CONCERNING DISTURBANCES PRODUCED BY A BODY MOVING IN A PLASMA

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An expression is derived for the Fourier components n_q of the disturbance to the electron density, produced by a body moving in a plasma, in the limit as the wave vector $q \to 0$. It is shown, in particular, that the exact expression for n_q contains terms proportional to $1/q$, which are absent in the first approximation of perturbation theory. The formulas are employed to calculate in various particular cases the effective cross section for scattering of electromagnetic waves of wave lengths considerably in excess of the characteristic dimension of the body.

1. FORMULATION OF THE PROBLEM

IN this paper we report on a theoretical investigation of the scattering of electromagnetic waves by a trail left by a moving body in an isotropic electron-ion plasma. This problem has become quite timely of late.[1-3]

We assume that the plasma is sufficiently rarefied, i.e., that the ion mean free path in the plasma is much greater than either the dimensions of the body or the length of the scattered electromagnetic wave:

$$l \gg \lambda, \ R_0. \tag{1}$$

The scattering of an electromagnetic wave by a body moving in a plasma consists of scattering on the body proper, for example a metal sphere, and scattering on the trail produced by the body, i.e., the region of perturbed electron concentration produced by the motion of the body. The scattering by the body itself is described by the usual formulas of diffraction theory and will not be calculated here. We note at once that since the dielectric constant of the trail is low, the scattering from the metallic body itself will be much greater than from a trail region of the same size. It is therefore clear that the trail can make a scattering contribution comensurate with the contribution of the body itself only if trail regions larger than the body participate in the scattering. In the scattering by structures that diminish slowly with distance, such as trails usually are, the regions effectively participating in the scattering have normally dimensions on the order of the wavelength (the contribution from greater distances is strongly reduced by the interference between waves scattered by the different parts of the trail). It is therefore clear beforehand that the trail can make a noticeable contribution to the scattering only at wavelengths much greater than the characteristic dimension of the body R_0:

$$\lambda \gg R_0. \tag{2}$$

We shall confine ourselves to this case throughout.

Since the body produces at large distances away from it only small changes in the dielectric constant (compared with unity), it is natural to employ perturbation theory for the calculation of the scattering. (The validity of the theory will be discussed later on.) Recognizing that the variation of the dielectric constant of the plasma is connected with the perturbation of the electron density δn by the formula

$$\delta\varepsilon(\mathbf{r}) = -\frac{4\pi e^2}{m\omega^2}\delta n(\mathbf{r})$$

(e and m are the electron charge and mass) we obtain by the well-known perturbation theory formula the amplitude of the scattered wave at distances large compared with the wavelength:

$$\mathbf{E}' = \frac{e^2}{m\omega^2\varepsilon}\frac{e^{ikR}}{R} n_{\mathbf{q}} [\mathbf{k}'[\mathbf{k}'\mathbf{E}_0]]. \tag{3}^*$$

Here $\mathbf{E}_0$ is the amplitude of the incident wave, $\mathbf{k}'$ is the wave vector of the scattered wave ($|\mathbf{k}'| = k = \sqrt{\epsilon}\,\omega/c$), ϵ is the dielectric constant of the plasma, and n_q is the Fourier component of the electron-density perturbation:

$$n_{\mathbf{q}} = \int \delta n(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}} d^3r, \quad \mathbf{q} = \mathbf{k}' - \mathbf{k}, \ |\mathbf{q}| = 2k \sin\frac{\psi}{2} \tag{4}$$

*$\mathbf{k}'[\mathbf{k}'\mathbf{E}_0] = \mathbf{k}' \times (\mathbf{k}' \times \mathbf{E}_0)$.

