ON THE KINETIC THEORY OF SHOCK WAVES

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The structure of a low-intensity shock wave in a monatomic gas is obtained at large distances from the wave front. The calculation is based on a kinetic equation with a simplified collision integral containing a constant collision time. It is shown that in this case various physical quantities approach their limiting values at infinity with a slower rate than in the hydrodynamic theory. Therefore, if the kinetic equation is replaced by a finite system of ordinary differential equations it is impossible in principle to obtain the correct asymptotic solutions of the kinetic equation.

1. INTRODUCTION

THE problem of the structure of the shock waves in a liquid and in a gas has been the subject of a large number of papers. However, as far as the author knows, not one of these papers is devoted to the investigation of the structure of the shock wave at large distances from the wave front (disregarding those calculations of the asymptotic values in which the structure of the shock wave is not taken into account). The reason for this is to be found in the circumstance that in these papers the kinetic equation is replaced by a finite system of ordinary differential equations. In the paper by Becker,[1] this system is the Navier-Stokes equation with account of the heat conductivity. Zoller,[2] using the method of Burnett,[3] wrote down a system of seven differential equations and investigated its properties. Grad,[4] by modifying the method of Burnett, derived a different system of differential equations and used it for the study of the shock waves.

Many other papers differ from the work of Becker, Zoller, and Grad either in that certain simplifying assumptions are introduced or in that they concentrate on the quantitative aspects of the problem or on the numerical solution of the basic system of differential equations.

The above-mentioned authors do not in any way attempt to simplify the collision integral in the kinetic equation. To the contrary, they try to write down the most accurate form possible. However, when the transition from the kinetic equation to the system of differential equations is made, only the coefficients of the equation depend on the form of the collision integral, so that the over-all picture changes only quantitatively when the form of the collision integral is altered.

The paper of Mott-Smith,[5] which stands by itself, also contains no answer to the problem of the asymptotic form of the shock wave, since it is not clear what is the character of the approximations underlying this work.

The aim of the present paper is the determination of the correct form of the asymptotic shock wave on the basis of the simplest kinetic equation, in which the collision integral is written in the radically simplified form (2.1). It will be shown that the hydrodynamic quantities approach their limiting values at great distances from the front of the shock wave as $C_1 \exp\{-C_2 |x|^{2/3}\}$ (C_1 and C_2 are certain constant coefficients). On the other hand, the solutions of the ordinary differential equations obtained in references 1, 2, and 4 approach their limiting values exponentially. This fact demonstrates that the replacement of the kinetic equation by a system of differential equations always leads to incorrect asymptotic values.

It is easy to see what the reason for this situation is. At large distances from the wave front, where all hydrodynamic quantities are close to their limiting values, one can linearize the differential equations. In this case the way in which the solution approaches its limiting value is determined by the smallest characteristic denominator. In going from one approximation to the next, the system of differential equations is changed, and in general the smallest characteristic denominator is changed, too. If the characteristic denominators obtained in this way go to zero, no approximation gives the correct asymptotic values. The asymptotic form derived in the present paper shows that this is indeed the case.

2. FORMULATION OF THE PROBLEM

In the present paper we consider the structure of a low intensity shock wave on the basis of the kinetic equation with collision integral. The latter is written in the form

$$J = (f_0 - f)/\tau, \tag{2.1}$$

where $f = f(x, \mathbf{v})$ is the distribution function (the x axis is directed along the direction of motion of the liquid perpendicular to the wave front), τ is the relaxation time, which is assumed constant, and $f_0(x, \mathbf{v})$ is the Maxwell distribution function,

$$f_0(x, \mathbf{v}) = \left[\frac{m}{2\pi kT(x)}\right]^{3/2} n(x) \exp\left[-m\frac{(v_x - u_x)^2 + v_y^2 + v_z^2}{2kT(x)}\right], \tag{2.2}$$

corresponding at every point to the average density $n(x)$, velocity $u(x)$, and energy $\frac{3}{2}n(x)kT(x)$:

$$n(x) = \int f(x, \mathbf{v})\, d\mathbf{v}, \qquad n(x)u(x) = \int v_x f(x, \mathbf{v})\, d\mathbf{v},$$
$$\frac{3}{2} n(x) kT(x) = \int \frac{m}{2}\left[(v_x - u(x))^2 + v_y^2 + v_z^2\right] f(x, \mathbf{v})\, d\mathbf{v}. \tag{2.3}$$

This form of the collision integral automatically ensures the fulfillment of the conservation laws for the number of particles, momentum, and energy in the use of the kinetic equation. On the other hand, if the deviation of the distribution function $f(x, \mathbf{v})$ from the Maxwellian form is small, the error caused by the replacement of the exact collision integral by the expression (2.1) is insignificant.

