POSITIONS OF THE SINGULARITIES OF CERTAIN FEYNMAN DIAGRAMS

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The position of the singularity is determined for diagrams that describe the scattering of two particles.

IN studying the scattering amplitude for two particles, one needs to investigate the distribution of the singularities of the Feynman diagrams. The purpose of the present paper is the determination of the positions of the singular points of certain Feynman diagrams. The position of the singular point of a diagram can be determined by the solution of the system of Landau equations, which have been investigated in a number of papers.[1-4]

Suppose we have a certain Feynman diagram (Fig. 1) with four free lines. Its singular points are determined by the system of Landau equations:

$$\sum q = p, \tag{1}$$

$$\sum q = 0, \tag{2}$$

$$\sum \alpha q = 0, \tag{3}$$

$$q^2 = m^2, \tag{4}$$

$$\sum \alpha = 1, \quad \alpha > 0, \tag{5}$$

where q are the internal four-momenta, p are the external four-momenta, m are the masses of the internal particles, and α are the Feynman parameters.

Equations (1) and (2) express the conservation of four-momentum at the external and internal vertices of the diagram.

The solution of the system of equations for a symmetrical diagram has a remarkable property: if a replacement $q_i \to \pm q_{i'}$ takes the diagram into itself, then

$$\alpha_i = \alpha_{i'}, \quad q_i q_k = \pm q_{i'} q_{k'}, \quad q_i q_{k'} = \pm q_{i'} q_k. \tag{6}$$

The sign of the scalar product is determined by the directions of the vectors in the diagram. As has been shown in a paper by Okun' and Rudik,[2] for diagrams with four free lines one still has to determine the values of the squares of the external four-momenta,

$$p_1^2 = M_1^2, \quad p_2^2 = M_2^2, \quad p_3^2 = M_3^2, \quad p_4^2 = M_4^2, \tag{7}$$

after which one can determine the connection between $W^2 = (p_1 + p_2)^2$ and $Q^2 = (p_1 + p_3)^2$ at the singular point.

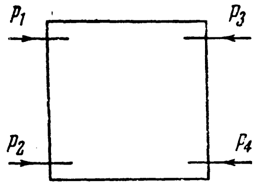

FIG. 1

Let us consider the diagram of Fig. 2. It is symmetrical with respect to the two transformations

$$q_1 \leftrightarrow -q_1, \quad q_2 \leftrightarrow -q_4, \quad q_5 \leftrightarrow q_6, \quad q_7 \leftrightarrow q_8, \quad q_3 \leftrightarrow -q_3;$$
$$q_1 \leftrightarrow -q_3, \quad q_2 \leftrightarrow -q_2, \quad q_4 \leftrightarrow -q_4, \quad q_5 \leftrightarrow q_7, \quad q_6 \leftrightarrow q_8.$$

Therefore $\alpha_1 = \alpha_3 = \alpha$, $\alpha_2 = \alpha_4 = \beta$, $\alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = 1$. The last relations can be written if we omit Eq. (5), as we shall do.

The equations (3) have the form

$$\alpha q_1 = q_5 - q_6, \qquad \beta q_4 = q_7 - q_5. \tag{8}$$

Since $q_1^2 = q_3^2 = p_1^2 = p_2^2 = p_3^2 = p_4^2 = \mu^2$ and $q_2^2 = q_4^2 = q_5^2 = q_6^2 = q_7^2 = q_8^2 = 1$, we can get from Eq. (8) the values of the scalar products by taking the squares of the equations: $q_{56} = 1 - \frac{1}{2}\alpha^2\mu^2$, $q_{57} = 1 - \frac{1}{2}\beta^2$, where $q_{ik} \equiv q_i q_k$.

If we multiply the equations (8) respectively by q_1 and q_4 and use the relations $q_{15} = -q_{16}$, $q_{47} = -q_{45}$, which follow from the symmetry of the diagram, we get $q_{15} = \frac{1}{2}\alpha\mu^2$, $q_{47} = \frac{1}{2}\beta$.

