METHOD OF SUCCESSIVE EXTENSION OF THE SPECTRAL FUNCTIONS IN THE MANDELSTAM REPRESENTATION

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A method is developed which makes it possible to extend the range of knowledge of the spectral functions and absorption parts, by a series of successive steps, from a knowledge of the absorption part of the scattering amplitudes specified in the physical region. By means of this method, the NN scattering amplitude can be expressed in terms of the $\pi\pi$ and πN scattering amplitudes, specified only in the physical regions.

ALTHOUGH representations of the Mandelstam type for the scattering amplitude have not been proved, it is quite probable that they exist. These representations, together with the conditions of unitarity in the two-particle representation, are employed by us in the present work as the basis for an approximate calculation of the amplitude of nucleon-nucleon scattering.

In the two-particle approximation to the unitarity condition, the amplitude of NN scattering is connected with the amplitudes of $\pi\pi$ and πN scattering. The purpose of our research is the development of a semiphenomenological method of constructing the NN scattering amplitude from the values of the $\pi\pi$ and πN amplitudes in the physical regions.

The work consists of two parts. In Part I a summary is given of Mandelstam's equations[1,2] for all possible channels of $\pi\pi$, πN and NN scattering. For simplicity, the spin-charge variables are neglected and subtraction is not considered.

In Part II it is shown how one can systematically find the spectral functions of the Mandelstam representation in ever wider regions according to the absorption parts of the amplitude given in the physical regions. It is very important that these regions grow quickly and that the spectral functions can be computed exactly in that part of them in which it is not necessary to take into account the graphs of perturbation theory corresponding to inelastic processes over two channels.
It is further shown how one can express the amplitude of NN scattering in terms of the absorption parts of the $\pi\pi$ and πN scattering amplitudes in the physical regions, and in terms of the absorption part of the amplitude of nucleon-antinucleon annihilation into two π mesons. The latter can be expressed in terms of the first two absorption parts.[3,4]

I. THE MANDELSTAM EQUATIONS FOR THE SPECTRAL FUNCTIONS

1. Scattering of Pions on Pions

Let us consider the reactions

$$\pi(q_1) + \pi(q_2) \to \pi(q_3) + \pi(q_4), \qquad \text{(A.I)}$$

$$\pi(q_1) + \pi(-q_4) \to \pi(q_3) + \pi(-q_2), \qquad \text{(A.II)}$$

$$\pi(q_1) + \pi(-q_3) \to \pi(-q_2) + \pi(q_4) \qquad \text{(A.III)}$$

[π(q) denotes a π meson with four-dimensional momentum q] and introduce the relativistically invariant quantities [the squares of the energies in the center of mass system (c.m.s.) for these three reactions]

$$\sigma_1 = (q_1 + q_2)^2, \quad \sigma_2 = (q_1 - q_4)^2, \quad \sigma_3 = (q_1 - q_3)^2, \qquad (1)$$

which obviously satisfies the relation

$$\sigma_1 + \sigma_2 + \sigma_3 = 4\mu^2, \qquad (2)$$

where μ — rest mass of the π meson.

We write down the double spectral representation for the amplitude A of the reactions considered here:

$$A = \frac{1}{\pi^2} \sum_{1 \leqslant i < k}^{3} \int_{4\mu^2}^{\infty} d\sigma_i' \int_{4\mu^2}^{\infty} d\sigma_k' \frac{A_{ik}(\sigma_i', \sigma_k')}{(\sigma_i' - \sigma_i)(\sigma_k' - \sigma_k)}, \qquad (3)$$

where the real spectral functions $A_{ik}(x, y)$ $(1 \leq i < k \leq 3)$ satisfy the relations*

*Actually, and this will be proved in Sec. 4, the functions $A_{ik}(x, y)$ are equal to zero in a much broader region than shown in the limits of integration of (3).

$$A_{12}(x, y) = A_{13}(x, y) = A_{23}(x, y); \quad (4)$$

$$A_{ik}(x, y) = A_{ik}(y, x) \qquad (1 \leqslant i < k \leqslant 3). \quad (5)$$

Furthermore, let us introduce the absorption parts of the amplitude

$$A_1(\sigma_1, \sigma_3) = \frac{1}{\pi}\int d\sigma_2' \frac{A_{12}(\sigma_1, \sigma_2')}{\sigma_2' - \sigma_2} + \frac{1}{\pi}\int d\sigma_3' \frac{A_{13}(\sigma_1, \sigma_3')}{\sigma_3' - \sigma_3}, \quad (6)$$

$$A_2(\sigma_2, \sigma_3) = \frac{1}{\pi}\int d\sigma_1' \frac{A_{12}(\sigma_1', \sigma_2)}{\sigma_1' - \sigma_1} + \frac{1}{\pi}\int d\sigma_3' \frac{A_{23}(\sigma_2, \sigma_3')}{\sigma_3' - \sigma_3}, \quad (7)$$

$$A_3(\sigma_3, \sigma_1) = \frac{1}{\pi}\int d\sigma_1' \frac{A_{13}(\sigma_1', \sigma_3)}{\sigma_1' - \sigma_1} + \frac{1}{\pi}\int d\sigma_2' \frac{A_{23}(\sigma_2', \sigma_3)}{\sigma_2' - \sigma_2}, \quad (8)$$

where the integration in the expressions for $A_i(x, y)$ $(i = 1, 2, 3)$ is carried out for fixed x over all real values of the variable of integration, at which the corresponding spectral function differs from zero.

It is clear that

$$A_i(x, y) = 0, \text{ if } x < 4\mu^2, \quad (9)$$

$$A_1(x, y) = A_2(x, y) = A_3(x, y), \quad (10)$$

$$A_i(x, y) = A_i(x, 4\mu^2 - x - y). \quad (11)$$

For derivation of the set of Mandelstam equations, let us write the unitarity condition in the approximation of elastic scattering, for example, for the reaction (A.III):*

$$A_{13}^{(3)}(\xi, \eta) = -\frac{1}{4\pi^2\eta^{1/2}(\eta/4 - \mu^2)^{1/2}} \int\limits_{(\xi_1 < \xi)} dx dy \frac{A_1^*(x, \eta) A_1(y, \eta)}{[(\xi - \xi_1)(\xi - \xi_2)]^{1/2}}, \quad (12)$$

$$\xi_{1,2} = \xi_{1,2}(\eta; x, y) = x + y + \frac{2xy}{\eta - 4\mu^2} \pm \frac{2}{\eta - 4\mu^2} \times [x^2 + (\eta - 4\mu^2)x]^{1/2}[y^2 + (\eta - 4\mu^2)y]^{1/2}, \quad (13)$$

and integration is carried out over all real x and y for which the absorption parts differ from zero.

