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VORTEX LINES IN AN IMPERFECT BOSE GAS

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It is shown that the vortex lines possessing a thickness which is inversely proportional to the square root of the gas density and of the intensity of the interaction may exist in Bose gases with weak repulsion between the atoms. The energy of a vortex line is computed. It is also shown that in the presence of a vortex line a branch appears in the energy spectrum of the gas which corresponds to oscillations of the vortex.

1. INTRODUCTION

As is known, according to Onsager[1] and Feynman,[2] vortex lines can exist in a superfluid, i.e., certain special lines, around which the superfluid part of the liquid rotates with a velocity

$$v = s\frac{\hbar}{mr}, \tag{1}$$

where m is the mass of the atom of the liquid, r is the distance to the special line, which is known as a vortex axis, and s is an integer. A certain energy is associated with each vortex line and can be computed in general form with logarithmic accuracy. To be precise, the energy per unit length of line is equal (here and in what follows we speak only about conditions at absolute zero of temperature)

$$E = s^2\frac{\pi\hbar^2 n}{m}\ln\frac{R}{r_0}, \tag{2}$$

where n is the number of atoms of liquid per unit volume, R is the characteristic dimension of the vessel, r_0 is a quantity which has the meaning of the length of the vortex axis (it is assumed that $R \gg r_0$). In the case of real helium, r_0 has the order of interatomic distances. It is seen from (2) that the vortex lines with $s > 1$ are energetically unfavorable, inasmuch as two lines with $s = 1$ have less energy than a single line with $s > 1$. Therefore, only lines with $s = 1$ can really exist. We shall consider only such in what follows.

Vortex lines are capable of performing vibrations. In this case, when the wavelength of such vibrations is much greater than r_0, these vibrations differ in no way from the vibrations of vortices in an ordinary ideal liquid, and have the dispersion law (see references 3—5)

$$\omega = \frac{\hbar k^2}{2m}\ln\frac{1}{kr_0} \qquad (kr_0 \ll 1) \tag{3}$$

(ω is the frequency, k is the wave vector of the vibration). Thus the existence of vortex lines in a liquid leads to the appearance of a new "one-dimensional" branch of elementary excitations.

The existence of vortex lines in rotating helium II was shown experimentally by Hall and Vinen, and cannot be doubted today. Nevertheless, it should be kept in mind that the vortices were predicted by Onsager and Feynman on the basis of semi-qualitative considerations, inasmuch as the wave function of the liquid employed by Feynman cannot be obtained from general principles. Therefore there is definite interest in investigating vortex lines on a simple model in which calculations can be carried through to the end. Such a model is the imperfect Bose gas with weak repulsion between atoms, which has been considered in detail by Bogolyubov.[6] The purpose of the present research is the investigation of vortex lines in such a gas. The method which we shall use is a generalization of the method of Bogolyubov to a spatially inhomogeneous state.

2. THE VORTEX LINE IN AN IMPERFECT BOSE GAS

We shall consider a Bose gas with weak pair repulsions between atoms. The Hamiltonian of such a system in the second quantization representation has the form

$$\hat{H} = \int\left\{-\frac{\hbar^2}{2m}\psi^+\Delta\psi + \frac{1}{2}\int U(|\mathbf{r}-\mathbf{r}_1|)\,\psi^+\psi_1^+\psi_1\psi\,d\tau_1\right\}d\tau. \tag{4}$$

We shall assume that the potential is short range, and that the gas is sufficiently rarefied, i.e., that the range of the potential is much less than the distance between particles. In this case, we can remove the ψ operator from under the integral sign over $d\tau_1$ and obtain*

$$\hat{H} = \int \left\{ -\frac{\hbar^2}{2m} \psi^+ \Delta \psi + \frac{1}{2} g \psi^+ \psi^+ \psi \psi \right\} d\tau, \quad g = \int U(r)\, d\tau. \tag{5}$$

The Hamiltonian (5) corresponds to the equation of motion for the Heisenberg operator ψ

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + g \psi^+ \psi \psi. \tag{6}$$

Just as was done by Bogolyubov,[6] we divide the operator into two parts:

$$\psi = a_0 + \vartheta, \qquad \vartheta \ll a_0, \tag{7}$$

where a_0 is a number, and ϑ is a small operator correction. However, inasmuch as the system is no longer homogeneous in space in the presence of a vortex, we shall assume that a_0 is not a constant, but is a certain function of the coordinates.†

