THEORY OF THE INTERACTION OF A CHARGED PARTICLE WITH A PLASMA IN A MAGNETIC FIELD

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The interaction of a nonrelativistic charged particle with an electron plasma in a magnetic field is studied by quantum field theoretic methods. The dielectric constant, frequencies, and damping coefficients for the longitudinal oscillations of the plasma in the magnetic field are calculated to first order in e^2. A general formula is obtained for the energy loss of a particle moving through a plasma. The case in which the particle moves with a velocity much greater than the mean thermal velocity of the electrons in the plasma is investigated in detail.

1. It is the purpose of the present paper to study the interaction of a nonrelativistic charged particle with an electron plasma in a constant uniform magnetic field.

Because the Coulomb forces are long-range forces, the energy losses of a charged particle which moves through a plasma are determined by remote collisions as well as short-range collisions. The contribution of the short-range collisions is usually found by the pair-collision method; the effect of long-range collisions is generally investigated through the use of the kinetic equation in conjunction with a self-consistent field. In some cases the plasma is regarded as an optically active medium and is characterized by a dielectric constant. The energy losses of a particle moving through a plasma in a magnetic field have been computed by these methods by A. Akhiezer and Faĭnberg (cf. reference 1), Kolomenskii,[2] and Sitenko and Stepanov.[3] However, these methods do not give an accurate description of the interaction of particles which are separated by distances of the order of the Debye radius, so that the results are only of logarithmic accuracy.

In the present paper the energy losses of a particle which passes through a plasma in a magnetic field are computed by quantum field-theoretical methods. In this way we are able to obtain a general formula for the energy losses and, in limiting cases, to find the factors which appear in the argument of the logarithm.*

*Quantum field theoretical methods have been used to study the interaction of a particle with a plasma (no magnetic field) in a paper by Larkin.[4]

2. The Hamiltonian of a system of charged particles which interact through Coulomb interaction with an external particle which moves through the system can be written in the form $\mathcal{H} + \mathcal{H}'(t)$, where $\mathcal{H}$ is the basic Hamiltonian,

$$\mathcal{H}'(t) = \int J_0(\mathbf{r}, t)\, a_0(\mathbf{r}, t)\, d\mathbf{r},$$

$$a_0(\mathbf{r}, t) = (4\pi)^{-1} \int |\mathbf{r} - \mathbf{r}'|^{-1} j_0(\mathbf{r}', t) d\mathbf{r}', \qquad (1)$$

$\mathcal{H}'$ is the interaction Hamiltonian, a_0 is the scalar-potential operator and j_0 and J_0 are operators describing the charge density of the system and of the incident particle; the operators are taken in the interaction representation.

Using $\mathcal{H}'(t)$, we form the scattering matrix

$$S = T \exp\left\{- i \int_{-\infty}^{\infty} \mathcal{H}'(t)\, dt\right\} \ldots$$

and determine the matrix elements, which couple the different states of the initial system, i.e., the medium plus the external particle. We characterize these states by the quantum numbers α and n, where $\alpha \equiv (\nu, p_z, q)$ represents the quantum numbers characterizing the incident particle in the magnetic field H, while n represents the ensemble of quantum numbers which describe the state of the medium with a given energy E_n and a definite number of particles N_n.

We shall assume that the velocity of the incident particle V is large $(e^2V^{-1} \ll 1)$ so that its interaction with particles in the medium can be analyzed by perturbation theory. In the linear approximation in $\mathcal{H}'$, the probability for transition of the system from a state α, n to a state α', n' is given by

$$2\pi\delta(E_n - E_{n'} + \mathscr{E}_\alpha - \mathscr{E}_{\alpha'}) \int dr\, dr' \langle\alpha'|\hat{J}_0(r)|\alpha\rangle\langle\alpha|\hat{J}_0(r')|\alpha'\rangle \times \langle n'|\hat{a}_0(r)|n\rangle\langle n|\hat{a}_0(r')|n'\rangle, \quad (2)$$

where $\mathscr{E}_\alpha = \mathscr{E}_{\nu,p_z} = \eta(\nu + 1/2)m/M + p_z^2/2M$ is the energy of the incident particle, M is its mass and $\eta = eH/mc$ is the Larmor frequency of the electron in the magnetic field H ($\hat{J}_0$ and $\hat{a}_0$ are operators in the Schrödinger representation).