To determine the quantity q_{58} we use the equation of conservation of four-momentum at the internal vertex, which we multiply by q_5:

$$q_{58} = -1 - q_{56} - q_{57} = -3 + \tfrac{1}{2}(\beta^2 + \alpha^2\mu^2).$$

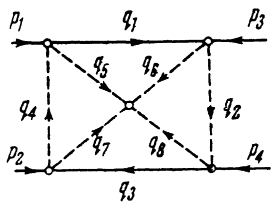

FIG. 2

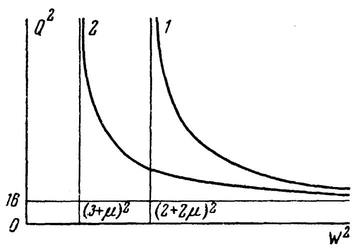

FIG. 3

The other quantities q_{ik} are determined from the equations obtained by multiplying (8) by various four-vectors q:

$$q_{18} = (2\beta^2 + \alpha^2\mu^2 - 8)/2\alpha, \quad q_{13} = (2\beta^2 + \alpha^2\mu^2 - 8)/\alpha^2,$$
$$q_{48} = (8 - 2\alpha^2\mu^2 - \beta^2)/2\beta, \quad q_{24} = (\beta^2 + 2\alpha^2\mu^2 - 8)/\beta^2,$$
$$q_{14} = (4 - \alpha^2\mu^2 - \beta^2)/\alpha\beta.$$

If we substitute the values of q_{14}, q_{15}, q_{45} in the equation $p_1^2 = \mu^2 = (q_1 + q_5 - q_4)^2$, we get a relation connecting α and β:

$$(2 + \alpha)\beta = 4 - \alpha^2\mu^2.$$

It is now easy to find the dependence of $W^2 = (p_1 + p_2)^2$ and $Q^2 = (p_1 + p_3)^2$ on α:

$$W^2 = \alpha^{-1}(2 + \alpha\mu^2)[8 + 2\alpha - (\alpha\mu)^2],$$
$$Q^2 = (8 + 2\alpha - \alpha^2\mu^2)^2/(4 - \alpha^2\mu^2), \tag{9}$$

where α varies from 0 to $2/\mu$, which follows from the fact that α and β are positive. The dependence of Q^2 on W^2 is shown in Fig. 3 (curve 1). If $\mu = 1$, the middle point $Q^2 = W^2 = 27$ is obtained for $\alpha = 1$.

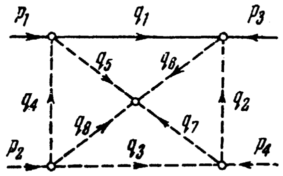

FIG. 4

Let us now study the singularities of the diagram shown in Fig. 4. It is invariant with respect to only one transformation:

$$q_1 \leftrightarrow -q_1, \quad q_2 \leftrightarrow q_4, \quad q_5 \leftrightarrow q_6, \quad q_7 \leftrightarrow q_8, \quad q_3 \leftrightarrow -q_3,$$

from which we have $\alpha_5 = \alpha_6 = \alpha$, $\alpha_7 = \alpha_8 = \beta$, $\alpha_2 = \alpha_4 = 1$. If we set $\alpha_3 = \gamma\beta$, then after some slight manipulation we can rewrite (3) as

$$q_4 = \beta q_8 - \alpha q_5, \quad \gamma q_3 = q_8 - q_7, \quad \gamma q_1 = \mu(q_5 - q_6). \tag{10}$$

The further solution of the system of equations is analogous to the previous case, and therefore we shall not present it. The answer is:

$$W^2 = (3 + \mu)^2 + (4 + 4\mu + 4\mu\gamma + \mu\gamma^2)\left(\frac{2}{\gamma} - 1\right),$$
$$Q^2 = \left(\frac{2+\gamma}{2-\gamma}\right)\left(\frac{4 + 4\mu + 2\mu\gamma - \mu\gamma^2}{1 + \mu + \mu\gamma}\right)^2. \tag{11}$$