The amplitude $A_{13}^{(3)}(\sigma_1, \sigma_3)$ represents the part of $A_{13}(\sigma_1, \sigma_3)$ in which the perturbation-theory graphs corresponding to elastic scattering with energy σ_3 and inelastic scattering with energy σ_1 make a contribution, so that we have

$$A_{13}^{(3)}(\sigma_1, \sigma_3) = 0, \text{ if } \sigma_1 < 16\mu^2 \text{ or } \sigma_3 < 4\mu^2. \quad (14)$$

Introducing the notation

$$A_{13}^{(3)}(\sigma_1, \sigma_3) = \chi(\sigma_1, \sigma_3), \quad (15)$$

we have, successively,

$$A_{13}(\sigma_1, \sigma_3) = \chi(\sigma_1, \sigma_3) + \chi(\sigma_3, \sigma_1) + X_A(\sigma_1, \sigma_3), \quad (16)$$

where $\chi(\sigma_3, \sigma_1)$ is the part of $A_{13}(\sigma_1, \sigma_3)$ in which the perturbation-theory graphs corresponding to elastic scattering with energy σ_1, and inelastic scattering with energy σ_3 make a contribution, while $X_A(\sigma_1, \sigma_3)$ $[X_A(\sigma_1, \sigma_3) = X_A(\sigma_3, \sigma_1)]$ is that part of $A_{13}(\sigma_1, \sigma_3)$ in which the perturbation-theory graphs make a contribution, corresponding to inelastic scattering both with energy σ_1 and with energy σ_3, so that, finally,

$$X_A(\sigma_1, \sigma_3) = 0, \text{ if } \sigma_1 < 16\mu^2 \text{ or } \sigma_3 < 16\mu^2. \quad (17)$$

Then substituting (16) in (6) and taking (4) into account, we get

$$A_1(\sigma_1, \sigma_3) = \frac{1}{\pi}\int dx\, [\chi(x, \sigma_1) + \chi(\sigma_1, x)]\left(\frac{1}{x - \sigma_2} + \frac{1}{x - \sigma_3}\right) + A_1^{uu}(\sigma_1, \sigma_3), \quad (18)$$

where

$$A_1^{uu}(\sigma_1, \sigma_3) = \frac{1}{\pi}\int dx\, X_A(\sigma_1, x)\left(\frac{1}{x - \sigma_2} + \frac{1}{x - \sigma_3}\right). \quad (19)$$

Equations (12), (15), (18), and (19) represent the desired set of Maxwell's equations in the case of the scattering of π mesons on π mesons.

2. Scattering of Pions on Nucleons

Now consider the reactions

$$\pi(q_1) + N(p_1) \to \pi(q_2) + N(p_2), \quad \text{(B.I)}$$

$$\pi(-q_2) + N(p_1) \to \pi(-q_1) + N(p_2), \quad \text{(B.II)}$$

$$N(p_1) + \bar{N}(-p_2) \to \pi(-q_1) + \pi(q_2) \quad \text{(B.III)}$$

[N(p) and $\bar{N}(p')$ denote respectively a nucleon with four-dimensional momentum p and an antinucleon with four-dimensional momentum p′] and introduce the relativistically invariant quantities (the squares of the energies in the c.m.s. for these three reactions)

$$s = (q_1 + p_1)^2, \quad \bar{s} = (p_1 - q_2)^2, \quad u = (p_1 - p_2)^2, \quad (20)$$

which obviously satisfy the relation

$$s + s + u = 2(m^2 + \mu^2), \quad (21)$$

where m — rest mass of the nucleon.

We write down the double spectral representation for the amplitude B of the reactions considered here:*

*In view of the symmetry of the reactions of (A.I), (A.II), and (A.III) one need not consider the unitarity conditions for the first two reactions.

*Actually, as will be proved in Sec. 5, the functions $B_{ik}(x, y)$ are equal to zero over a much wider region.

$$B = \frac{g^2}{m^2 - s} + \frac{g^2}{m^2 - \bar{s}} + \frac{1}{\pi^2}\int\limits_{(m+\mu)^2}^{\infty} ds' \int\limits_{(m+\mu)^2}^{\infty} d\bar{s}' \frac{B_{12}(s', \bar{s}')}{(s' - s)(\bar{s}' - \bar{s})}$$

$$+ \frac{1}{\pi^2}\int\limits_{(m+\mu)^2}^{\infty} ds' \int\limits_{4\mu^2}^{\infty} du' \frac{B_{13}(s', u')}{(s' - s)(u' - u)}$$

$$+ \frac{1}{\pi^2}\int\limits_{(m+\mu)^2}^{\infty} d\bar{s}' \int\limits_{4\mu^2}^{\infty} du' \frac{B_{23}(\bar{s}', u)}{(\bar{s}' - \bar{s})(u' - u)}, \tag{22}$$

where g is the interaction constant of π mesons with nucleons, and the real spectral functions $B_{ik}(x, y)$ $(1 \le i < k \le 3)$ satisfy the relations

$$B_{12}(x, y) = B_{12}(y, x), \quad B_{13}(x, y) = B_{23}(x, y). \tag{23}$$

We further introduce the absorption parts of the amplitude

$$B_1(s, u) = \frac{1}{\pi}\int d\bar{s}' \frac{B_{12}(s, \bar{s}')}{\bar{s}' - \bar{s}} + \frac{1}{\pi}\int du' \frac{B_{13}(s, u')}{u' - u}, \tag{24}$$

$$B_2(\bar{s}, u) = \frac{1}{\pi}\int ds' \frac{B_{12}(s', \bar{s})}{s' - s} + \frac{1}{\pi}\int du' \frac{B_{23}(\bar{s}, u')}{u' - u}, \tag{25}$$

$$B_3(u, s) = \frac{1}{\pi}\int ds' \frac{B_{13}(s', u)}{s' - s} + \frac{1}{\pi}\int d\bar{s}' \frac{B_{23}(\bar{s}', u)}{\bar{s}' - \bar{s}}, \tag{26}$$

where integration in the expressions for $B_i(x, y)$ $(i = 1, 2, 3)$ is carried out for fixed x over all real values of the variable of integration for which the corresponding spectral function differs from zero.