Substituting (7) in (6), and combining terms of zero order, we get

$$i\hbar \frac{\partial a_0}{\partial t} = -\frac{\hbar^2}{2m} \Delta a_0 + g\,|a_0|^2 a_0. \tag{8}$$

Assuming that a_0 depends on time according to the law $e^{-iE_0 t/\hbar}$, we get

$$-\frac{\hbar^2}{2m} \Delta a_0 - E_0 a_0 + g\,|a_0|^2 a_0 = 0. \tag{9}$$

An equation of such a form has already been considered by Ginzburg and the author[9] in connection with the phenomenological theory of superfluidity close to the λ point. (Of course, the coefficients of the equation there have an entirely different meaning.)

The vortex lines correspond to the solution of (9) possessing a symmetry relative to some axis. Transforming to cylindrical coordinates r, φ, z, and setting

$$a_0 = e^{i\varphi} F(r),$$

we obtain the following equation for F:

$$-\frac{\hbar^2}{2m} \frac{1}{r} \frac{d}{dr} r \frac{dF}{dr} + \frac{\hbar^2}{2mr^2} F - E_0 F + gF^3 = 0. \tag{10}$$

As was shown in reference 9, Eq. (10) has a solution which vanishes as $r \to 0$, and which approaches a certain constant $\sqrt{n_0}$ as $r \to \infty$; n_0 obviously has the meaning of density of particles in the condensed state and at an infinite distance from the line; with accuracy up to terms $\sim g^{3/2}$, it is identical with the total density of the gas. It follows from (10) that the constant E_0 is related to n_0 by the formula

$$E_0 = gn_0. \tag{11}$$

Taking this into account, we find that a_0 can be written in the form

$$a_0 = \sqrt{n_0} \exp i(\varphi - gn_0 t/\hbar)\, \psi_0(r/r_0), \tag{12}$$

$$r_0 = \hbar / \sqrt{2mgn_0}. \tag{13}$$

$\psi_0(\xi)$ is a real function satisfying the equation

$$\frac{1}{\xi} \frac{d}{d\xi} \xi \frac{d\psi_0}{d\xi} + \left(1 - \frac{1}{\xi^2}\right)\psi_0 - \psi_0^3 = 0, \quad \psi_0(0) = 0, \quad \psi_0(\infty) = 1. \tag{14}$$

The graph of the function ψ_0 was given earlier by Ginzburg and the author.[9] For $\xi \to 0$, we have $\psi_0 \sim \xi$. For large $\xi \gg 1$,

$$\psi_0(\xi) \approx 1 - {}^1\!/_2 \xi^2. \tag{15}$$

Equation (12) describes the vortex line with $s = 1$.

The quantity r_0 obviously has the meaning of the length of the vortex line. It is seen from (12) that the mean velocity of particles around the line is equal to $\hbar/mr$, as it ought to be.

As was shown in reference 6, the smallness of the amplitude of scattering of particles $mg/4\pi\hbar^2$, in comparison with the mean distance between particles, serves as the condition for applicability of the approximation (7):

$$mg/\hbar^2 \ll n_0^{-1/3} \approx n^{-1/3}. \tag{16}$$

n is the number density of the particles. Upon satisfaction of the conditions (16),

$$r_0 \gg n^{-1/3}. \tag{17}$$

Thus the condition for applicability of these formulas reduces to the requirement that the length of the vortex be large in comparison with the interatomic distances. This means that in the rarefied gas the vortex line is a macroscopic formation. The same condition is also necessary for applicability of the equations previously introduced.[9] However, in this case, this condition was guaranteed in reference 9 by the closeness to the λ point, while in our case it was guaranteed by the rarefied character of the gas.

The resultant formulas make it possible to compute the energy of the vortex line with non-logarithmic accuracy. For this purpose it is nec-

*It is actually impossible to use Eq. (5) directly in calculations to higher orders of g. This is connected with the necessity of carrying out a renormalization of the scattering amplitude of particles (see reference 7). However, we shall not calculate terms of higher order.