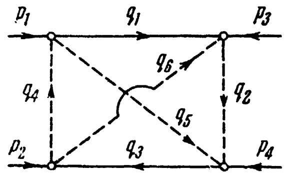

FIG. 5

From the condition that α is positive we find that γ must vary over the range from 0 to 2. For $\mu = 1$ Eq. (11) coincides with Eq. (9). The relation between W^2 and Q^2 is shown in Fig. 3 (curve 2).

Let us now go on to the diagram of Fig. 5, which admits two symmetry transformations:

$$q_1 \leftrightarrow -q_1, \quad q_4 \leftrightarrow -q_2, \quad q_3 \leftrightarrow -q_3, \quad q_4 \leftrightarrow -q_6;$$
$$q_2 \leftrightarrow -q_3, \quad q_4 \leftrightarrow -q_4, \quad q_2 \leftrightarrow -q_2, \quad q_5 \leftrightarrow q_6.$$

From these we have $\alpha_1 = \alpha_3 = \alpha$, $\alpha_2 = \alpha_4 = \beta$, $\alpha_5 = \alpha_6 = 1$. The relations (3) now reduce to a single equation:

$$\alpha q_1 + \beta q_2 = q_5. \tag{12}$$

If we multiply this equation by the various q_i, then by using the symmetry conditions we can find all the q_{ik} from the relations so obtained. Substituting the q_{ik} in W^2 and Q^2, we have

$$W^2 = 2(1 + 1/\alpha)^2(1 + \alpha^2\mu^2 - \beta^2),$$
$$Q^2 = 2(1 + 1/\beta)^2(1 - \alpha^2\mu^2 + \beta^2). \tag{13}$$

The equation that connects α and β is a cubic:

$$2\alpha\beta + \alpha^2\mu^2 + \beta^2 = 1 + \beta(\beta^2 - \alpha^2\mu^2 - 1) + \alpha(\alpha^2\mu^2 - \beta^2 - 1).$$

If we divide this equation by $(1 + \beta)^3$ and write

$$x = \alpha\mu/(1 + \beta), \; y = (1 - \beta)/(1 + \beta)$$

we get the simpler equation

$$\mu y^2 = (1 + \mu x - x^2)x.$$

Taking into account the fact that α is positive, we find that x and y can vary only over the ranges $x \geq 0$ and $|y| \leq 1$. The plot of the relation between Q^2 and W^2 is shown in Fig. 6.

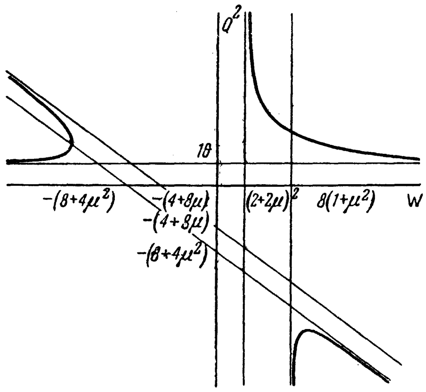

FIG. 6

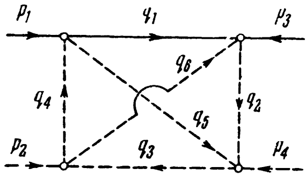

FIG. 7

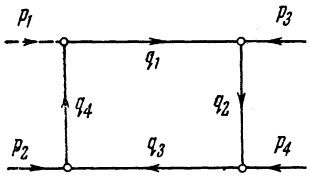

FIG. 8

Let us consider the diagram of Fig. 7. It has one symmetry transformation:

$$q_1 \leftrightarrow -q_1, \qquad q_3 \leftrightarrow -q_3, \qquad q_4 \leftrightarrow q_2, \qquad q_5 \leftrightarrow q_6$$

and consequently $\alpha_5 = \alpha_6 = 1$, $\alpha_2 = \alpha_4 = \beta$.