The relations (2.3) together with the kinetic equation

$$v_x \partial f/\partial x = \frac{1}{\tau}(f_0 - f) \tag{2.4}$$

form a system of four equations with the four unknown functions f, n, u, and T. To this we must add the boundary conditions. For a shock wave, these conditions are that as $x \to \pm\infty$ the functions $n(x)$, $u(x)$, and $T(x)$ approach certain limits which we denote by $n_\pm$, $u_\pm$, and $T_\pm$. The distribution function f approaches the Maxwell function.

The system (2.3), (2.4) can be solved by making use of the smallness of the relative discontinuities

$$\varepsilon_1 = (n_+ - n_-)/n_-, \qquad \varepsilon_2 = (u_+ - u_-)/u_-,$$
$$\varepsilon_3 = (T_+ - T_-)/T_-. \tag{2.5}$$

Here it turns out that the results obtained for small distances from the wave front [$x \sim u_- \tau \epsilon^{-1}$; see formula (6.4)] are in agreement with the results of hydrodynamic theory with account of the coefficients of viscosity and heat conductivity. At distances much larger than $u_- \tau \epsilon^{-1}$, the kinetic approach [see formula (7.4)] leads to the conclusion that the hydrodynamic quantities approach their limiting values at a slower rate than in the hydrodynamic theory. This fact is explained physically by the presence of fast particles which penetrate to large distances from the wave front with almost no collisions. It is clear, therefore, that the structure of the shock wave far away from its front is determined by the dependence of the relaxation time τ on the velocity at large velocities. It appears that the hydrodynamic theory predicts correctly only the structure of the steepest part of a low intensity shock wave.

3. DERIVATION OF THE BASIC SYSTEM OF INTEGRAL EQUATIONS

The formal solution of the kinetic equation (2.4) has the form

$$f(x, \mathbf{v}) = \frac{1}{\tau v_x} \int_{-S_{v_x}}^{x} f_0(\xi, \mathbf{v}) \exp\left[\frac{\xi - x}{\tau v_x}\right] d\xi, \tag{3.1}$$

where $S_{v_x} \equiv \infty \operatorname{sgn} v_x$: $S_{v_x} = +\infty$ if $v_x > 0$, and $S_{v_x} = -\infty$ if $v_x < 0$.

Using (3.1), we can eliminate f from the equations (2.3). Introducing the dimensionless variables

$$n' = n/n_-, \qquad T' = T/T_-,$$
$$u' = u/u_-, \qquad x' = x/\tau u_-, \tag{3.2}$$

we obtain the basic system of nonlinear integral equations

$$n'(x') = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} dv \int_{-S_v}^{x'} \frac{n'(\xi)}{\sqrt{T'(\xi)}} \frac{1}{v} \exp\left[\frac{\xi - x'}{v} - \alpha\frac{(v - u'(\xi))^2}{T'(\xi)}\right] d\xi,$$

$$n'(x')\, u'(x') = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} dv \int_{-S_v}^{x'} \frac{n'(\xi)}{\sqrt{T'(\xi)}} \exp\left[\frac{\xi - x'}{v} - \alpha\frac{(v - u'(\xi))^2}{T'(\xi)}\right] d\xi,$$

$$\frac{3}{2\alpha} n'(x')\, T'(x') + n'(x')\, u'^2(x')$$
$$= \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} dv \int_{-S_v}^{x'} \frac{n'(\xi)}{\sqrt{T'(\xi)}}\, v \exp\left[\frac{\xi - x'}{v} - \alpha\frac{(v - u'(\xi))^2}{T'(\xi)}\right] d\xi$$
$$+ \frac{1}{\sqrt{\alpha\pi}} \int_{-\infty}^{\infty} dv \int_{-S_v}^{x'} n'(\xi) \sqrt{T'(\xi)} \frac{1}{v} \exp\left[\frac{\xi - x'}{v} - \alpha\frac{(v - u'(\xi))^2}{T'(\xi)}\right] d\xi, \quad \alpha = \frac{mu_-^2}{2kT_-}. \tag{3.3}$$