In order to obtain the total probability for transition of the particle from a state characterized by energy $\mathscr{E}_{\nu p_z}$ to a state with $\mathscr{E}_{\nu' p_z'}$, we sum Eq. (2) over final states of the medium and average over initial states using the density matrix $\rho_0 = \exp\{\beta(\Omega + \mu N - \mathscr{H})\}$ (β is the reciprocal temperature, Ω is the thermodynamic potential, N is the operator describing the number of particles, μ is the chemical potential); we also sum over all values of the variable q' and average over all values of the variable q. In terms of Fourier components we have

$$W_{\nu p_z, \nu' p_z'} = 2\pi \int dk\, \Phi(k, \mathscr{E}_{\nu p_z} - \mathscr{E}_{\nu' p_z'})\, U_{\nu p_z, \nu' p_z'}(k), \quad (3)$$

where $\Phi(\mathbf{k}, \omega)$ and $U_{\nu p_z, \nu' p'}(\mathbf{k})$ are the components of the Fourier functions

$$\Phi(r_1 - r_2; \omega) = \sum_{n, n'} \exp\{\beta(\Omega + \mu N_n - E_n)\} \langle n'|\hat{a}_0(r_1)|n\rangle \times \langle n|\hat{a}_0(r_2)|n'\rangle\, \delta(E_n - E_{n'} + \omega), \quad (4)$$

$$U_{\nu p_z, \nu' p_z'}(r_1 - r_2) = \sum_{q, q'} \langle\alpha'|\hat{J}_0(r_1)|\alpha\rangle\langle\alpha|\hat{J}_0(r_2)|\alpha'\rangle. \quad (5)$$

The energy lost by the particle per unit time is related to W by the expression

$$-\frac{d}{dt} E_{\nu p_z} = \sum_{\nu', p'_z} (\mathscr{E}_{\nu p_z} - \mathscr{E}_{\nu' p'_z})\, W_{\nu p_z, \nu' p'_z}. \quad (6)$$

3. We now calculate the function $U_{\nu p_z, \nu' p_z'}(\mathbf{k})$. In this calculation we make use of the expression for the matrix element of the operator $\hat{J}_0(\mathbf{r})$

$$\langle\alpha'|\hat{J}_0(r)|\alpha\rangle = e\Psi^*_{\alpha'}(r)\,\Psi_\alpha(r),$$

where the $\Psi_\alpha(\mathbf{r})$ are the wave functions for the particle in the magnetic field

$$\Psi_\alpha(r) = (2\pi)^{-1} \exp(ip_z z + iqy)\, \varphi_\nu(x - q/m\eta) \quad (7)$$

($\varphi_\nu(x)$ is the wave function describing an oscillator of frequency η).

Noting that

$$\int_{-\infty}^{\infty} d\xi \exp(i\xi m\eta y)\, \varphi_\nu\left(\xi + \frac{x}{2}\right)\varphi_\nu\left(\xi - \frac{x}{2}\right) = \exp\left\{-m\eta\frac{x^2 + y^2}{4}\right\} L_\nu\left(m\eta\frac{x^2 + y^2}{2}\right), \quad (8)$$

($L_\nu(x)$ is the Laguerre polynomial), we write Eq. (5) in the form

$$U_{\nu p_z, \nu' p_z'}(r) = e^2 \frac{(m\eta)^2}{(2\pi)^4} \exp\{i(p_z - p_z')z\} \times \exp\left\{-m\eta\frac{x^2 + y^2}{2}\right\} L_\nu\left(m\eta\frac{x^2 + y^2}{2}\right) L_{\nu'}\left(m\eta\frac{x^2 + y^2}{2}\right).$$

Taking the Fourier transform we have

$$U_{\nu p_z, \nu' p_z'}(k) = e^2 m\eta\, (2\pi)^{-2}\, \delta(p_z - p_z' - k_z)\, \Lambda_{\nu\nu'}(k_t/\sqrt{2m\eta}), \quad (9)$$

where $k_t = \sqrt{k_x^2 + k_y^2}$ and

$$\Lambda_{\nu\nu'}(a) = \int_0^\infty ds\, J_0(2a\sqrt{s})\, L_\nu(s)\, L_{\nu'}(s)\, e^{-s}. \quad (10)$$

In order to compute the function $\Phi(\mathbf{k}, \omega)$ we relate it to the retarded Green's function for a scalar photon in the medium

$$D(r_1 - r_2, t_1 - t_2) = -i\theta(t_1 - t_2)\, \mathrm{Sp}\{\rho_0 [a_0(r_1, t_1), a_0(r_2, t_2)]\}, \quad \theta(t) = 0;\ 1 \text{ for } t \leqslant 0. \quad (11)$$