The equations (3) are as follows:

$$\beta q_4 + \alpha_1 q_1 = q_6, \quad \beta q_4 + \alpha_3 q_3 = -q_5.$$

Solving this system of equations by the method presented above, we find that α_1, α_3, and β are connected by the relations

$$\alpha_1^2\mu^2(1+\beta) + \alpha_1[(1+\beta)^2 - \alpha_3^2] = (1+\beta)(1-\beta)^2,$$
$$\alpha_3^2(1+\beta) + \alpha_3[(1+\beta)^2 - \alpha_1^2\mu^2] = (1+\beta)(1-\beta)^2.$$

These equations become much simpler if we replace α_1, α_3, and β by variables x, y, z defined by

$$x = \alpha_1\mu/(1+\beta), \quad y = (1-\beta)/(1+\beta),$$
$$z = \alpha_3/(1+\beta).$$

We have

$$\mu x^2 + x(1-z^2) = \mu y^2, \; z^2 + z(1-x^2) = y^2.$$

From this we easily find that

$$z = \frac{1+\mu x}{\mu + x}\, x, \qquad y^2 = \frac{(\mu + 2x - x^3)(1+\mu x)\, x}{(\mu + x)^2}.$$

The condition that α be positive requires that $x \geq 0$, $z \geq 0$, $-1 \leq y \leq 1$. We can now find the relation between Q^2 and W^2:

$$W^2 = \left(\mu + \frac{2x}{1+y}\right)^2 + \frac{2y}{xz}\left(\mu + \frac{2x}{1+y}\right)\left(1 + \frac{2z}{1+y}\right)$$
$$+ \left(1 + \frac{2z}{1+y}\right)^2, \qquad Q^2 = 16\,\frac{1 + y^2 - x^2 - z^2}{(1-y^2)^2}. \tag{14}$$

The curve (14) is of the same nature as the curve (13), of course with different values of the asymptotes.

We note a distinguishing feature of these diagrams: they have for the asymptotes not only lines on which W^2 or Q^2 is equal to the sum of masses of intermediate particles, but also other lines.

Let us now consider diagrams for which singularities exist only under definite conditions, for example a square in which the masses of all the particles are equal to 1, except that there is one external particle with the mass μ (Fig. 8). This diagram has no symmetry transformations. If we set $W^2 = 2 + 2F$ and $Q^2 = 2 + 2\Phi$, then F and Φ are connected by the relation

$$4F(\Phi^2 - 1) = (3 - \mu^2)(\Phi + 1)$$
$$\pm 4\sqrt{(\Phi+1)(\Phi - 0.5)(\Phi - \Phi_1)(\Phi - \Phi_2)},$$
$$4\Phi_1 = \mu^2 - 2 + \sqrt{3\mu^2(4 - \mu^2)},$$
$$4\Phi_2 = \mu^2 - 2 - \sqrt{3\mu^2(4 - \mu^2)}.$$

For what values of μ does the diagram have a singular point? From the condition that the Feynman parameters α be positive it follows that for $\mu^2 > 4$ there is no singular point, and for $\mu^2 < 4$ there is one. In Fig. 9 curve 1 shows the relation

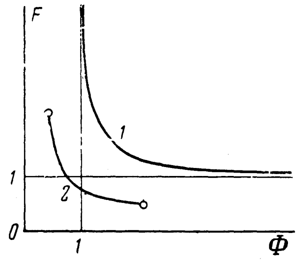

FIG. 9

between Q^2 and W^2 for $\mu^2 \leq 3$, and curve 2 for $3 < \mu^2 < 4$.