(k is the wave vector of the incident wave and ψ is the scattering angle, i.e., the angle between k and k'). The effective cross section of scattering in a solid-angle element do is given by

$$d\sigma = \frac{1}{16\pi^2 e^2}\left(\frac{\omega_0}{\omega}\right)^4 \frac{|n_{\mathbf{q}}|^2}{n_0^2} k^4 \sin^2\psi_1 do \qquad (5)$$

(ψ_1 is the angle between k' and $\mathbf{E}_0$, $\omega_0^2 = 4\pi n_0 e^2/m$, and n_0 is the unperturbed electron density). For comparison we give here the formula for the effective cross section of scattering by a metal sphere of radius R_0 for $\lambda \gg R_0$:

$$d\sigma_m = \left(\frac{\omega}{c}\right)^4 \frac{\sigma_0^3}{\pi^3} \sin^2\psi_1 do \qquad (\sigma_0 = \pi R_0^2). \qquad (6)$$

As can be seen from (5), the problem of calculating the effective electromagnetic-wave scattering cross section reduces to the calculation of the Fourier components of the variation of the electron density. The perturbations produced in a plasma by a rapidly moving body of dimensions greater than the Debye radius were considered in detail by A. V. Gurevich.[2] To calculate $n_{\mathbf{q}}$ for this case we could therefore, in principle, use Gurevich's results. It turns out, however, that it is simpler to determine $n_{\mathbf{q}}$ directly from the kinetic equation. This yields a more rigorous solution for the problem, for it permits evaluation of certain effects not considered in reference 2, which become significant when q is small.

2. DERIVATION OF THE GENERAL FORMULA

We consider in the present paper only the case when the velocity V_0 of the body is much less than the thermal velocity of the electrons:

$$V_0 \ll \sqrt{kT/m}. \qquad (7)$$

Under these conditions the electron density is directly connected with the electric potential φ by means of the Boltzmann distribution function

$$n = n_0 e^{e\varphi/kT}. \qquad (8)$$

On the other hand, the distribution of the ions should be determined simultaneously with the potential. We shall solve this problem with the aid of the kinetic equation for the ion velocity and coordinate distribution function. Inasmuch as the distribution function is independent of the time in a frame fixed in the body, the equation in this frame has the form

$$\frac{\partial f}{\partial \mathbf{r}}\mathbf{v} - \frac{\partial f}{\partial \mathbf{v}}\frac{1}{M}\frac{\partial}{\partial \mathbf{r}}(e\varphi + U) = 0. \qquad (9)$$

[By virtue of condition (1) we can neglect collisions between particles.] Here M is the ion mass, v the ion velocity in the frame in which the body is at rest, and U the energy of interaction between the ions and the surface of the body. Introducing the ion velocity in the system at rest

$$\mathbf{u} = \mathbf{v} + \mathbf{V}_0,$$

we can rewrite (9) as

$$\frac{\partial f}{\partial \mathbf{r}}(\mathbf{u} - \mathbf{V}_0) - \frac{1}{M}\frac{\partial f}{\partial \mathbf{u}}\frac{\partial}{\partial \mathbf{r}}(U + e\varphi) = 0. \qquad (10)$$

The electric potential φ satisfies, with allowance for (8), the equation

$$\Delta\varphi = -4\pi e\left(\int f d^3u - n\right) = -4\pi e\left(\int f d^3u - n_0 e^{e\varphi/kT}\right). \qquad (11)$$

In Eq. (10) it is convenient to separate from f the coordinate-independent part f_0, which is equal to the distribution function at an infinite distance from the body:

$$f = f' + f_0, \qquad f_0 = n_0 (M/2\pi kT)^{3/2} \exp\{-Mu^2/2kT\}.$$

Substituting this in (10) we get

$$\frac{\partial f'}{\partial \mathbf{r}}(\mathbf{u} - \mathbf{V}_0) - \frac{1}{M}\frac{\partial f_0}{\partial \mathbf{u}}\frac{\partial}{\partial \mathbf{r}}(U + e\varphi) - \frac{1}{M}\frac{\partial f'}{\partial \mathbf{u}}\frac{\partial}{\partial \mathbf{r}}(U + e\varphi) = 0. \qquad (12)$$