For this purpose we use the spectral representation of the components of the Fourier function D

$$D(k, \omega) = \int_{-\infty}^{\infty} \frac{\rho(k, \omega')}{\omega' + \omega + i0} d\omega',$$

where

$$\rho(k, \omega) = \sum_{n, n'} [\exp\{\beta(\Omega + \mu N_n - E_n)\} - \exp\{\beta(\Omega + \mu N_{n'} - E_{n'})\}] \times \langle n|\hat{a}_0(k)|n'\rangle\langle n'|\hat{a}_0(k)|n\rangle\, \delta \times (E_n - E_{n'} - \mu N_n + \mu N_{n'} + \omega)$$
$$= (e^{\beta\omega} - 1) \sum_{n, n'} \exp\{\beta(\Omega + \mu N_{n'} - E_{n'}\} \langle n|\hat{a}_0(k)|n'\rangle \times \langle n'|\hat{a}_0(k)|n\rangle\, \delta(E_n - E_{n'} + \omega)$$

(we may note that the only nonvanishing matrix elements for the potential a_0 are those between states with the same number of particles: $N_n = N_{n'}$). Comparing this expression with Eq. (4) we obtain the following relation for Φ

$$\Phi(k, \omega) = (e^{-\beta\omega} - 1)^{-1} \rho(k, -\omega) = \frac{2}{2\pi}(1 - e^{-\beta\omega})^{-1}\, \mathrm{Im}\, D(k, \omega). \quad (12)$$

The function $D(\mathbf{k}, \omega)$ is the analytic continuation of the temperature Green's function $\mathscr{D}(\mathbf{k}, k_4)$:

$$\mathscr{D}(k, k_4) = \frac{1}{2} \int_{-\beta}^{\beta} d\tau\, \mathrm{Sp}\{\rho_0 T_\tau [a_0(k, \tau)\, a_0(k, 0)]\}\, e^{ik_4\tau} \quad (13)$$

[$a_0(\mathbf{k}, \tau) = e^{\tau(\mathscr{H} - \mu N)}\, \hat{a}_0(\mathbf{k})\, e^{-\tau(\mathscr{H} - \mu N)}$] and is related to the temperature polarization operator $\mathscr{P}$ by the expression

$$\mathscr{D}(\mathbf{k}, k_4) = -\{k^2 - \mathscr{P}(\mathbf{k}, k_4)\}^{-1}. \quad (14)$$

Introducing the dielectric constant $\epsilon(\mathbf{k}, \omega) = 1 + \kappa(\mathbf{k}, \omega)$, where

$$\varkappa(\mathbf{k}, \omega) = -k^{-2}\mathscr{P}(\mathbf{k}, i\omega - 0), \quad (15)$$

we write Eq. (12) in the form

$$\Phi(\mathbf{k}, \omega) = \frac{1}{\pi} k^{-2}(1 - e^{-\beta\omega})^{-1} \operatorname{Im} \frac{\varkappa(\mathbf{k}, \omega)}{1 + \varkappa(\mathbf{k}, \omega)}. \quad (16)$$

Substituting Eqs. (9) and (16) in Eq. (6), we obtain the following general formula for the energy loss of a charged particle which moves through a system of charged particles in a magnetic field:

$$-\frac{d}{dt} E_{\nu p_z} = \frac{2e^2 m\eta}{(2\pi)^2} \sum_{\nu'} \int_{-\infty}^{\infty} \frac{\omega d\omega}{1 - e^{-\beta\omega}} \int \frac{d\mathbf{k}}{k^2} \Lambda_{\nu\nu'}\left(\frac{k_t}{\sqrt{2m\eta}}\right)$$
$$\times \operatorname{Im} \frac{\varkappa(\mathbf{k}, \omega)}{1 + \varkappa(\mathbf{k}, \omega)} \delta(\mathscr{E}_{\nu' p_z} - \mathscr{E}_{\nu', p_z - k_z} - \omega). \quad (17)$$

