$$\varphi(0, k_1) = 0, \qquad \tfrac{1}{2} < k_1^2 \leqslant 1. \qquad (18)$$

This behavior of the proposed commutation function, which falls off like $1/s$ as $s \to \infty$ and which vanishes on the average as $s \to 0$, corresponds very closely to the conditions imposed by Heisenberg[3] on the commutation function of the nonlinear spinor field. Therefore our commutation function can be employed in analogous problems.

Thus, for D(s) we can take a radially-symmetrical solution of the form (16b). With this the integral equation (14) can be simplified by approximately replacing the elliptic functions with trigonometric ones. We then obtain for the weighting function the expression

$$\rho(k, \omega) \approx \frac{\text{const}}{[a_k, a_{-k}]_-} \int \cos\left(\frac{\ln(s_0/s)}{\sqrt{2k_1^2 - 1}}\right) \cos(k_\mu x_\mu) \frac{d^4x}{s}, \quad (19)$$

where $[a_k, a_{-k}]_-$ can be determined from (3) and (11).

The results obtained can subsequently be used for specific calculations, by obvious generalization to the case of other boson and spinor nonlinear fields and by establishing a connection with the Greenians and other singular functions.

Note added in proof (March 12, 1961). A similar suggestion is made by H. Mitter in the case of a spinor nonlinear field. We are grateful to him and to H. P. Dürr for supplying us with a preprint.

[1] D. F. Kurgelaidze, JETP **38**, 842 (1959), Soviet Phys. JETP **9**, 594 (1959).

[2] A. Borgart, Trudy, First All-Union Inter-University Theoretical Conference, Uzhgorod, 1958.

[3] Heisenberg, Dürr, Mitter, Schlieder, and Yamazaki, Z. Naturforsch. **14a**, 441 (1959).

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