It is clear that

$$B_i(x, y) = 0 \text{ for } x < (m+\mu)^2 \ (i = 1, 2);$$

$$B_3(x, y) = 0, \text{ for } x < 4\mu^2; \tag{27}$$

$$B_1(x, y) = B_2(x, y),$$

$$B_3(x, y) = B_3(x, 2(m^2 + \mu^2) - x - y). \tag{28}$$

For derivation of the set of Mandelstam equations, we write the condition for unitarity in the approximation of elastic scattering first for the reaction (B.I):

$$B_{13}^{(1)}(s, u) = -\frac{1}{8\pi^2 k\sqrt{s}}\left\{\int\limits_{(u_1 < u)} dxdy \frac{B_3^*(x, s)B_3(y, s)}{[(u - u_1)(u - u_2)]^{1/2}}\right.$$

$$\left. + \int\limits_{(u_3 < u)} dxdy \frac{\widetilde{B}_2^*(x, u(s, x))\widetilde{B}_2(y, u(s, y))}{[(u - u_3)(u - u_4)]^{1/2}}\right\}, \tag{29}$$

$$B_{12}^{(1)}(s, \bar{s}) = -\frac{1}{8\pi^2 k\sqrt{s}}\int\limits_{(\bar{s}_1 < \bar{s})} dxdy$$

$$\times \frac{\widetilde{B}_2^*(x, u(s, x))B_3(y, s) + \widetilde{B}_2(x, u(s, x))B_3^*(y, s)}{[(\bar{s} - \bar{s}_1)(\bar{s} - \bar{s}_2)]^{1/2}}, \tag{30}$$

where

$$u_{1,2} = u_{1,2}(s; x, y) = x + y + \frac{xy}{2k^2}$$

$$\pm \frac{1}{2k^2}(x^2 + 4k^2x)^{1/2}(y^2 + 4k^2y)^{1/2}, \tag{31}$$

$$u_{3,4} = u_{3,4}(s; x, y) = x + y + \frac{1}{2k^2}[xy - (m^2 - \mu^2)^2$$

$$+ (x + y - 2\alpha)(s - 4k^2 - 2\alpha) \pm \Delta(s, x)\,\Delta(s, y)], \tag{32}$$

$$\bar{s}_{1,2} = \bar{s}_{1,2}(s; x, y) = x + y + \frac{1}{2k^2}[xy + y(s - 4k^2 - 2\alpha)$$

$$\pm (y^2 + 4k^2y)^{1/2}\Delta(s, x)], \tag{33}$$

$$\Delta(s, x) = [(\alpha - x)^2 - 2(s - 2k^2 - 2\alpha)(\alpha - x)$$

$$+ 4\alpha k^2 + 4m^2\mu^2]^{1/2}, \quad \alpha = m^2 + \mu^2, \tag{34}$$

$$\widetilde{B}_2(\bar{s}, u) = \pi g^2\delta(\bar{s} - m^2) + B_2(\bar{s}, u),$$

$$u(s, \bar{s}) = 2(m^2 + \mu^2) - s - \bar{s}; \tag{35}$$

k is connected with s by the relation $s = [(k^2 + m^2)^{1/2} + (k^2 + \mu^2)^{1/2}]^2$, and the integration is carried out over all real x and y for which the corresponding absorption parts are different from zero.

$B_{13}^{(1)}(s, u)$ is the part of $B_{13}(s, u)$ in which the graphs of perturbation theory corresponding to elastic scattering with energy s and inelastic scattering with energy u make a contribution, so that we have

$$B_{13}^{(1)}(s, u) = 0, \text{ if } s < (m + \mu)^2 \text{ or } u < 16\mu^2. \tag{36}$$

$B_{12}^{(1)}(s, \bar{s})$ is the part of $B_{12}(s, \bar{s})$ in which the graphs of perturbation theory corresponding to elastic scattering with energy s and inelastic scattering with energy $\bar{s}$ make a contribution, so that we have

$$B_{12}^{(1)}(s, \bar{s}) = 0, \text{ if } s < (m + \mu)^2 \text{ or } \bar{s} < (m + 2\mu)^2. \tag{37}$$

We now write down the unitarity condition in the approximation of elastic scattering for the reaction (B.III):*

$$B_{13}^{(3)}(s, u) = -\frac{1}{4\pi^2 u^{1/2}(u/4 - \mu^2)^{1/2}}$$

$$\times \int\limits_{(s_1 < s)} dxdy \frac{\widetilde{B}_1(x, u)A_1(y, u)}{[(s - s_1)(s - s_2)]^{1/2}}, \tag{38}$$

where

$$s_{1,2} = s_{1,2}(u; x, y) = x + y + \frac{2}{u - 4\mu^2}\{y(x - m^2 + \mu^2)$$

$$\pm [(x - m^2 + \mu^2)^2 + (u - 4\mu^2)x]^{1/2}[y^2 + (u - 4\mu^2)y]^{1/2}\}; \tag{39}$$

$$\widetilde{B}_1(x, u) = \pi g^2\delta(x - m^2) + B_1(x, u), \tag{40}$$

and the integration is carried out over all real x and y for which the absorption parts differ from zero.

The amplitude of $B_{13}^{(3)}(s, u)$ is that part of $B_{13}(s, u)$ in which the graphs of perturbation theory corresponding to elastic scattering with energy u and inelastic scattering with energy s make a contribution such that

*In view of the symmetry of the reactions (B.I) and (B.II), we need not consider the unitarity condition for the reaction (B.II). We shall assume that the unitarity condition for the reaction (B.III) is valid also in the non-physical region of energy $4\mu^2 < u < 4m^2$.

$$B_{13}^{(3)}(s, u) = 0, \text{ if } s < (m+2\mu)^2 \text{ or } u < 4\mu^2. \quad (41)$$

Introducing the notation

$$B_{13}^{(1)}(s, u) = \psi_1(s, u), \quad B_{12}^{(1)}(s, \bar{s}) = \psi_2(s, \bar{s}), \quad B_{13}^{(3)}(s, u) = \psi_3(s, u), \quad (42)$$

we obtain

$$B_{13}(s, u) = \psi_1(s, u) + \psi_3(s, u) + X_B(s, u), \quad (43)$$

$$B_{12}(s, \bar{s}) = \psi_2(s, \bar{s}) + \psi_2(\bar{s}, s) + Y_B(s, \bar{s}). \quad (44)$$

Here $\psi_2(\bar{s}, s)$ is the part of $B_{12}(s, \bar{s})$ in which the graphs of perturbation theory corresponding to elastic scattering with energy $\bar{s}$ and inelastic scattering with energy s make a contribution; $X_B(s, u)$ is the part of $B_{13}(s, u)$ in which the graphs of perturbation theory corresponding to inelastic scattering with energy s and with energy u make a contribution; $Y_B(s, \bar{s})$ $[Y_B(s, \bar{s}) = Y_B(\bar{s}, s)]$ is the part of $B_{12}(s, \bar{s})$ in which the graphs of perturbation theory corresponding to inelastic scattering with energy s and with energy $\bar{s}$ make a contribution, so that certainly

$$X_B(s, u) = 0, \text{ if } s < (m+2\mu)^2 \text{ or } u < 16\mu^2, \quad (45)$$

$$Y_B(s, \bar{s}) = 0, \text{ if } s < (m+2\mu)^2 \text{ or } \bar{s} < (m+2\mu)^2. \quad (46)$$