4. We now compute the electric susceptibility κ of a plasma in a magnetic field. Equation (15) relates κ to the reduced polarization operator $\mathscr{P}(\mathbf{k}, k_4)$; this operator is calculated by means of graphs. In the first approximation in e^2

$$\mathscr{P}(\mathbf{r}_1 - \mathbf{r}_2; k_4) = \frac{2e^2}{\beta} \sum_{p_4} \mathscr{G}(\mathbf{r}_1, \mathbf{r}_2; p_4)\, \mathscr{G}(\mathbf{r}_2, \mathbf{r}_1; p_4 - k_4),$$
$$p_4 = \frac{2n+1}{\beta}\pi, \quad n = 0, \pm 1, \quad (18)$$

where the Green's function for the electron in the magnetic field $\mathscr{G}(\mathbf{r}_1, \mathbf{r}_2, p_4)$ is of the form

$$\mathscr{G}(\mathbf{r}_1, \mathbf{r}_2; p_4) = \sum_{\alpha} \Psi_\alpha(\mathbf{r}_1) \frac{1}{\varepsilon_\alpha - \mu + ip_4} \Psi^*_\alpha(\mathbf{r}_2), \quad (19)$$

$\epsilon_\alpha \equiv \epsilon_{\nu, p_z} = \eta(\nu + 1/2) + p_z^2/2m$ and the function $\Psi_\alpha(\mathbf{r})$ is given by (7) (we neglect the interaction between the electron spin and the magnetic field).

Using Eq. (8), we write Green's function for the electron in the form

$$\mathscr{G}(\mathbf{r}_1, \mathbf{r}_2; p_4) = \frac{m\eta}{(2\pi)^2} \exp\left\{im\eta \frac{x_1 + x_2}{2}(y_1 - y_2)\right\}$$
$$\times \sum_{\nu=0}^{\infty} \int_{-\infty}^{\infty} dp_z \exp\{ip_z(z_1 - z_2)\}$$
$$\times \exp\left\{-m\eta \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{4}\right\}$$
$$\times L_\nu\left\{m\eta \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{2}\right\} \frac{1}{\varepsilon_{\nu p_z} - \mu + ip_4}. \quad (20)$$

Substituting (20) in (18) and taking Fourier components we have

$$\mathscr{P}(\mathbf{k}, k_4) = \frac{2e^2 m\eta}{(2\pi)^2} \sum_{\nu, \nu'} \int_{-\infty}^{\infty} dp_z \Lambda_{\nu\nu'}\left(\frac{k_t}{\sqrt{2m\eta}}\right)$$
$$\times \frac{1}{\beta} \sum_{p_4} \frac{1}{\varepsilon_{\nu p_z} - \mu + ip_4} \frac{1}{\varepsilon_{\nu', p_z - k_z} - \mu + i(p_4 - k_4)}.$$

The summation over p_4 is carried out using the relation

$$\frac{1}{\beta} \sum_{p_4} (\varepsilon_\alpha - \mu + ip_4)^{-1} = \frac{1}{2} - n_\alpha,$$

where $n_\alpha = n_{\nu p_z} = [\exp\{\beta(\epsilon_{\nu p_z} - \mu)\} + 1]^{-1}$ is the Fermi distribution function. Taking account of the relation between κ and $\mathscr{P}$ (15), we obtain the final expression for the electric susceptibility of the electron gas in the magnetic field:

$$\varkappa(\mathbf{k}, \omega) = -\frac{2e^2 m\eta}{(2\pi)^2} \frac{1}{k^2}$$
$$\times \sum_{\nu, \nu'} \int_{-\infty}^{\infty} dp_z \Lambda_{\nu\nu'}\left(\frac{k_t}{\sqrt{2m\eta}}\right) \frac{n_{\nu p_z} - n_{\nu', p_z - k_z}}{\varepsilon_{\nu p_z} - \varepsilon_{\nu', p_z - k_z} - \omega - i0}. \quad (21)$$

In a weak magnetic field ($k_t^2/2m\eta \gg 1$ or $k_t\bar{p}/2m\eta \gg 1$; $\bar{p}$ is the mean momentum of the plasma electrons) the summation over ν and ν' is carried out by means of Eq. (A.2) of the Appendix. We obtain the following familiar expression for $\kappa(\mathbf{k}, \omega)$

$$\varkappa(\mathbf{k}, \omega) = -\frac{2e^2}{(2\pi)^3} \frac{1}{k^2} \int d\mathbf{p} \frac{n_\mathbf{p} - n_{\mathbf{p}-\mathbf{k}}}{\varepsilon_\mathbf{p} - \varepsilon_{\mathbf{p}-\mathbf{k}} - \omega - i0}, \quad (22)$$

where $\epsilon_p = p^2/2m$ and $n_p = [\exp\{\beta(\epsilon_p - \mu)\} + 1]^{-1}$.