†Inhomogeneous ground states of a Bose system were investigated by Gross[8] without connection to the problem of vortex lines.

essary to substitute a_0 from (12) in (5) in place of ψ and integrate over the volume. In this case, the term proportional to the volume gives simply the energy of the homogeneous gas, while the term which depends logarithmically on the volume is the energy of the vortex. Numerical integration, results of which are taken from the work of Ginzburg and the author,[9] gives the following expression for the unit length of line located in the center of a cylindrical vessel of radius R:

$$E = (\pi\hbar^2 n_0 / m) \ln (1{,}46R / r_0). \tag{18}$$

3. VIBRATIONS OF THE VORTEX LINE

In the present section we shall show that there is actually a branch in the energy spectrum of the Bose gas in the presence of a vortex line with the dispersion law (3). To find the energy spectrum of a weakly perturbed state, it is necessary to substitute ψ in (6) with accuracy up to terms $\sim \vartheta$. We have

$$i\hbar \frac{\partial\vartheta}{\partial t} = -\frac{\hbar^2}{2m}\Delta\vartheta + 2g\,|a_0|^2\vartheta + a_0^2\vartheta^+. \tag{19}$$

We write ϑ in the form

$$\vartheta = e^{i(\varphi - E_0 t/\hbar)} \sum_{l,k} b_l(k,\ r)\, e^{i(kz + l\varphi)} \quad (l = 0,\ \pm 1,\ \pm 2,\ \ldots).$$

Further, substituting in (19) and taking (12) into account, we obtain a set of linear equations for $b_l(k)$ and $b^*_{-l}(-k)$:

$$i\hbar \frac{\partial b_l(k)}{\partial t} = -\frac{\hbar^2}{2m}\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{(l+1)^2}{r^2} - k^2\right] b_l(k) + gn_0(2\psi_0^2 - 1)\, b_l(k) + gn_0\psi_0^2 b^*_{-l}(-k),$$

$$-i\hbar \frac{\partial b^*_{-l}(-k)}{\partial t} = -\frac{\hbar^2}{2m}\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{(l-1)^2}{r^2} - k^2\right] b^*_{-l}(-k) + gn_0(2\psi_0^2 - 1)\, b^*_{-l}(-k) + gn_0\psi_0^2 b_l(k). \tag{20}$$

As is known, the energy levels of each linear quantum-mechanical system are determined by the characteristic frequencies of the corresponding classical problem. Therefore, in finding the energy spectrum, it suffices for us to find the characteristic frequencies of the system (20), assuming that $b_l(k)$ and $b^*_{-l}(-k)$ are simply certain classical functions of the coordinates. Among the solutions of the system (20) there are functions which decay exponentially as $r \to \infty$, and functions which oscillate as $r \to \infty$. The latter clearly describe phonons which are scattered by the vortex. The first correspond to the excited states of the vortex line itself.

A complete investigation of the characteristic values of the system (20) is very complicated, and there is scarcely any direct relation to the properties of real vortex lines in helium II. Actually, the "macroscopic" vortex in the gas probably has a much higher degree of freedom and energy levels than the vortex in helium having an atomic thickness. We therefore limit ourselves only to showing that among the solutions of (20) there are actually long wavelength vibrations with the dispersion law (3).

First of all we must determine precisely how the values of l correspond to the vibrations (3). For this purpose we note that as $k \to 0$ these vibrations simply transform into a displacement of the vortex as a whole. In this case a_0 is changed evidently by the value

$$\vartheta_0 = \delta\mathbf{r}\nabla a_0 = \delta r\left[\cos(\varphi - \varphi_0)\frac{\partial a_0}{\partial r} - \frac{1}{r}\sin(\varphi - \varphi_0)\frac{\partial a_0}{\partial\varphi}\right], \tag{21}$$

where $\delta\mathbf{r} = \delta\mathbf{r}(\delta r, \varphi_0)$ = displacement vector. Therefore our equations should admit a solution of the form (21) for k = 0.

Equation (21) contains terms $\sim e^{i\varphi}e^{\pm i\varphi}$, i.e., terms with $l = \pm 1$. Thus the vibrations of interest to us are described by Eqs. (20) with $l = \pm 1$. We write down equations with $l = -1$, assuming that b_{-1}, $b_1^* \sim e^{-i\omega t}$, and transform to dimensionless variables. We obtain

$$\left(\frac{1}{\xi}\frac{d}{d\xi}\xi\frac{d}{d\xi} - \frac{4}{\xi^2} - \varkappa^2\right) f + (2\psi_0^2 - 1)\, f + \psi_0^2 f_1 = -\varepsilon f,$$

$$\left(\frac{1}{\xi}\frac{d}{d\xi}\xi\frac{d}{d\xi} - \varkappa^2\right) f_1 + (2\psi_0^2 - 1)\, f_1 + \psi_0^2 f = \varepsilon f_1. \tag{22}$$

