ON THE PROBLEM OF COMPUTING MOMENTS OF INERTIA OF NUCLEI

S. T. BELYAEV

Submitted to JETP editor, September 15, 1960

J. Exptl. Theoret. Phys. (U.S.S.R.) **40**, 672-675 (February, 1961)

By using the generalized canonical transformation method, an expression is obtained for the moment of inertia of a nucleus which takes nucleon pairing into account. The result is essentially the same as that obtained previously by Migdal using the Green's function method.

THE moment of inertia of a deformed nucleus, when we take into account superfluidity (i.e., the Cooper pairing of nucleons), was calculated earlier[1] within the framework of adiabatic perturbation theory. As was shown by Migdal[2] using the Green's function method, including the rotational energy by means of perturbation theory is not sufficient, since the rotation changes the Cooper pairs and thus gives rise to an additional term in the moment of inertia. In the present paper we shall show that a consistent application of the canonical transformation method leads to this same result.

The nucleons in the deformed self-consistent field are described by the Hamiltonian

$$H = \sum_{\nu_1}(\varepsilon_{\nu_1} - \lambda)\, a^+_{\nu_1} a_{\nu_1} - \frac{1}{2}\sum \langle 12\,|\,G\,|\,2'1'\rangle\, a^+_1 a^+_2 a_{2'} a_{1'}, \quad (1)$$

where $a^+_{\nu_1}(a_{\nu_1})$ are the operators for creation (annihilation) of a nucleon in the state $\nu_1 \equiv 1$; $\epsilon_{\nu_1} \equiv \epsilon_1$ is the energy of this single-particle state, and λ is the chemical potential for the system.

The nucleon pairing can be taken into account by introducing quasiparticles by means of the canonical transformation

$$a_\nu = u_\nu \alpha_\nu + v_\nu \alpha^+_{\tilde\nu}, \quad \alpha_\nu = u_\nu a_\nu - v_\nu a^+_{\tilde\nu}, \quad (2)$$

where $\tilde\nu$ denotes the state which is the time reversed state to ν (and has the same energy),* while the coefficients of the transformation satisfy the conditions

$$u_{\tilde\nu} = u_\nu, \quad v_{\tilde\nu} = -v_\nu, \quad u^2_\nu + v^2_\nu = 1 \quad (3)$$

and are chosen so that the terms with $\alpha\alpha$ and $\alpha^+\alpha^+$ in the Hamiltonian go to zero after the transformation. It is known that this requirement is equivalent to a minimization of the quasiparticle vacuum state (defined by $\alpha\Psi_0 = 0$):

$$\Psi_0 = \prod_{|\nu|}(u_\nu + v_\nu a^+_\nu a^+_{\tilde\nu})\,|\,0\rangle. \quad (4)$$

*If $|\nu\rangle = |nljm\rangle$, $|\tilde\nu\rangle = (-1)^{j+m}|nlj-m\rangle$.

Calculation of average values using the function (4) is equivalent to averaging independently over pairs of operators,* for example

$$\langle a^+_1 a^+_2 a_{2'} a_{1'}\rangle_0 = \langle a^+_1 a_{1'}\rangle_0 \langle a^+_2 a_{2'}\rangle_0 - \langle a^+_1 a_{2'}\rangle_0 \langle a^+_2 a_{1'}\rangle_0 + \langle a^+_1 a^+_2\rangle_0 \langle a_{2'} a_{1'}\rangle_0, \quad (5)$$

where only the "diagonal" expressions

$$\langle a^+_1 a_1\rangle_0 = v^2_1, \quad \langle a^+_1 a^+_{\tilde 1}\rangle_0 = \langle a_{\tilde 1} a_1\rangle_0 = u_1 v_1 \quad (6)$$

are different from zero.

To find the moment of inertia, we look for the lowest state of the system with a fixed projection of the angular momentum on an axis perpendicular to the nuclear axis. To do this, we add to the Hamiltonian the term

$$H_\omega = -\omega \hat J_x = -\omega \Sigma \langle 1\,|\,j_x\,|\,2\rangle\, a^+_1 a_2, \quad (7)$$

and then determine the Lagrange multiplier ω from the condition $\langle \hat J_x\rangle = J_x$.