In a strong magnetic field ($k_t^2/2m\eta \ll 1$ and $k_t\bar{p}/2m\eta \ll 1$) we use Eq. (A.2) of the Appendix to compute $\kappa(\mathbf{k}, \omega)$. In this case we have when $k \ll \bar{p}$

$$\operatorname{Re}\varkappa(\mathbf{k}, \omega) = -\frac{\Omega^2}{\omega^2}\cos^2\theta - \frac{\Omega^2}{\omega^2 - \eta^2}\sin^2\theta,$$
$$\operatorname{Im}\varkappa(\mathbf{k}, \omega) = \Omega^2 \frac{(2\pi m\beta)^{1/2}\omega}{4\pi k^3 \cos\theta}\left\{\exp\left\{-\frac{\beta m}{2}\left(\frac{\omega}{k\cos\theta}\right)^2\right\}\left(1 - \frac{k^2\sin^2\theta}{m\beta\eta^2}\right)\right.$$
$$+ \frac{k^2 \sin^2\theta}{2m\beta\eta^2}\left[\exp\left\{-\frac{\beta m}{2}\left(\frac{\omega - \eta}{k\cos\theta}\right)^2\right\}\right.$$
$$\left.\left.+ \exp\left\{-\frac{\beta m}{2}\left(\frac{\omega + \eta}{k\cos\theta}\right)^2\right\}\right]\right\}, \quad (23)$$

where Ω is the electron Langmuir frequency

$$\Omega^2 = \frac{e^2 N}{m}, \quad N = \frac{2m\eta}{(2\pi)^2} \sum_\nu \int_{-\infty}^{\infty} dp_z n_{\nu p_z}$$

and θ is the angle formed by the vector $\mathbf{k}$ and the magnetic field [the second equation in (23) is obtained under the assumption that the electron gas is nondegenerate: $\beta\mu \ll 1$; the first expression is obtained under the assumption that $\omega/k \gg \bar{p}/m$].

The poles of the function $D(\mathbf{k}, \omega)$ determine the dispersion relation for the characteristic longitudinal oscillations of the plasma. From (14) and (15) we have $D(\mathbf{k}, \omega) = -\{k^2\epsilon(\mathbf{k}, \omega)\}^{-1}$; thus we obtain the dispersion equation

$$\varepsilon(\mathbf{k}, \omega) \equiv 1 + \varkappa(\mathbf{k}, \omega) = 0. \quad (24)$$

The oscillation frequencies and damping factors $\omega = \omega_{1,2} + i\gamma_{1,2}$ are then

$$\omega_{1,2}^2 = \frac{1}{2}(\Omega^2 + \eta^2) \pm \frac{1}{2}\sqrt{(\Omega^2 + \eta^2)^2 - 4\Omega^2\eta^2\cos^2\theta},$$

$$\gamma_{1,2} = \Omega^2 \frac{(2\pi m\beta)^{3/2}\omega_{1,2}^2}{8\pi k^3\cos\theta}\left[1 + \frac{\Omega^2\eta^2\sin^2\theta}{(\omega_{1,2}^2 - \eta^2)^2}\right]^{-1}$$
$$\times\left\{\exp\left\{-\frac{\beta m}{2}\left(\frac{\omega_{1,2}}{k\cos\theta}\right)^2\right\} + \frac{k^2\sin^2\theta}{2m\beta\eta^2}\right.$$
$$\times\left[\exp\left\{-\frac{\beta m}{2}\left(\frac{\omega_{1,2} - \eta}{k\cos\theta}\right)^2\right\}\right.$$
$$\left.\left. + \exp\left\{-\frac{\beta m}{2}\left(\frac{\omega_{1,2} + \eta}{k\cos\theta}\right)^2\right\}\right]\right\}, \quad (25)$$

in agreement with the results obtained in references 5 and 6.

5. We now find the energy losses of a particle which moves through a nondegenerate plasma ($\beta\mu \ll 1$) with a velocity V which is much larger than the mean thermal velocity of the electrons in the plasma. If $\eta/\Omega \ll m^i/m$ (m^i is the ion mass), then the interaction of the particle with the ions in the plasma can be neglected. It will be assumed that the magnetic field satisfies the inequality $\beta\eta \ll 1$.