Finally, substituting (43), (44) in (24), (26) and taking (23) into account, we obtain

$$B_1(s, u) = \frac{1}{\pi}\int d\bar{s}' \frac{\psi_2(s, \bar{s}') + \psi_2(\bar{s}', s)}{\bar{s}' - \bar{s}} + \frac{1}{\pi}\int du' \frac{\psi_1(s, u') + \psi_3(s, u')}{u' - u} + B_1^{uu}(s, u), \quad (47)$$

$$B_3(u, s) = \frac{1}{\pi}\int dx\,[\psi_1(x, u) + \psi_3(x, u)]\left(\frac{1}{x-s} + \frac{1}{x-\bar{s}}\right) + B_3^{uu}(u, s), \quad (48)$$

where

$$B_1^{uu}(s, u) = \frac{1}{\pi}\int d\bar{s}' \frac{Y_B(s, \bar{s}')}{\bar{s}' - \bar{s}} + \frac{1}{\pi}\int du' \frac{X_B(s, u')}{u' - u}, \quad (49)$$

$$B_3^{uu}(u, s) = \frac{1}{\pi}\int dx X_B(x, u)\left(\frac{1}{x-s} + \frac{1}{x-\bar{s}}\right). \quad (50)$$

Equations (29), (30), (38), and (47) — (50) are the desired set of Mandelstam equations in the case of the scattering of π mesons on nucleons.

3. Scattering of Nucleons on Nucleons

Finally, we shall consider the reactions

$$N(n_1) + N(p_1) \to N(n_2) + N(p_2), \quad \text{(C.I)}$$

$$N(n_1) + \bar{N}(-p_2) \to \bar{N}(-p_1) + N(n_2), \quad \text{(C.II)}$$

$$N(n_1) + \bar{N}(-n_2) \to \bar{N}(-p_1) + N(p_2) \quad \text{(C.III)}$$

and introduce the relativistically invariant quantities (the squares of the energy in the c.m.s. for the three reactions)

$$w = (n_1 + p_1)^2, \quad \bar{t} = (n_1 - p_2)^2, \quad t = (n_1 - n_2)^2, \quad (51)$$

which obviously satisfy the relation

$$w + \bar{t} + t = 4m^2. \quad (52)$$

We now write down the double spectral representation for the amplitude C of the reactions considered here:*

$$C = \frac{g^2}{\mu^2 - \bar{t}} + \frac{g^2}{\mu^2 - t} + \frac{1}{\pi^2}\int_{4m^2}^{\infty} dw' \int_{4\mu^2}^{\infty} d\bar{t}' \frac{C_{12}(w', \bar{t}')}{(w' - w)(\bar{t}' - \bar{t})} + \frac{1}{\pi^2}\int_{4m^2}^{\infty} dw' \int_{4\mu^2}^{\infty} dt' \frac{C_{13}(w', t')}{(w' - w)(t' - t)} + \frac{1}{\pi^2}\int_{4\mu^2}^{\infty} d\bar{t}' \int_{4\mu^2}^{\infty} dt' \frac{C_{23}(\bar{t}', t')}{(\bar{t}' - \bar{t})(t' - t)}, \quad (53)$$

where the real spectral functions $C_{ik}(x, y)$ $(1 \le i < k \le 3)$ satisfy the relations

$$C_{12}(x, y) = C_{13}(x, y), \quad C_{23}(x, y) = C_{23}(y, x). \quad (54)$$

We introduce the absorption parts of the amplitude

$$C_1(w, t) = \frac{1}{\pi}\int d\bar{t}' \frac{C_{12}(w, \bar{t}')}{\bar{t}' - \bar{t}} + \frac{1}{\pi}\int dt' \frac{C_{13}(w, t')}{t' - t}, \quad (55)$$

$$C_2(\bar{t}, w) = \frac{1}{\pi}\int dw' \frac{C_{12}(w', \bar{t})}{w' - w} + \frac{1}{\pi}\int dt' \frac{C_{23}(\bar{t}, t')}{t' - t}, \quad (56)$$

$$C_3(t, w) = \frac{1}{\pi}\int dw' \frac{C_{13}(w', t)}{w' - w} + \frac{1}{\pi}\int d\bar{t}' \frac{C_{23}(\bar{t}', t)}{\bar{t}' - \bar{t}}, \quad (57)$$

where the integration in the expression for $C_i(x, y)$ $(i = 1, 2, 3)$ is carried out for fixed x over all real values of the variable of integration for which the corresponding spectral function is different from zero.

It is clear that

$$C_1(x, y) = 0, \text{ if } x < 4m^2; \quad C_i(x, y) = 0, \text{ or } x < 4\mu^2 \ (i = 2, 3); \quad (58)$$

$$C_1(x, y) = C_1(x, 4m^2 - x - y); \quad C_2(x, y) = C_3(x, y). \quad (59)$$

For derivation of the set of Mandelstam equations, we write down the unitarity condition initially for the reaction (C.I) in the elastic scattering approximation:

$$C_{13}^{(1)}(w, t) = -\frac{1}{4\pi^2 w^{1/2}(w/4 - m^2)^{1/2}} \times \int_{(t_1 < t)} dx dy \frac{\tilde{C}_3^*(x, w)\tilde{C}_3(y, w)}{[(t - t_1)(t - t_2)]^{1/2}}, \quad (60)$$

$$t_{1,2} = t_{1,2}(w; x, y) = x + y + \frac{2}{w - 4m^2}\{xy \pm [x^2 + x(w - 4m^2)]^{1/2} \times [y^2 + y(w - 4m^2)]^{1/2}\}, \quad (61)$$

*In reality, as will be shown in Sec. 6, the functions $C_{ik}(x, y)$ $(1 \le i < k \le 3)$ are equal to zero in a wider region than shown in (53).

$$\widetilde{C}_3(t, w) = \pi g^2 \delta(t - \mu^2) + C_3(t, w), \tag{62}$$

and the integration is carried out over all real x and y for which the absorption parts are different from zero.