Here

$$\xi = r/r_0, \quad \varkappa = kr_0, \quad \varepsilon = \hbar\omega/E_0, \quad b_1^*(k) = e^{-i\omega t}\sqrt{n_0} f,$$
$$b_{-1}(-k) = e^{-i\omega t}\sqrt{n_0} f_1.$$

It is quite simple to establish the fact that, as it should follow from (21), Eqs. (22) satisfy the regular functions

$$f^0 = (d\psi_0/d\xi - \psi_0/\xi), \qquad f_1^0 = (d\psi_0/d\xi + \psi_0/\xi) \tag{23}$$

for $\kappa = 0$ and $\epsilon = 0$ at $\xi = 0$.

We transform to a solution (22) for $\kappa \ll 1$. We initially consider distances $\xi \ll 1/\kappa$. In this case, we can neglect κ^2 in the equations and look for a solution in the form

$$f = f^0 + \varepsilon f', \quad f_1 = f_1^0 + \varepsilon f_1, \tag{24}$$

where on the right hand side of the equations, we can substitute f^0 and f_1^0 for f and f_1:

$$\left(\frac{1}{\xi}\frac{d}{d\xi}\xi\frac{d}{d\xi} - \frac{4}{\xi^2}\right) f' + (2\psi_0^2 - 1)\, f' + \psi_0^2 f_1' = -f^0,$$

$$\frac{1}{\xi}\frac{d}{d\xi}\xi\frac{d}{d\xi} f_1' + (2\psi_0^2 - 1)\, f_1' + \psi_0^2 f' = f_1^0. \tag{25}$$

We directly verify that the system (25) has a regular solution at zero of the form

$$f' = \frac{1}{2}\xi\psi_0, \quad f_1' = -\frac{1}{2}\xi\psi_0. \tag{26}$$

We shall not set forth the construction details of this solution. We only note that it is connected with the existence in Eq. (8) of solutions describing the motion of the gas as a whole together with the vortex filament, i.e., in the final analysis with Galilean invariance of the initial equations.

We now consider the region of separation $\xi \gg 1$. It is seen from (23) that at such distances

$$f - f_1 \ll f + f_1. \tag{27}$$

This inequality has a simple physical meaning. It is easy to see that the quantity $f + f_1$ is proportional to the perturbation of the density around the vortex, while $f - f_1$ is the perturbation of the velocity potential. The second perturbations fall off more rapidly than the first. We subtract the second equation of (22) from the first and everywhere write

$$f + f_1 = 0, \quad \psi_0^2 = 1 - 1/\xi^2.$$

As a result we get

$$\left(\frac{1}{\xi}\frac{d}{d\xi}\xi\frac{d}{d\xi} - \frac{1}{\xi^2} - \varkappa^2\right)(f - f_1) = 0. \tag{28}$$

The solution of Eq. (28) which vanishes at infinity has the form

$$f - f_1 = CK_1(\varkappa\xi), \tag{29}$$

where K is a modified Bessel function of second order (see, for example, reference 10, page 30).

At $\xi \ll 1/\kappa$ we find from (29), with accuracy up to terms $\sim \kappa \ln \kappa$,

$$f - f_1 \approx C\left(1/\varkappa\xi + \frac{1}{2}\xi\varkappa \ln \varkappa\right). \tag{30}$$

On the other hand, it follows from (29) and (26) that for $\xi \gg 1$

$$f - f_1 \approx -(2/\xi - \varepsilon\xi). \tag{31}$$

For $1 \ll \xi \ll 1/\kappa$, Eqs. (28) and (29) should coincide, whence we find

$$\varepsilon = -\varkappa^2 \ln \varkappa$$

for, transforming to dimensional quantities,

$$\omega = \frac{\hbar k^2}{2m}\ln\frac{1}{kr_0} = \frac{\hbar k^2}{2m}\ln\frac{\sqrt{2gmn_0}}{k\hbar},$$

which is identical with (3).

We note in closing that perturbations with the dispersion law (3) are not entirely stable in the sense that any perturbation with $k \neq 0$ possesses a finite lifetime. This is connected with the fact that an arbitrary perturbation with $k \neq 0$ can under such a dispersion law decay into three perturbations with lesser k. (Decay into two perturbations in the one-dimensional case is impossible because of the zero statistical weight of such a process.) Of course, this observation applies to the vibrations of vortices both in a gas and in a liquid.

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