We divide the integral over **k** in Eq. (17) into two regions: $k < k_1$ and $k > k_1$, where k_1 is chosen so that $k_0 \ll k_1 \ll \bar{p}$ ($k_0 = \sqrt{e^2N\beta}$ is the reciprocal of the Debye radius). In calculating the contribution from short-range collisions ($k > k_1$) the curvature of the Larmor orbit of the incident particle can be described approximately by the momentum **p**. Using Eq. (A.2) we have

$$\left(-\frac{dE_{\mathbf{p}}}{dt}\right)_c = \frac{2e^2}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{\omega d\omega}{1 - e^{-\beta\omega}}$$
$$\times\int_{k_1}^{\infty} dk \int do_{\mathbf{k}}\, \mathrm{Im}\frac{\varkappa(\mathbf{k},\omega)}{1 + \varkappa(\mathbf{k},\omega)}\delta\left(\frac{p^2}{2M} - \frac{(\mathbf{p} - \mathbf{k})^2}{2M} - \omega\right).$$

(We assume that the unit volume in the number space ν, p_z corresponds to the volume $(2\pi m\eta)^{-1}$ in **p**-space).

The electrical susceptibility $\varkappa(\mathbf{k}, \omega)$ in this region of wave vectors being considered is given by Eq. (22); noting that $\varkappa$ is of order k_0^2/k^2, we can neglect this quantity compared with unity. Neglecting terms of order $\bar{p}/mV$, we have

$$\left(-\frac{dE_{\mathbf{p}}}{dt}\right)_c = \frac{e^2\Omega^2}{4\pi V}\ln\frac{2mMV}{k_1(M + m)}. \quad (26)$$

We now compute the contribution due to long-range collisions ($k < k_1$). If one of the following conditions is satisfied:

$$\sin\alpha \ll \frac{m}{M}\frac{\eta}{max\{\Omega, \eta\}} \quad \text{or} \quad \sin\alpha \gg \frac{m}{M}\frac{\eta}{max\{\Omega, \eta\}}$$

(α is the angle between the velocity of the incident particle and the magnetic field), then the state of the incident particle can be described approximately by the momentum **p**. (The first condition corresponds to motion of the particle along the magnetic field, while the second means that the curvature of the Larmor orbit of the particle is small). Using (A.1) and (A.2) we have

$$\left(-\frac{dE_{\mathbf{p}}}{dt}\right)_s = \frac{2e^2}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{\omega d\omega}{1 - e^{-\beta\omega}}$$
$$\times\int_0^{k_1} dk\int do_{\mathbf{k}}\,\mathrm{Im}\frac{\varkappa(\mathbf{k},\omega)}{1 + \varkappa(\mathbf{k},\omega)}\delta\left(\frac{\mathbf{pk}}{M} - \omega\right).$$

Noting that the main contribution in this integral is given by the circuit around the poles of the integrand lying in the lower half plane of the complex variable ω close to the points $\omega_{1,2}$ [cf. Eq. (25)] we have

$$-\left(\frac{dE_{\mathbf{p}}}{dt}\right)_s = \frac{e^2}{(2\pi)^2 V}\int_{-\infty}^{\infty}\frac{\omega d\omega}{1 - e^{-\beta\omega}}\int_0^{k_1}\frac{dk}{k}\int do_{\mathbf{k}}\,|\omega^2 - \eta^2|$$
$$\times\,\delta\left(\omega^2 - \eta^2 - \Omega^2 + \frac{\Omega^2\eta^2}{\omega^2}\cos^2\theta\right)$$
$$\times\,\delta\left(\cos\alpha\cos\theta + \sin\alpha\sin\theta\cos\varphi - \frac{\omega}{Vk}\right)$$

(θ and φ are the polar angles of the vector **k**). Finally, carrying out the integration over dk and $do_{\mathbf{k}}$, and neglecting terms of order $\bar{p}/mV$, we have

$$\left(-\frac{dE_{\mathbf{p}}}{dt}\right)_s = \frac{e^2\Omega^2}{4\pi V}\left\{f_1\left(\alpha, \frac{\Omega}{\eta}\right)\ln\frac{k_1V}{\eta} - f\left(\alpha, \frac{\Omega}{\eta}\right)\right\},$$

where

$$f_1(\alpha, u) = \frac{1}{\pi u^2}\left\{\int_0^{z_1} g(z)\,dz - \int_{z_2}^{z_3} g(z)\,dz\right\},$$

$$g(z) = \frac{z(1 - z)}{\sqrt{z(z - z_1)(z - z_2)(z_3 - z)}},$$

$$f(\alpha, u) = \frac{1}{\pi u^2}\left\{\int_0^{z_1} g(z)\ln z\,dz - \int_{z_2}^{z_3} g(z)\ln z\,dz\right\},$$

$$z_{1,2} = \frac{1 + u^2}{2} \mp \frac{1}{2}\sqrt{(1 + u^2)^2 - 4u^2\sin^2\alpha}, \qquad z_3 = 1 + u^2. \quad (27)$$