The system (3.3) must be supplemented by the boundary conditions

$$n'(-\infty) = 1, \quad u'(-\infty) = 1, \quad T'(-\infty) = 1;$$
$$n'(+\infty) = 1 + \varepsilon_1, \quad u'(+\infty) = 1 + \varepsilon_2,$$
$$T'(+\infty) = 1 + \varepsilon_3. \tag{3.4}$$

The relaxation time τ does not enter in the relations (3.3) and (3.4). Therefore, it determines only the scale.

It is easily shown that the discontinuities ϵ_1, ϵ_2, and ϵ_3 must satisfy the well-known relations of shock wave theory in order that the system (3.3) with the conditions (3.4) be soluble. We shall therefore regard all discontinuities as known in the following.

4. THE CASE OF LOW-INTENSITY SHOCK WAVES ($\epsilon_j \ll 1$)

The basic system (3.3) is conveniently rewritten in the form

$$\varphi_i [y(x), \alpha] = \int_{-\infty}^{\infty} K_i(x - \xi, y(\xi), \alpha)\, d\xi, \qquad i = 1, 2, 3, \tag{4.1}$$

where

$$y = \{y_1, y_2, y_3\}, \quad y_1 = n', \quad y_2 = u', \quad y_3 = T';$$
$$\varphi_1 = y_1, \quad \varphi_2 = y_1 y_2, \quad \varphi_3 = (3/2\alpha) y_1 y_3 + y_1 y_2^2;$$

$$K_1(x, y, \alpha) = \sqrt{\frac{\alpha}{\pi}}\, y_1 y_3^{-1/2} \int_0^{S_x} \exp\left[-\frac{x}{v} - \alpha \frac{(v - y_2)^2}{y_3}\right] \frac{dv}{v},$$

$$K_2(x, y, \alpha) = \sqrt{\frac{\alpha}{\pi}}\, y_1 y_3^{-1/2} \int_0^{S_x} \exp\left[-\frac{x}{v} - \alpha \frac{(v - y_2)^2}{y_3}\right] dv,$$

$$K_3(x, y, \alpha) = \frac{1}{\sqrt{\pi}} \int_0^{S_x} y_1 \left(\sqrt{\frac{\alpha}{y_3}}\, v + \sqrt{\frac{y_3}{\alpha}} \frac{1}{v}\right) \exp\left[-\frac{x}{v} - \alpha \frac{(v - y_2)^2}{y_3}\right] dv.$$

Our aim is the solution of this system of integral equations with the assumption that the discontinuities $\epsilon_j \ll 1$. For this purpose we carry out the following transformations. We set

$$y_j(x) = 1 + \varepsilon_j \theta(x) + \varepsilon_j \mu_j(x); \quad \theta(x) = 1,$$
$$x > 0; \quad \theta(x) = 0, \ x < 0$$

It is easily seen that the functions $\mu_j(x)$ do not exceed unity in order of magnitude and go to zero as $x \to \pm\infty$.

Let us expand both sides of (4.1) in powers of ϵ_j and discard all terms containing ϵ_j in third or higher order. The resulting relations are then Fourier-transformed. We shall use the notation

$$\bar{\mu}_j(k) = \int_{-\infty}^{\infty} \mu_j(x) e^{ikx} dx, \qquad \overline{\mu_j^+}(k) = \int_0^{\infty} \mu_j(x) e^{ikx} dx,$$

$$\overline{\mu_j^-}(k) = \bar{\mu}_j(k) - \overline{\mu_j^+}(k),$$

$$\bar{\mu}_{jl}(k) = \int_{-\infty}^{\infty} \mu_j(x) \mu_l(x) e^{ikx} dx, \quad \overline{\mu_{jl}^+}(k) = \int_0^{\infty} \mu_j(x) \mu_l(x) e^{ikx} dx.$$