When the term H_ω is added to the Hamiltonian, the selection rule forbidding production of pairs of quasiparticles $\alpha^+\alpha^+$ is violated, and cannot be reestablished by simply changing the coefficients u_ν and v_ν in (2). To eliminate the terms in $\alpha\alpha$ and $\alpha^+\alpha^+$ in the total Hamiltonian $H' = H + H_\omega$, we make a further canonical transformation of more general form:[3]

$$\alpha_\nu \rightarrow \alpha_\nu(\omega) + \sum_{\nu'} f_{\nu\nu'} \alpha^+_{\tilde\nu'}(\omega). \quad (8)$$

The coefficients $f_{\nu\nu'}$ are related to the rotational term H_ω ($f_{\nu\nu'} \sim \omega$). They may be assumed to be small, and are included only in the first nonvanishing order. From the condition for the transformation (8) to be canonical, and from the symmetry with respect to change in sign of ω, it follows that

$$f_{\nu\nu'} + f_{\tilde\nu'\tilde\nu} = 0, \quad f_{\nu\nu} = 0. \quad (9)$$

For these new quasiparticles $\alpha_\nu(\omega)$ (corrected for rotation), the vacuum state differs from (4) and,

*Equation (5) shows that the canonical transformation method is equivalent to the generalized Hartree-Fock method.[3]

as we find from (8), it has the form

$$\Psi_\omega=\exp\left\{\frac{1}{2}\sum_{\nu\nu'} f_{\nu\nu'}\alpha_\nu^+\alpha_{\tilde{\nu}'}^+\right\}\Psi_0$$

$$=\exp\left\{\frac{1}{2}\sum_{\nu\nu'} f_{\nu\nu'}(u_\nu a_\nu^+ - v_\nu a_{\tilde{\nu}})(u_{\nu'}a_{\tilde{\nu}'}^+ + v_{\nu'}a_{\nu'})\right\}\Psi_0. \quad (10)$$

We note that the function (10) is not normalized:

$$(\Psi_\omega,\Psi_\omega)=\exp\left(\frac{1}{2}\sum_{\nu\nu'}|f_{\nu\nu'}|^2\right).$$

To determine the coefficients $f_{\nu\nu'}$, we demand that the average value of the total Hamiltonian H′ be a minimum in the state (10). Relation (5) is still valid for averages calculated with the function (10), but in addition to the diagonal quantities

$$\langle a_1^+ a_1\rangle = v_1^2 + (u_1^2 - v_1^2)\sum_2 |f_{12}|^2,$$

$$\langle a_1^+ a_{\tilde{1}}^+\rangle = \langle a_{\tilde{1}} a_1\rangle = u_1 v_1 - 2u_1 v_1 \sum_2 |f_{12}|^2, \quad (11)$$

the nondiagonal expressions

$$\langle a_1^+ a_2\rangle = (u_1 v_2 - v_1 u_2) f_{12}^*,$$

$$\langle a_1^+ a_{\tilde{2}}^+\rangle = \langle a_{\tilde{2}} a_1\rangle^* = (u_1 u_2 + v_1 v_2) f_{12}^* \quad (12)$$

are also different from zero. In (11) and (12) we have used the relation

$$f_{12}^* + f_{21} = 0, \quad (13)$$

which can be verified by direct computation.

Calculating the average values of H and H_ω using (11) and (12), we get

$$\langle H\rangle = W_0 + \frac{1}{2}\sum_{12}(E_1+E_2)|f_{12}|^2$$

$$-\frac{1}{2}\sum\langle 1\tilde{2}|G|\tilde{2}'1'\rangle(u_1u_2+v_1v_2)(u_{1'}u_{2'}+v_{1'}v_{2'})f_{12}^* f_{1'2'}$$

$$+\frac{1}{2}\sum(\langle 1\tilde{1}'|G|\tilde{2}'2\rangle$$

$$-\langle 1\tilde{1}'|G|2\tilde{2}'\rangle)(u_1v_2-v_1u_2)(u_{1'}v_{2'}-v_{1'}u_{2'})f_{12}^* f_{1'2'}, \quad (14)$$

$$\langle H_\omega\rangle = -\omega\langle \hat{J}_x\rangle = -\omega\sum\langle 1|j_x|2\rangle(u_1v_2-v_1u_2)f_{12}^*, \quad (15)$$

where W_0 is the term not containing f, and E is the energy of the quasiparticles;

$$E_1 = \sqrt{(\tilde{\varepsilon}_1-\lambda)^2+\Delta_1^2},$$

$$\tilde{\varepsilon}_1 = \varepsilon_1 - \sum_2(\langle 12|G|21\rangle - \langle 12|G|12\rangle)v_2^2,$$

$$\Delta_1 = \sum_2\langle 1\tilde{1}|G|\tilde{2}2\rangle u_2 v_2. \quad (16)$$

The quantity Δ (the gap in the quasiparticle spectrum) characterizes the pairing of the particles. The parameters u and v can be expressed in terms of Δ:

$$u_1^2 = \frac{1}{2}[1+(\tilde{\varepsilon}_1-\lambda)/E_1], \quad v_1^2 = \frac{1}{2}[1-(\tilde{\varepsilon}_1-\lambda)/E_1].$$