Since $f_1(\alpha, u) \equiv 1$ (cf. Sec. 3 of the Appendix), the energy loss of the particle due to long-range collisions is:

$$\left(-\frac{dE_{\mathbf{p}}}{dt}\right)_s = \frac{e^2\Omega^2}{4\pi V}\left\{\ln\frac{k_1V}{\eta} - f\left(\alpha, \frac{\Omega}{\eta}\right)\right\}. \quad (28)$$

Adding this expression to Eq. (26) for $(-dE_{\mathbf{p}}/dt)_c$ we have

$$-\frac{dE_{\mathbf{p}}}{dt} = \frac{e^2\Omega^2}{4\pi V}\left\{\ln\frac{2mMV^2}{(M + m)\eta} - f\left(\alpha, \frac{\Omega}{\eta}\right)\right\}. \quad (29)$$

6. If the particle moves along the magnetic field ($\sin\alpha \ll 1$) then $f(\alpha, u) = \ln\sqrt{1+u^2}$ and the total loss is given by

$$-\frac{dE_p}{dt} = \frac{e^2\Omega^2}{4\pi V}\ln\frac{2mMV^2}{(M+m)\sqrt{\Omega^2+\eta^2}}. \tag{30}$$

This formula has been obtained (to within logarithmic accuracy) by Sitenko and Stepanov.[3*]

For a strong magnetic field ($\eta \gg \Omega$)

$$f(\alpha, u) \to \frac{1}{4}\sin^2\alpha\left\{\frac{1}{4} + \ln\frac{u^2\sin^2\alpha}{4}\right\} \quad (u \ll 1),$$

and the energy-loss formula assumes the form

$$-\frac{dE_p}{dt} = \frac{e^2\Omega^2}{4\pi V}\left\{\ln\frac{2mMV^2}{(M+m)\eta} - \frac{\sin^2\alpha}{4}\ln\frac{\Omega^2\sin^2\alpha}{4\eta^2} - \frac{\sin^2\alpha}{4}\right\}. \tag{31}$$

If a heavy particle ($M \gg m$) moves perpendicularly to the magnetic field ($\alpha = \pi/2$), then

$$-\frac{dE_p}{dt} = \frac{e^2\Omega^2}{4\pi V}\left\{\ln\frac{4mV^2}{\sqrt{2\Omega\eta}} - \frac{1}{4}\right\}. \tag{32}$$

In the case of a weak magnetic field ($\eta \ll \Omega$) $f = (\alpha, \Omega/\eta) = \ln(\Omega/\eta)$ and

$$-\frac{dE_p}{dt} = \frac{e^2\Omega^2}{4\pi V}\ln\frac{2mMV^2}{(M+m)\Omega}. \tag{33}$$

This formula agrees with the energy loss formula obtained by Larkin.[4]

In conclusion the author wishes to thank A. I. Akhiezer for a discussion of this work.

APPENDIX

1. We compute the asymptote of the function $\Lambda_{\nu\nu'}(a)$ [cf. Eq. (10)] as $a \to 0$. Expanding the Bessel function $J_0(2a\sqrt{s})$ in a Taylor series we have

$$\Lambda_{\nu\nu'}(a) = \sum_{k=0}^{\infty}(-1)^k a^{2k}(k!)^{-2}\lambda_{\nu\nu'}^{(k)},$$

$$\lambda_{\nu\nu'}^{(k)} = \int_0^{\infty} ds\, s^k e^{-s} L_\nu(s) L_{\nu'}(s).$$