$C_{13}^{(1)}(w, t)$ consists of two components:

$$C_{13}^{(1)}(w, t) = \Lambda(w, t) + \Phi_1(w, t). \tag{63}$$

The first component corresponds to the case in which both variables of integration are equal to μ^2. It is clear that

$$\Lambda(w,t) = -(g^4/4) w^{-1/2}(w/4 - m^2)^{-1/2}[t - t_1(w; \mu^2, \mu^2)]^{-1/2} \times [t - t_2(w; \mu^2, \mu^2)]^{-1/2} \tag{64}$$

for those w and t for which $\Lambda(w, t) \neq 0$.* This component of $\Lambda(w, t)$ is the part of $C_{13}(w, t)$ in which the graphs of perturbation theory corresponding to elastic scattering with energy w and with energy t make a contribution.

The second component $\Phi_1(w, t)$ is the part of $C_{13}(w, t)$ in which the graphs of perturbation theory corresponding to elastic scattering with energy w and inelastic scattering with energy t make a contribution, so that we have

$$\Phi_1(w, t) = 0, \quad \text{if} \quad w < 4m^2 \quad \text{or} \quad t < 9\mu^2. \tag{65}$$

We write the unitarity condition now for the reaction (C.III) in the approximation of elastic scattering:†

$$C_{13}^{(3)}(w, t) = -\frac{1}{4\pi^2 t^{1/2}(t/4 - \mu^2)^{1/2}} \int_{(w_1 < w)} dx\, dy \frac{\widetilde{B}_1^*(x, t) \widetilde{B}_1(y, t)}{[(w - w_1)(w - w_2)]^{1/2}},$$

$$C_{23}^{(3)}(\xi, \eta) = C_{13}^{(3)}(\xi, \eta), \tag{66}$$

where

$$w_{1,2} = w_{1,2}(t, x, y) = x + y + \frac{2}{t - 4\mu^2}\{(x - m^2 + \mu^2)(y - m^2 + \mu^2) \pm [(x - m^2 + \mu^2)^2 + x(t - 4\mu^2)]^{1/2}[(y - m^2 + \mu^2)^2 + y(t - 4\mu^2)]^{1/2}\}, \tag{67}$$

and the integration is carried out over all real x and y for which the absorption parts differ from zero.

The amplitude $C_{13}^{(3)}(w, t)$ consists of two components:

$$C_{13}^{(3)}(w, t) = \Lambda'(w, t) + \Phi_2(w, t). \tag{68}$$

*The boundary of the region where $\Lambda(w, t) = 0$ will be established in Sec. 6, but naturally we have $\Lambda(w, t) = 0$ if $w < 4m^2$ or $t < 4\mu^2$.

†In view of the symmetry of the reactions (C.II) and (C.III), we need not consider the unitarity condition for the reaction (C.II). We shall assume that the unitarity condition for the reaction (C.III) is also valid even in the non-physical energy region $4\mu^2 < t < 4m^2$.

The first component corresponds to the case in which both variables of integration are equal to m^2. It is obvious that

$$\Lambda'(w,t) = -(g^4/4) t^{-1/2}(t/4 - \mu^2)^{-1/2}[w - w_1(t; m^2, m^2)]^{-1/2} \times [w - w_2(t; m^2, m^2)]^{-1/2} \tag{69}$$

for those w and t for which $\Lambda'(w, t) \neq 0$. It is easy to show, therefore, that

$$\Lambda'(w, t) = \Lambda(w, t). \tag{70}$$

The second component in (68) is the part of $C_{13}(w, t)$ in which the graphs of perturbation theory corresponding to elastic scattering with energy t and inelastic scattering with energy w make a contribution, such that we have

$$\Phi_2(w, t) = 0, \quad \text{if} \quad w < (2m + \mu)^2 \quad \text{or} \quad t < 4\mu^2. \tag{71}$$

Taking into account Eqs. (54), (63), (66), (68), and (70) we have, consequently,

$$C_{13}(w, t) = \Lambda(w, t) + \Phi_1(w, t) + \Phi_2(w, t) + X_C(w, t), \tag{72}$$

$$C_{23}(\bar{t}, t) = \Lambda(\bar{t}, t) + \Lambda(t, \bar{t}) + \Phi_2(\bar{t}, t) + \Phi_2(t, \bar{t}) + Y_C(\bar{t}, t), \tag{73}$$

where $X_C(w, t)$ is the part of $C_{13}(w, t)$ in which the graphs of perturbation theory corresponding to inelastic scattering with energy w and with energy t make a contribution, and $Y_C(\bar{t}, t)$ $[Y_C(\bar{t}, t) = Y_C(t, \bar{t})]$ is the part of $C_{23}(\bar{t}, t)$ in which the graphs of perturbation theory corresponding to inelastic scattering with energy $\bar{t}$ and with energy t make a contribution, so that we certainly have

$$X_C(w, t) = 0, \ \text{if} \ w < (2m + \mu)^2 \ \text{or} \ t < 9\mu^2, \tag{74}$$

$$Y_C(\bar{t}, t) = 0, \ \text{if} \ \bar{t} < 9\mu^2 \ \text{or} \ t < 9\mu^2. \tag{75}$$

Finally, substituting (72) and (73) in (57), we obtain

$$C_3(t, w) = C_3^e(t, w) + C_3^u(t, w) + C_3^{uu}(t, w), \tag{76}$$

where

$$C_3^e(t, w) = \frac{1}{\pi}\int dx\, [\Phi_2(x, t) + \Lambda(x, t)] \left(\frac{1}{x - w} + \frac{1}{x - \bar{t}}\right), \tag{77}$$

$$C_3^u(t, w) = \frac{1}{\pi}\int dx \left[\frac{\Phi_1(x, t)}{x - w} + \frac{\Phi_2(t, x) + \Lambda(t, x)}{x - \bar{t}}\right], \tag{78}$$

$$C_3^{uu}(t, w) = \frac{1}{\pi}\int dx \left[\frac{X_C(x, t)}{x - w} + \frac{Y_C(x, t)}{x - \bar{t}}\right]. \tag{79}$$

Equations (60), (66) and (76) — (79) are the desired set of Mandelstam equations in the case of scattering of nucleons on nucleons.

II. METHOD OF SUCCESSIVE EXTENSION OF THE RANGE OF KNOWLEDGE OF SPECTRAL FUNCTIONS

4. Scattering of Pions on Pions

We assume that the partial amplitudes $h_l(\nu^2)$ are given us for the reactions A.I), so that in its physical part

$$A_1(\sigma_1, \sigma_3) = \sum_l \operatorname{Im} h_l(\nu^2) P_l(\cos\chi), \quad (80)$$

where l is the angular momentum, and ν^2 and χ are the square of the momentum and the scattering angle for this reaction in the c.m.s., connected with the variables σ_1, σ_2, σ_3 by the relations

$$\sigma_1 = 4\mu^2 + 4\nu^2, \quad \sigma_2 = -2\nu^2(1 + \cos\chi), \quad \sigma_3 = -2\nu^2(1 - \cos\chi). \quad (81)$$

We continue analytically the expression on the right side of (80) in $\cos\chi$ for ν^2 = const, finding $A_1(\sigma_1, \sigma_3)$ in the entire region where

$$A_{13}(\sigma_1, \sigma_3) = 0, \quad A_{12}(\sigma_1, \sigma_2) = 0. \quad (82)$$

We shall call this region the region of zero approximation of the absorption part.