We solve the resultant system of equations with respect to $\bar{\mu}_j(k)$ by expressing the latter in terms of the $\bar{\mu}_{jl}(k)$. We find

$$\varepsilon_\gamma \bar{\mu}_\gamma(k) = \frac{1}{ik}\left\{\varepsilon_\gamma + \frac{1}{2D(k, \alpha)} \sum \varepsilon_j \varepsilon_l B_{jl\gamma}(k)\right\}$$
$$- \frac{1}{D(k, \alpha)} \sum_{j, l} \varepsilon_j \varepsilon_l \overline{\mu_j^+}(k) B_{jl\gamma}(k)$$
$$- \frac{1}{2D(k, \alpha)} \sum_{j, l} \bar{\mu}_{jl}(k) B_{jl\gamma}(k), \tag{4.2}$$

where the functions $B_{jl\gamma}(k)$ and $D(k, \alpha)$ are constructed in the following way. Let us introduce the notation

$$K_{ij}(x, y, \alpha) = \frac{\partial}{\partial y_j} K_i(x, y, \alpha),$$

$$K_{ijl}(x, y, \alpha) = \frac{\partial^2}{\partial y_i \partial y_l} K_i(x, y, \alpha),$$

$$G_{ij}(k, \alpha) = \int_{-\infty}^{\infty} K_{ij}(x, 1, \alpha) e^{ikx} dx,$$

$$G_{ijl}(k, \alpha) = \int_{-\infty}^{\infty} K_{ijl}(x, 1, \alpha) e^{ikx} dx,$$

$$D(k, \alpha) = \det \left| \frac{G_{ij}(k, \alpha) - G_{ij}(0, \alpha)}{ik} \right|. \tag{4.3}$$

We denote the minors of this determinant by $A_{ij}(k, \alpha)$. Furthermore,

$$B_{jl\gamma}(k) = \sum_m A_{m\gamma}(k, \alpha_0) \frac{G_{mjl}(k, \alpha_0) - G_{mjl}(0, \alpha_0)}{ik}. \tag{4.4}$$

Here α_0 is the critical value of the parameter α corresponding to a "shock wave" of zero intensity. For such a wave, the velocity of the liquid u is equal to the velocity of sound c, and therefore $\alpha_0 = mc_-^2/2kT_- = 5/6$. The same value of α_0 is obtained below from the condition of solubility of the problem under consideration. In view of the fact that we are considering small discontinuities, the difference $\alpha - \alpha_0$ is small, and we have therefore replaced α by α_0 in the quadratic terms in (4.4).

The elements of the determinant (4.3) have a comparatively simple form. For example, the element $D_{11}(k, \alpha)$ is equal to

$$D_{11}(k, \alpha) = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} \frac{v e^{-\alpha(v-1)^2}}{1 - ikv} dv. \tag{4.5}$$

In the derivation of the relations (4.2) we have made use of the identity

$$\varphi_i(y, \alpha) \equiv \int_{-\infty}^{\infty} K_i(x - \xi; y, \alpha)\, d\xi,$$

which has a simple physical meaning: a current with constant density n, velocity u, and temperature T is possible for arbitrary values of n, u, and T.

Let us consider Eq. (4.2) in more detail. In order that the functions $\bar{\mu}_\gamma(k)$ remain finite for $k = 0$, we must require

$$\varepsilon_\gamma + \frac{1}{2D(0,\ \alpha)} \sum_{j,l} \varepsilon_j \varepsilon_l B_{jl\gamma}(0) = 0. \tag{4.6}$$

It is easily seen that these conditions coincide up to terms of third order with the usual conditions for the discontinuities of the hydrodynamic quantities in a shock wave.

It follows from (4.6) that $D(0, \alpha) \sim \epsilon$. This implies that all terms in (4.2) have the same order of magnitude for small k. This is the explanation for the necessity of including the quadratic terms even in the case of a shock wave of low intensity. As was to be expected, the problem of the shock wave is nonlinear already in the first nonvanishing approximation.