Taking account of the fact that

$$\int_0^{\infty} ds\, e^{-s} L_\nu(s) L_{\nu'}(s) = \delta_{\nu\nu'}$$

and using the recurrence formula for the Laguerre polynomials[8]

$$sL_\nu(s) = -(\nu+1)L_{\nu+1}(s) + (2\nu+1)L_\nu(s) - \nu L_{\nu-1}(s),$$

we can find the successive coefficients $\lambda_{\nu\nu'}^{(k)}$ and compute $\Lambda_{\nu\nu'}(a)$ to the required degree of accuracy. Keeping the first two terms we have

$$\Lambda_{\nu\nu'}(a) = \delta_{\nu\nu'} + a^2\{(\nu+1)\delta_{\nu+1,\nu'} - (2\nu+1)\delta_{\nu\nu'} + \nu\delta_{\nu-1,\nu'}\} \tag{A.1}$$

We may note that the expansion of the function $\Lambda_{\nu\nu'}(a)$ as $a \to 0$ actually reduces (for $\nu \neq 0$) to the quantity $a^2\nu$.

2. We now investigate a sum of the form

$$\sum_{\nu'}\Lambda_{\nu\nu'}\left(\frac{k_t}{\sqrt{2m\eta}}\right)F(\nu, \nu') \quad \text{for} \quad \frac{k_t^2(2\nu+1)}{2m\eta} \gg 1.$$

Using the asymptote for the Laguerre polynomials[8]

$$\exp\left(-\frac{s}{2}\right)L_\nu(s) = J_0(2\sqrt{s(\nu+{}^1\!/_2)}) + O(\nu^{-3/4}),$$

we have

$$\Lambda_{\nu\nu'}\left(\frac{k_t}{\sqrt{2m\eta}}\right) \to m\eta\int_0^{\infty}\rho\, d\rho\, J_0(k_t\rho)\, J_0(p_t\rho)\, J_0(p_t'\rho) = \frac{2m\eta}{2\pi}\frac{1}{\Delta},$$

where Δ is the area of the triangle formed by from the segments k_t, p_t and p_t'; if this triangle cannot be formed, then $\Delta^{-1} = 0$ [we use the notation $p_t^2 = 2m\eta(\nu + 1/2)$, $p_t'^2 = 2m\eta(\nu' + 1/2)$]. Noting that the quantity Δ^{-1} can be written in the form

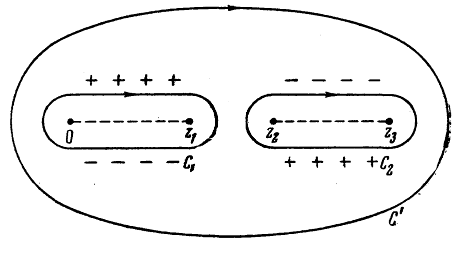
$$\Delta^{-1} = \frac{1}{2}\int_0^{2\pi} d\varphi\, \delta(\mathbf{p}_t' - \mathbf{p}_t + \mathbf{k}_t),$$

where φ is the angle between the plane vectors $\mathbf{p}_t'$ and $\mathbf{k}_t$, we have

$$\sum_{\nu'}\Lambda_{\nu\nu'}\left(\frac{k_t}{\sqrt{2m\eta}}\right)F(\nu, \nu') \to \frac{1}{2\pi}\int d\mathbf{p}_t'\delta(\mathbf{p}_t' - \mathbf{p}_t + \mathbf{k}_t)\, F(\nu, \nu')\Big|_{\substack{\nu+1/2 = p_t^2/2m\eta,\\ \nu'+1/2 = p_t'^2/2m\eta.}} \tag{A.2}$$

3. We compute the function $f_1(\alpha, u)$ [cf. Eq. (27)]. In the plane of the complex variable z we take a cut along the real axis along the segments $(0, z_1)$ and (z_2, z_3). The function $g(z)$ is a single-valued analytic function in the divided planes; we define this function such that $g(z) > 0$ along the upper edge of the cut $(0, z)$. The quantity $f_1(\alpha, u)$ can be written in the form

$$f_1(\alpha, u) = \frac{1}{2\pi u^2}\int_C [g(z) - i]\, dz,$$



*It has come to the author's attention that Eq. (30) has also been obtained by V. Gurevich and Firsov by quantum field-theoretical methods.[7]

where the contour C consists of the two closed contours C_1 and C_2 (cf. figure) and the integrand vanishes at infinity. Using Cauchy's theorem we replace the integration over the contour C by integration over the infinitely remote contour C′. Since the residue of the function $[g(z) - i]$ is $i[1 - (z_1 + z_2 + z_3)/2]$ at an infinitely remote point, we find $f_1(\alpha, u) = 1$.

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152