It is easy to find where the equations of (82) are satisfied. In view of (4), it is sufficient for us to establish where $A_{13}(\sigma_1, \sigma_3) = 0$ or where $\chi(\sigma_3, \sigma_1) + \chi(\sigma_1, \sigma_3) = 0$. But on the basis of (9), (12) and (13) it can be established that $\chi(\sigma_3, \sigma_1) = 0$ if $\sigma_3 \le \xi_1(\sigma_1; 4\mu^2, 4\mu^2)$. Therefore, $A_{13}(\sigma_1, \sigma_3) = 0$ in the region where

$$\sigma_3 \leqslant \xi_1(\sigma_1; 4\mu^2, 4\mu^2), \quad \sigma_1 \leqslant \xi_1(\sigma_3; 4\mu^2, 4\mu^2).$$

We shall denote the boundary of this region by $\sigma_1 = \sigma_1^{(0)}(\sigma_3)$ or $\sigma_3 = \sigma_1^{(0)}(\sigma_1)$.

Knowing the absorption part $A_1(\sigma_1, \sigma_3)$ in the region of its zero approximation, we can find $\chi(\sigma_3, \sigma_1)$ by Eq. (12) in the region where $\sigma_3 \le \xi_1(\sigma_1; \sigma_1^{(0)}(\sigma_1), 4\mu^2)$. Consequently, we can find the spectral function $\chi(\sigma_3, \sigma_1) + \chi(\sigma_1, \sigma_3)$ in the region where

$$\sigma_3 \leqslant \xi_1(\sigma_1; \sigma_1^{(0)}(\sigma_1), 4\mu^2), \quad \sigma_1 \leqslant \xi_1(\sigma_3; \sigma_1^{(0)}(\sigma_3), 4\mu^2).$$

We shall call this region the region of zero approximation of the spectral function $A_{13}(\sigma_1, \sigma_3)$, and shall denote its boundary by $\sigma_1 = \sigma_1^{(1)}(\sigma_3)$ or $\sigma_3 = \sigma_1^{(1)}(\sigma_1)$.

On the basis of (4), we can now find the spectral function $A_{12}(\sigma_1, \sigma_2)$ in the region of its zero approximation. The region of zero approximation of the spectral functions $A_{13}(\sigma_1, \sigma_3)$ and $A_{12}(\sigma_1, \sigma_2)$ of course includes within itself the region of zero approximation of the absorption part $A_1(\sigma_1, \sigma_3)$.

We shall show how we can now find $A_1(\sigma_1, \sigma_3)$ in the entire region of zero approximation of the spectral functions $A_{13}(\sigma_1, \sigma_3)$ and $A_{12}(\sigma_1, \sigma_2)$. The latter we shall therefore call the region of first approximation of the absorption part $A_1(\sigma_1, \sigma_3)$.

With this purpose, we introduce the auxiliary function $A_1'(\sigma_1, \sigma_3)$, which will appear on the right side of Eq. (6) if we set $A_{13}(\sigma_1, \sigma_3) = 0$ and $A_{12}(\sigma_1, \sigma_2) = 0$ outside the regions of zero approximation of the spectral functions $A_{13}(\sigma_1, \sigma_3)$ and $A_{12}(\sigma_1, \sigma_2)$, respectively. We shall consider the function $A_1'(\sigma_1, \sigma_3)$ to be known. Further, we shall consider the function $A_1(\sigma_1, \sigma_3) - A_1'(\sigma_1, \sigma_3)$. For σ_1 = const, this function is already analytic in σ_3 in the region of zero approximation of the spectral functions $A_{13}(\sigma_1, \sigma_3)$ and $A_{12}(\sigma_1, \sigma_2)$. Therefore, it can be obtained there by means of analytic continuation in $\cos\chi$ for ν^2 = const from the region of zero approximation of the absorption part $A_1(\sigma_1, \sigma_3)$. It can then be obtained in the region of zero approximation of the spectral functions $A_{13}(\sigma_1, \sigma_3)$, and $A_{12}(\sigma_1, \sigma_2)$ and the absorption part $A_1(\sigma_1, \sigma_3)$.

By extending the region in which the absorption part $A_1(\sigma_1, \sigma_3)$ is known, we can by Eq. (12) extend the region in which the spectral functions $A_{13}(\sigma_1, \sigma_3)$ and $A_{12}(\sigma_1, \sigma_2)$ are known which, in turn, makes it possible again to extend the region in which the absorption part $A_1(\sigma_1, \sigma_3)$ is known.

It is clear that the spectral function $A_{13}(\sigma_1, \sigma_3)$ can be found exactly by such a successive extension of the region of its knowledge only when $X_A(\sigma_1, \sigma_3) = 0$.

5. Scattering of Pions on Nucleons

We shall first assume that the partial amplitudes $f_l(k^2)$ are known to us for the reaction (B.I), so that in its physical region

$$B_1(s, u) = \sum_l \operatorname{Im} f_l(k^2) P_l(\cos\varphi), \quad (83)$$

where l is the angular momentum, and k^2 and φ are the squares of the momentum and scattering angles for this reaction in the c.m.s., connected with the variables s and u by the relations

$$s = [(k^2 + m^2)^{1/2} + (k^2 + \mu^2)^{1/2}]^2, \quad u = -2k^2(1 - \cos\varphi). \quad (84)$$

Second, we shall assume that the analytic continuations of the partial amplitudes $g_l(q^2)$ are given to us for the reaction (B.III) when $q^2 > 0$, so that we have for $q^2 > 0$ and $\cos\vartheta \approx 0$,*

*It will be established below for what values of $\cos\vartheta$ is (85) valid.

$$B_3(u, s) = \sum_l \operatorname{Im} g_l(q^2) P_l(\cos\vartheta), \quad (85)$$

where l is the angular momentum and q^2 and ϑ are the squares of the momentum of the meson and the scattering angle for this reaction in the c.m.s., connected with the variables s and u by the relations

$$s = m^2 - \mu^2 - 2q^2 + 2q(q^2 + \mu^2 - m^2)^{1/2}\cos\vartheta,$$
$$u = 4q^2 + 4\mu^2. \quad (86)$$

By continuing the expression on the right side of Eq. (83) analytically in $\cos\varphi$ for k^2 = const, and the expression on the right side of Eq. (85) in $\cos\vartheta$ for q^2 = const, we find $B_1(s,u)$ over the entire region where

$$B_{13}(s, u) = 0, \quad B_{12}(s, \bar{s}) = 0^* \quad (87)$$

and $B_3(u,s)$ over the entire region where

$$B_{13}(s, u) = 0, \quad B_{23}(\bar{s}, u) = 0. \quad (88)$$

We shall call these regions respectively the regions of zero approximation of the absorption part $B_1(s,u)$ and of the absorption part $B_3(u,s)$.