We note finally that the relation $D(0, \alpha) \sim \epsilon$ implies $D(0, \alpha_0) = 0$. This equation leads to the value of α_0 mentioned above.

5. METHOD OF SUCCESSIVE APPROXIMATIONS

The quadratic terms in Eq. (4.2) are not equally important as the linear terms for all values of k, but only for small k, when $D(k, \alpha)$ is not very different from $D(0, \alpha)$, which has the same order of magnitude as the discontinuities ϵ. It is therefore reasonable to represent all coefficients in (4.2) in the form of a sum of their approximate asymptotic expressions for small k and certain corrections of higher order of smallness. Thus we rewrite Eq. (4.2) in the form

$$-\frac{1}{ik}\left[1 + \frac{1}{2\left[D(\alpha) + ikD_1(\alpha)\right]} \sum_{j,\,l} \frac{\varepsilon_j \varepsilon_l}{\varepsilon_\gamma} B_{jl\gamma}(0)\right] + \bar{\mu}_\gamma(k)$$
$$+ \frac{1}{D(\alpha) + ikD_1(\alpha)} \sum_{j,\,l} \frac{\varepsilon_j \varepsilon_l}{\varepsilon_\gamma} B_{jl\gamma}(0) \overline{\mu_j^+}(k)$$
$$+ \frac{1}{2\left[D(\alpha) + ikD_1(\alpha)\right]} \sum_{j,\,l} \frac{\varepsilon_j \varepsilon_l}{\varepsilon_\gamma} B_{jl\gamma}(0)\, \bar{\mu}_{jl}(k) = F_\gamma(k), \tag{5.1}$$

where

$$F_\gamma(k) = \frac{1}{\varepsilon_\gamma} \sum_{j,\,l} \varepsilon_j \varepsilon_l \left[\frac{B_{jl\gamma}(k)}{D(k, \alpha)} - \frac{B_{jl\gamma}(0)}{D(\alpha) + ikD_1(\alpha)}\right]$$
$$\times \left[\frac{1}{2ik} - \overline{\mu_j^+}(k) - \frac{1}{2}\bar{\mu}_{jl}(k)\right];$$

$$D(k, \alpha) \approx D(\alpha) + ikD_1(\alpha), \qquad |k| \ll 1. \tag{5.2}$$

It can be shown that the function $F_\gamma(k)$ must be neglected, since it is of higher order of smallness than the terms on the left-hand side of (5.1). We recall that terms of this order have already been neglected earlier.

In reality, however, we cannot discard the function $F_\gamma(k)$, because it is not analytic in the point $k = 0$ on the real axis, in contrast to the coefficients on the left-hand side of (5.1). The presence of a nonanalytic function in the equation leads to a nonanalytic solution. On the other hand, it is well known that the Fourier transform of a nonanalytic function goes to zero as $x \to \pm\infty$ at a considerably slower rate than the Fourier transform of a function which is analytic on the whole real axis.

Neglecting the small term on the right-hand side would thus lead to a serious distortion of the form of the shock wave at large distances from the wave front. The terms neglected earlier have an analytic part which is of the same order of smallness as the function $F_\gamma(k)$, while the remaining nonanalytic part is of even higher order. By keeping the right-hand side of (5.1) we therefore are kept from unwittingly going beyond the accuracy of the approximation.

To solve Eq. (5.1) we first neglect the right-hand side and find the first approximation. We then substitute on the right-hand side of (5.1) the functions μ_j found in first approximation and obtain an equation for the second approximation. There is no sense in trying to improve the accuracy any further in the framework of the Eq. (5.1), since this equation is itself not sufficiently exact for this purpose. We therefore confine ourselves to the above-mentioned two approximations.