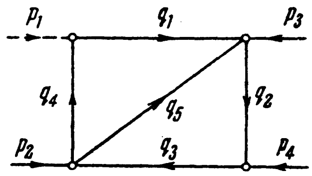

FIG. 10

Let us now consider the same square, but with a diagonal (Fig. 10). Before seeking the equations for W^2 and Q^2, let us find the equations that connect the α_i. We have

$$\alpha_1^2 + \alpha_4^2 + (2 - \mu^2)\alpha_1\alpha_4 = 1,$$
$$\alpha_2^2 + \alpha_3^2 + \alpha_2\alpha_3 = 1, \qquad \alpha_1 + \alpha_2 = \alpha_3 + \alpha_4. \tag{15}$$

It follows from the equations (15) that $(4 - \mu^2)\alpha_1\alpha_4 = 3\alpha_2\alpha_3$, but since $\alpha > 0$, we must have $\mu^2 < 4$, which is indeed the condition for the existence of a singular point. The expressions for W^2 and Q^2 in terms of the α_i are:

$$W^2 = 1 + (1 + \alpha_1 + \alpha_4)(1 + \alpha_2 + \alpha_3)(1 + \alpha_3 - \alpha_2)\alpha_1^{-1}\alpha_3^{-1},$$
$$Q^2 = 1 + (1 + \alpha_1 + \alpha_4)(1 + \alpha_2 + \alpha_3)(1 - \alpha_3 + \alpha_2)\alpha_2^{-1}\alpha_4^{-1}.$$

The curve for this relation is shown in Fig. 11.

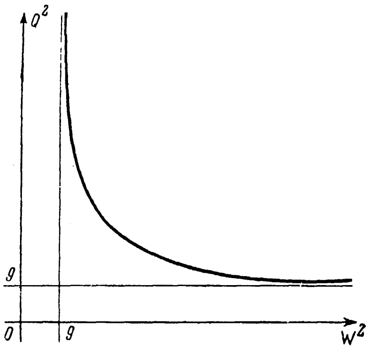

FIG. 11

The next diagram differs from that just considered in having the diagonal through different vertices (Fig. 12). We again investigate the conditions under which there is a solution with positive α. The system of equations for the α_i is more complicated:

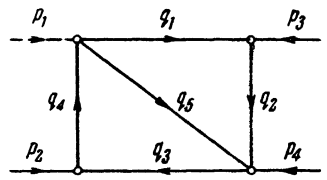

FIG. 12

$$\alpha_1^2 + \alpha_2^2 + \alpha_1\alpha_2 = 1, \quad \alpha_3^2 + \alpha_4^2 + \alpha_3\alpha_4 = 1,$$

$$(1 + \alpha_1 + \alpha_2)(1 + \alpha_3 + \alpha_4)(\alpha_1 - \alpha_2 + \alpha_4 - \alpha_3) = (\mu^2 - 1)\,\alpha_1\alpha_4. \quad (16)$$

It is convenient to go over to new variables x and y: $\alpha_2 = \alpha_1 x - 1, \quad \alpha_3 = \alpha_4 y - 1.$

Then instead of Eqs. (16) we get equations for x and y:

$$\frac{2 + 2x - x^2}{1 + x + x^2} + \frac{2 + 2y - y^2}{1 + y + y^2} = \frac{\mu^2 - 1}{(1 + x)(1 + y)},$$

$$\alpha_1 = \frac{1 + 2x}{1 + x + x^2}, \quad \alpha_2 = \frac{x^2 - 1}{1 + x + x^2},$$

$$\alpha_3 = \frac{y^2 - 1}{1 + y + y^2}, \quad \alpha_4 = \frac{1 + 2y}{1 + y + y^2}. \quad (17)$$

The condition for the α_i to be positive requires that x, $y \geq 1$. With such x and y a solution of (17) exists only under the condition that $\mu^2 \leq 9$. Consequently, this diagram (Fig. 12) can have a singular point for $\mu^2 \leq 9$. The dependence of Q^2 on W^2 is of the form shown by curve 1 in Fig. 9 for $\mu^2 < 7$, and by curve 2 for $\mu^2 > 7$.

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