It is easy to find where Eqs. (87) and (88) are satisfied. On the basis of (27), (29) — (35), and (38) — (40) it can be proved that $\psi_1(s,u) = 0$ if $u \le u_1(s; 4\mu^2, 4\mu^2)$ [here, as is easy to show, $u \le u_3(s; m^2, m^2)$] $\psi_2(s,\bar{s}) = 0$ if $\bar{s} \le \bar{s}_1(s; m^2, 4\mu^2)$; $\psi_3(s,u) = 0$ if $s \le s_1(u; m^2, 4\mu^2)$. Therefore, $B_{13}(s,u) = 0$ in the region where

$$u \leqslant u_1(s;\ 4\mu^2,\ 4\mu^2), \qquad s \leqslant s_1(u;\ m^2,\ 4\mu^2),$$

and $B_{12}(s,u) = 0$ in the region where

$$\bar{s} \leqslant \bar{s}_1(s;\ m^2,\ 4\mu^2), \qquad s \leqslant \bar{s}_1(\bar{s};\ m^2,\ 4\mu^2).$$

The boundaries of these regions will be denoted by $s = s^{(0)}(u)$ or $u = u^{(0)}(s)$ and by $\bar{s} = \bar{s}^{(0)}(s)$ or $s = s^{(0)}(\bar{s})$.

Knowing the absorption parts $B_1(s,u)$, $B_3(u,s)$, $B_2(\bar{s},u)$ [the latter by virtue of (28)] in the region of their zero approximation, and the absorption part $A_1(y,u)$ in the region of its n-th approximation ($n \ge 0$), we can by Eqs. (29), (30) and (38) find $\psi_1(s,u)$ in the region where $u \le u_1(s; u^{(0)}(s), 4\mu^2)$ [here, as is easy to show, $u \le u_3(s; m^2, m^2)$ all the same]; $\psi_2(s,\bar{s})$ — in the region where $\bar{s} \le \bar{s}_1(s; s^{(0)}(s), 4\mu^2)$ and $\bar{s} \le \bar{s}_1(s; m^2, u^{(0)}(s))$, and $\psi_3(s,u)$ — in the region where $s \le s_1(u; s^{(0)}(u), 4\mu^2)$, and $s \le s_1(u; m^2, \sigma_1^{(n)}(u))$. Consequently, we can find the spectral functions $\psi_1(s,u) + \psi_3(s,u)$ in the region where

$$u \leqslant u_1(s;\ u^{(0)}(s),\ 4\mu^2),\ s \leqslant s_1(u;\ s^{(0)}(u),\ 4\mu^2),$$
$$s \leqslant s_1(u;\ m^2,\ \sigma^{(n)}(u)),$$

and the spectral function $\psi_2(s,\bar{s}) + \psi_2(\bar{s},s)$ in the region where

$$\bar{s} \leqslant \bar{s}_1(s;\ \bar{s}^{(0)}(s),\ 4\mu^2), \qquad \bar{s} \leqslant \bar{s}_1(s;\ m^2,\ u^{(0)}(s)),$$
$$s \leqslant \bar{s}_1(\bar{s};\ \bar{s}^{(0)}(\bar{s}),\ 4\mu^2), \qquad s \leqslant \bar{s}_1(\bar{s};\ m^2,\ u^{(0)}(\bar{s})).$$

We shall call these regions respectively the regions of zero approximation of the spectral function $B_{13}(s,u)$ and the spectral function $B_{12}(s,\bar{s})$, and the boundaries of them we shall denote by $s = s^{(1)}(u)$ or $u = u^{(1)}(s)$ and by $\bar{s} = \bar{s}^{(1)}(s)$ or $s = s^{(1)}(\bar{s})$.

On the basis of (23), we can now find the spectral function $B_{23}(\bar{s},u)$ in the region of its zero approximation. The region of zero approximation of the spectral functions $B_{13}(s,u)$ and $B_{12}(s,\bar{s})$ naturally contain within themselves the region of the zero approximation of the absorption part $B_1(s,u)$, while the region of zero approximation of the spectral functions $B_{13}(s,u)$ and $B_{23}(\bar{s},u)$ contain within themselves the region of zero approximation of the absorption part $B_3(u,s)$.

Continuing in the same fashion as in the case of the scattering of π mesons on π mesons, we can now find $B_1(s,u)$ throughout the entire region of zero approximation of the spectral functions $B_{13}(s,u)$ and $B_{12}(s,\bar{s})$, and also $B_3(u,s)$ throughout the entire region of the zero approximation of the spectral functions $B_{13}(s,u)$ and $B_{23}(\bar{s},u)$. We shall therefore call these latter the regions of first approximation of the absorption part $B_1(s,u)$ and the absorption part $B_3(u,s)$, respectively.

Extending the region in which the absorption parts $B_1(s,u)$ and $B_3(u,s)$ are known, we can, by Eqs. (29), (30), and (38), extend the region in which the spectral functions $B_{13}(s,u)$, $B_{12}(s,\bar{s})$ and $B_{23}(\bar{s},u)$ are known; in turn this makes it possible again to extend the region in which the absorption parts $B_1(s,u)$ and $B_3(u,s)$ are known.

It is clear that the spectral functions $B_{13}(s,u)$ and $B_{12}(s,\bar{s})$ can be found exactly in such a stepwise extension of the region of their knowledge only when $X_B(s,u) = 0$ and $Y_B(s,\bar{s}) = 0$.

6. Scattering of Nucleons on Nucleons

Making use of the unitarity condition for the reaction (C.III), we shall have for $4\mu^2 < t < 9\mu^{2}$* and $\cos\psi \approx 0$:†

*And only in the part of this region which is symmetric in $\cos\varphi$, if $B_1(s,u)$ is analytically continued in the form of an expansion in Legendre polynomials.

*We again assume the validity of the unitarity condition for the reaction (C.III) in the non-physical region.