6. FIRST APPROXIMATION (AT SMALL DISTANCES FROM THE WAVE FRONT)

As already mentioned, we must neglect the right-hand side of (5.1) for the calculation of the functions μ_j in the first approximation. As a first consequence, we find that then all functions μ_j are identical. Indeed, the coefficients $A_{m\gamma}(0, \alpha_0)$ entering in the expression (4.4) for $B_{jl\gamma}(0)$ are the cofactors of the elements of the determinat $D(0, \alpha_0)$, which is equal to zero. They can therefore be written in the form of a product,

$$A_{m\gamma}(0, \alpha_0) = \rho_m \sigma_\gamma, \tag{6.1}$$

where

$$\rho_m = A_{m1}(0, \alpha_0) / A_{11}(0, \alpha_0), \quad \sigma_\gamma = A_{1\gamma}(0, \alpha_0) / A_{11}(0, \alpha_0).$$

It follows from (6.1), (4.4), and (4.6) that the ratios

$$A_{m\gamma}(0, \alpha_0) / \sigma_\gamma, \qquad B_{jl\gamma}(0) / \sigma_\gamma, \qquad \varepsilon_\gamma / \sigma_\gamma, \qquad B_{jl\gamma}(0) / \varepsilon_\gamma$$

are independent of the index γ. The coefficients of Eq. (5.1), therefore, do not depend on the index

γ. Hence all the functions $\bar{\mu}_\gamma$ are identical in this approximation.

With this in mind, we rewrite Eq. (5.1) with the help of (4.6) and obtain

$$-1+\overline{\mu_\gamma^+}(ik-\delta)-\overline{\mu_\gamma^{+2}}\delta=-\overline{\mu_\gamma^-}(ik+\delta)+\overline{\mu_\gamma^{-2}}\delta,$$
$$\delta=D(\alpha)/D_1(\alpha)\approx\alpha-\alpha_0=\alpha-\tfrac{5}{6}. \tag{6.2}$$

The left-hand side of (6.2) is analytic and bounded in the upper half-plane, and the right-hand side is analytic and bounded in the lower half-plane. According to the Liouville theorem, each side of (6.2) is therefore equal to a constant:

$$-1+\overline{\mu_\gamma^+}(ik-\delta)-\overline{\mu_\gamma^{+2}}\delta=C,$$
$$-\overline{\mu_\gamma^-}(ik+\delta)+\overline{\mu_\gamma^{-2}}\delta=C. \tag{6.3}$$

The first of these relations can be rewritten in the form

$$-1-\mu_\gamma(+0)-\int_0^\infty e^{ikx'}[\delta\mu_\gamma(x')$$
$$+\mu'_\gamma(x')+\delta\mu^2_\gamma(x')]\,dx'=C.$$

The integral in this relation goes to zero as $k\to\pm\infty$. We therefore obtain

$$-1-\mu_\gamma(+0)=C,\qquad \mu'_\gamma+\delta\mu_\gamma+\delta\mu^2_\gamma=0,\qquad x>0.$$

In the same way we obtain from the second equation in (6.3)

$$-\mu_\gamma(-0)=C,\qquad \mu'_\gamma+\delta\mu_\gamma-\delta\mu^2_\gamma=0,\qquad x<0.$$

It follows from these relations that the function $\nu_\gamma(x)=\theta(x)+\mu_\gamma(x)$ is continuous at $x=0$. With the appropriate choice of the separation constant C we have then

$$\nu_\gamma(x)=\frac{1}{1+e^{-\delta x}}.$$

In terms of the dimensional variables this formula has the following form

$$\frac{n(x)-n_-}{n_-}=\frac{u(x)-u_-}{u_-}=\frac{T(x)-T_-}{T_-}$$
$$=\left[1+\exp\left(-x\,\frac{m(u_-^2-c_-^2)}{2kT_-\tau u_-}\right)\right]^{-1} \tag{6.4}$$

This solution satisfies the conditions at infinity only if $\delta>0$ or, which is the same thing, $u_->c_-$. Thus the relative velocity of the gas in front of the shock wave is larger than the velocity of sound.

The relation (6.4) coincides with the well known formula[6] describing the structure of the shock wave in the "approximation of the kinetic coefficients," i.e., under the assumption that the average velocity, density, and temperature change slowly in the diffuse front of the shock wave.