Taking the variation of $\langle H\rangle + \langle H_\omega\rangle$ with respect to f_{12}^* [and using (9) and (13)], we get an integral equation for determining f_{12}:

$$(E_1+E_2)f_{12} - \sum_{1'2'}\langle 1\tilde{2}|G|\tilde{2}'1'\rangle(u_1u_2+v_1v_2)$$

$$\times(u_{1'}u_{2'}+v_{1'}v_{2'})f_{1'2'} + \sum_{1'2'}(\langle 1\tilde{1}'|G|\tilde{2}'2\rangle$$

$$-\langle 1\tilde{1}'|G|2\tilde{2}'\rangle)(u_1v_2-v_1u_2)(u_{1'}v_{2'}-v_{1'}u_{2'})f_{1'2'}$$

$$=\omega(u_1v_2-v_1u_2)\langle 1|j_x|2\rangle. \quad (17)$$

Using (17), we can rewrite (14) in the form

$$\langle H\rangle = W_0 + \frac{1}{2}\omega^2\sum(u_1v_2-v_1u_2)\langle 1|j_x|2\rangle f_{12}^*\omega^{-1} \quad (18)$$

or, eliminating ω by using (15), we get

$$\langle H\rangle = W_0 + \frac{1}{2}J_x^2/\mathcal{J},$$

where the moment of inertia $\mathcal{J}$ is defined by the expression

$$\mathcal{J} = \sum_{12}\langle 2|j_x|1\rangle(u_1v_2-v_1u_2)f_{12}\omega^{-1}. \quad (19)$$

It is convenient to write $\mathcal{J}$ as a sum of individual terms. From (17) and (19) we find

$$\mathcal{J} = \mathcal{J}^{(1)} + \mathcal{J}^{(2)} + \mathcal{J}^{(3)}, \quad \mathcal{J}^{(1)} = \sum_{12}\frac{|\langle 1|j_x|2\rangle|^2}{E_1+E_2}(u_1v_2-v_1u_2)^2, \quad (20)$$

$$\mathcal{J}^{(2)} = \sum_{121'2'}\frac{\langle 2|j_x|1\rangle}{E_1+E_2}\langle 1\tilde{2}|G|\tilde{2}'1'\rangle(u_1v_2-v_1u_2)$$

$$\times(u_1u_2+v_1v_2)(u_{1'}u_{2'}+v_{1'}v_{2'})f_{1'2'}\omega^{-1}, \quad (21)$$

$$\mathcal{J}^{(3)} = -\sum_{121'2'}\frac{\langle 2|j_x|1\rangle}{E_1+E_2}(\langle 1\tilde{1}'|G|\tilde{2}'2\rangle$$

$$-\langle 1\tilde{1}'|G|2\tilde{2}'\rangle)(u_1v_2-v_1u_2)^2(u_{1'}v_{2'}-v_{1'}u_{2'})f_{1'2'}\omega^{-1}. \quad (22)$$

Taking the rotation into account by means of adiabatic perturbation theory gives only the first term $\mathcal{J}^{(1)}$.[1] This method is equivalent to including only the diagonal corrections to the average values (11), which naturally can be done without going beyond the scope of the transformation (2). The inclusion of the nondiagonal values (12) leads to the additional terms. The term $\mathcal{J}^{(2)}$ (which coincides with that found by Migdal[2]) takes account of the influence of rotation on the pairing. The additional term $\mathcal{J}^{(3)}$ describes the change in the self-consistent field of the nucleons as a result of the rotation. In the absence of pairing ($\Delta = 0$), the term $\mathcal{J}^{(1)}$ is equal to the rigid value of the moment of inertia, and decreases with increasing Δ, going to zero for $\Delta \to \infty$. The term $\mathcal{J}^{(2)}$ goes to zero for $\Delta = 0$, and reaches the hydrodynamic value for $\Delta \to \infty$. $\mathcal{J}^{(3)}$ gives a correction of order $A^{-1/3}$ to $\mathcal{J}$.

[1] S. T. Belyaev, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **31**, No. 11 (1959).
[2] A. B. Migdal, JETP **37**, 249 (1959), Soviet Phys. JETP **10**, 176 (1960).
[3] N. N. Bogolyubov, Usp. Fiz. Nauk **67**, 549 (1959), Soviet Phys. Uspekhi **2**, 236 (1959).

Translated by M. Hamermesh
104