†It will be established below for what values of $\cos\psi$ is (89) valid.

$$C_3(t, w) = C_3^e(t, w) = \frac{q}{(q^2+\mu^2)^{1/2}} \sum_l |g_l(q^2)|^2 P_l(\cos\psi), \quad (89)$$

where q^2 and ψ are the squares of the momentum of the meson and the angle of scattering for this reaction in the c.m.s., which are connected with the variables w and t by the relations

$$w = -2(q^2 + \mu^2 - m^2)(1 + \cos\psi), \quad t = 4q^2 + 4\mu^2. \quad (90)$$

Analytically continuing the expression on the right hand side of Eq. (89) in $\cos\psi$ for $q^2 = \text{const}$, we find $C_3(t, w) = C_3^e(t, w)$ over the entire region where $4\mu^2 < t < 9\mu^2$ and where

$$C_{13}(w, t) = 0, \quad C_{23}(\bar{t}, t) = 0. \quad (91)$$

We shall call this region the region of zero approximation for the absorption part $C_3(t, w)$.

It is easy to establish where the equations of (91) are satisfied. On the basis of (58) and (60) — (63) it can be established that $\Lambda(w, t) = 0$ if $t \le t_1(w; \mu^2, \mu^2)$. Therefore, $C_{13}(w, t) = 0$ in the region where $t \le t_1(w; \mu^2, \mu^2)$, while $C_{23}(\bar{t}, t) = 0$ in the region in which $t \le t_1(\bar{t}; \mu^2, \mu^2)$ and $\bar{t} \le t_1(t; \mu^2, \mu^2)$.

Knowing the absorption parts $B_1(s, t)$ in the region of its n-th approximation ($n \ge 0$), we can find $\Phi_2(w, t)$ from the first of Eqs. (66) in the region in which $w \le w_1(t; s^{(n)}(t), m^2)$. The functions $\Phi_2(w, t)$ can thus be assumed to be given in the case of scattering of nucleons on nucleons. But then we can find the spectral function $C_{13}(w, t)$ in the region where $4\mu^2 < t < 9\mu^2$ and $w \le w_1(t; s^{(n)}(t), m^2)$, and the spectral function $C_{23}(\bar{t}, t)$ in the region where $4\mu^2 < t < 9\mu^2$ and $\bar{t} \le w_1(t; s^{(n)}(t), m^2)$. These regions, which are identical for sufficiently high n with the region $4\mu^2 < t < 9\mu^2$, we shall call the regions of zero approximation of the spectral function $C_{13}(w, t)$ and the spectral function $C_{23}(\bar{t}, t)$, respectively. These regions of course include the region of zero approximation of the absorption part $C_3(t, w)$.

Proceeding as above, we can now find $C_3(t, w)$ throughout the entire region of the zero approximation of the spectral functions $C_{13}(w, t)$ and $C_{23}(\bar{t}, t)$. We shall also call the latter the region of first approximation of the absorption part $C_3(t, w)$.

Knowing the absorption part $C_3(t, w)$ for sufficiently large n in the region of its first approximation, we can, by Eq. (60), find $\Phi_1(w, t)$ throughout the entire region where $9\mu^2 < t < 16\mu^2$. We shall call this region the region of first approximation of the spectral functions $C_{13}(w, t)$ and $C_{23}(\bar{t}, t)$.

Making use of Eqs. (77) and (78), we can now find $C_3^e(t, w)$ and $C_3^u(t, w)$ in the region of first approximation of spectral functions $C_{13}(w, t)$ and $C_{23}(\bar{t}, t)$. We shall also call the latter the region of second approximation of the absorption part $C_3(t, w)$.

Extending the region in which the absorption part $C_3(t, w)$ is known, we can by Eq. (60) extend the region in which the spectral functions $C_{13}(w, t)$ and $C_{23}(\bar{t}, t)$ are known; in turn, this again enables us to extend the region in which the absorption part $C_3(t, w)$ is known.

It is clear that the spectral functions $C_{13}(w, t)$ and $C_{23}(\bar{t}, t)$ can be found exactly in such a stepwise extension of the region of their knowledge only when $X_C(w, t) = 0$ and $Y_C(\bar{t}, t) = 0$.

7. Discussion

By slightly altering the method set forth in Sec. 4, we can proceed to construct integral equations for the amplitudes of the reactions A, B, and C in their physical regions.

Assuming also that the absorption part $C_1(w, t)$ is known in the physical region of the reaction (C.I), and proceeding just as in the case of $\pi\pi$ and πN scattering, we could have successively found the values for the absorption parts of the reactions A, B, and C over wider and wider regions by means of a knowledge of them in the physical regions. In this case, we could have completely discarded the unknown parts of the spectral functions X and Y, restricting ourselves to the problem of determining that part of the amplitude which does not contain graphs which are inelastic in both channels. Then expressing the amplitude in terms of the absorption part, we could have obtained the integral equations for the amplitudes in the physical regions.

Such equations require simplification for practical use. In particular, rewriting the equations of Mandelstam for a fixed angle of scattering for any reaction, and then transforming to the partial amplitudes, we can easily obtain an equation similar to that given by Cini and Fubini.[5] The series of additional terms appearing in this case both on the right and the left of the cut make the approximation of Cini and Fubini doubtful. The obtaining of more convincing estimates appears possible to us in our approach to the construction of integral equations; these estimates are connected with the neglect of isolated components. These problems, however, go beyond the framework of the present paper.

Keeping in mind the calculation which is now under way of a real case in which spin and charge variables are present, we shall make a concluding remark relative to the capability of inclusion of the phases of nucleon-antinucleon annihilation into two π mesons. It appears most advantageous to us to connect them with the phases of $\pi\pi$ and πN scattering by means of a solution by the method of Muskhelishvili[6] of an integral equation for the amplitude of πN scattering for a fixed angle of scattering of the reaction (B.III). Thus one can choose the phases of the nucleon-antinucleon annihilation into two π mesons from the phase of $\pi\pi$ scattering and from the absorption part $B_1(s, u)$. The latter can be expressed through the pole term* and through the phase shifts of $\pi\pi$ and πN scattering.

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*Only this pole term was considered by Mandelstam.[4]

[1] S. Mandelstam, Phys. Rev. **112**, 1844 (1958).
[2] K. A. Ter-Martirosyan, JETP **39**, 827 (1960), Soviet Phys. JETP **12**, 575 (1961).
[3] Efremov, Meshcheryakov, and Shirkov, Equations for Meson-nucleon Scattering at Low Energies, Preprint, Joint Institute for Nuclear Research, 1960.
[4] S. Mandelstam, Phys. Rev. Lett. **4**, 84 (1960).
[5] M. Cini and S. Fubini, Theory of Low Energy Scattering in Field Theory, CERN preprint, 1960.
[6] N. I. Muskhelishvili, Сингулярные интегральные уравнения, (Singular Integral Equations), Gostekhizdat, 1946.

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