7. STRUCTURE OF THE WAVE AT LARGE DISTANCES

Let us now take account of the function $F_\gamma(k)$ standing on the right-hand side of Eq. (5.1). For this purpose we set

$$\mu_j(k)=\bar{\mu}_{j0}(k)+\bar{\rho}_j(k),$$

where $\bar{\mu}_{j_0}(k)$ are the approximate solutions of (5.1) obtained in Sec. 5, and $\bar{\rho}_j(k)$ are corrections due to the inclusion of right-hand side of (5.1). As has already been shown, the functions $\bar{\rho}_j(k)$ are considerably smaller than the $\bar{\mu}_{j_0}(k)$ for all real values of k. Therefore, the inclusion of the correction terms represents a surpassing of the accuracy at small distances. At large distances, on the other hand, the functions $\rho_j(x')$ play the major role, since the functions $\mu_{j_0}(x')$ go to zero very rapidly.

With this in mind, we calculate the function $\rho_j(x')$ only for large values of x'. To be definite, we consider the region in front of the shock wave ($x'<0$). Then

$$\mu_j(x')\approx\rho_j(x')=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{\rho}_j(k)\,e^{-ikx'}\,dk,\qquad |x'|\gg\delta^{-2}. \tag{7.1}$$

The value of this integral for large x' is determined by the behavior of the function $\bar{\rho}_j(k)$ near its lowest singular point in the upper half-plane. This point is the branch point $k=0$. Indeed, the integral (4.5) representing the function $D_{11}(k,\alpha)$ coincides in the left half-plane with one analytic function and in the right half-plane with another. All other functions entering in the expression for $F_\gamma(k)$ have the same character. Thus the function $F_\gamma(k)$ coincides in the right half-plane with one analytic function, $F_\gamma^+(k)$, and in the left half-plane with another, $F_\gamma^-(k)$.

The function $\bar{\rho}_\gamma^-(k)$ is analytic in the lower half-plane. With the help of (5.1), it can be continued through the positive half-axis into the right half-plane and through the negative half-axis into the left half-plane. Since the first continuation involves the function $F_\gamma^+(k)$ and the second continuation the function $F_\gamma^-(k)$, the values of $\bar{\rho}_\gamma^-(k)$ are different on the left and right sides of the imaginary axis. The difference between these values is equal to

$$\delta\overline{\rho_\gamma^-}(k)=F_\gamma^+(k)-F_\gamma^-(k).$$

Thus, if we deform the contour of integration in (7.1) in such a way as to make it run along the imaginary axis, we obtain

$$\mu_j(x') = \frac{1}{2\pi}\int_0^{i\infty} \delta\overline{\rho_j^-}(k)\, e^{-ikx'}\, dk, \qquad x' < 0, \qquad |x'| \gg \delta^{-2}. \tag{7.2}$$

Computing this integral by the method of steepest descent, we find

$$\mu_j(x') = \frac{25}{54\sqrt{3}} \frac{|x'|^{5/3}}{\alpha - \alpha_0} \exp\left[-\frac{3}{2}(2\alpha x'^2)^{1/3} - 2(\alpha^2 x')^{1/3} - \frac{4\alpha}{9}\right]. \tag{7.3}$$

In the dimensional variables this relation takes the form

$$\frac{n(x) - n_-}{n_-} = \frac{u(x) - u_-}{u_-} = \frac{T(x) - T_-}{T_-}$$
$$= \frac{25}{54\sqrt{3}}\left(\frac{x}{\tau u_-}\right)^{5/3} \frac{2kT_-}{m(u_-^2 - c_-^2)} \exp\left[-\frac{3}{2}\left(\frac{m}{kT_-}\frac{x^2}{\tau^2}\right)^{1/3}\right.$$
$$\left. - 2\left(\frac{m^2 u_-^3}{4k^2 T_-^2}\frac{x}{\tau}\right)^{1/3} - \frac{2}{9}\frac{mu_-^2}{kT_-}\right],$$
$$x < 0, \qquad |x/\tau c_-| \gg [2kT_-/m(u_-^2 - c_-^2)]^3. \tag{7.4}$$

Roughly speaking, the difference between the values of n, u, and T and their limiting values goes to zero with increasing |x| as exp(− const $|x|^{2/3}$), while the hydrodynamic theory leads to a decrease of the type exp(− const |x|).

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