To solve the system (11) and (12) we change over to Fourier components in the coordinates. Equation (12) in Fourier components becomes

$$i\mathbf{q}(\mathbf{u} - \mathbf{V}_0) f'_{\mathbf{q}} - \frac{e}{M}\frac{\partial f_0}{\partial \mathbf{u}} i\mathbf{q}\varphi_{\mathbf{q}} - \frac{1}{(2\pi)^3 M}\int i\mathbf{q}_1 (U_{\mathbf{q}_1} + e\varphi_{\mathbf{q}_1})\frac{\partial f'_{\mathbf{q}-\mathbf{q}_1}}{\partial u} d^3q_1 - i\frac{\mathbf{q}}{M}\frac{\partial f_0}{\partial \mathbf{u}} U_{\mathbf{q}} = 0. \qquad (13)$$

We now recognize that we are interested only in long waves, i.e., in small values of q, and let q tend to 0. Since $U_{\mathbf{q}}$ certainly tends to a constant limit as $\mathbf{q} \to 0$, the last term in (13) can be discarded. (The first two terms must be retained, since, as will be shown later, $f'_{\mathbf{q}}$ and $\varphi_{\mathbf{q}}$ tend to infinity as $\mathbf{q} \to 0$.) In the third term we can also put $\mathbf{q} = 0$. For this purpose it is necessary that the integral in this term converge. But, as we shall see later, the quantities $f'_{\mathbf{q}}$ and $\varphi_{\mathbf{q}}$ are proportional to $1/q$ as $q \to 0$, and consequently this integral converges. Finally we obtain as $q \to 0$

$$i\mathbf{q}(\mathbf{u} - \mathbf{V}_0) f'_{\mathbf{q}} - \frac{e}{M}\frac{\partial f_0}{\partial \mathbf{u}} i\mathbf{q}\varphi_{\mathbf{q}} = I(\mathbf{u}), \qquad (14)$$

$$I(\mathbf{u}) = \frac{1}{M}\int i\mathbf{q}(U_{\mathbf{q}} + e\varphi_{\mathbf{q}})\frac{\partial f'_{-\mathbf{q}}}{\partial \mathbf{u}}\frac{d^3q}{(2\pi)^3} = \frac{1}{M}\int \frac{\partial f'}{\partial \mathbf{u}}\frac{\partial}{\partial \mathbf{r}}(U + e\varphi) d^3r. \qquad (15)$$

Our problem is thus subdivided into two stages — calculation of the function $I(\mathbf{u})$ for the given body and the solution of (14). We note that $I(\mathbf{u})$ is proportional to the product of the potential by the correction to the distribution function, and consequently differs from zero only in the second perturbation-theory approximation in the potential. It is evident therefore that the results of Kraus and Watson,[3] who carried out the calculations in first approximation, are certainly incorrect at large distances.

We change now to Fourier components in (11). We note here that in the expansion of $n_k = n_0(e^{e\varphi/kT} - 1)_q$ in powers of $e\varphi/kT$,

$$n_{\mathbf{q}} = n_0\left(\frac{e}{kT}\varphi_{\mathbf{q}} + \frac{e^2}{2k^2T^2}\int \varphi_{\mathbf{q}_1}\varphi_{\mathbf{q}-\mathbf{q}_1}\frac{d^3q_1}{(2\pi)^3} + \ldots\right), \tag{16}$$

the first term varies as $1/q$, while the others tend to constants and can be neglected. Therefore

$$n_{\mathbf{q}} = n_0 e\varphi_{\mathbf{q}}/kT \tag{17}$$

and instead of (11) we have as $q \to 0$

$$\varkappa^2\varphi_{\mathbf{q}} = 4\pi e\int f'_{\mathbf{q}} d^3u, \tag{18}$$

where $\kappa = \sqrt{4\pi n_0 e^2/kT} = 1/\sqrt{2}R_D$, with R_D the Debye radius.

We note that for small $\mathbf{q}$ the system of inhomogeneous nonlinear equations (11) and (12) is reduced to a system of linear inhomogeneous equations (14) and (18). This result has a simple physical meaning: when $\mathbf{q}$ is small, the large distances from the body, at which the deviations of the distribution function from equilibrium are small, become significant. It follows from (14) and (18) that f'_q and φ_q are actually proportional to $1/q$, as pointed out earlier.

Let us divide both halves of (14) by $i\mathbf{q}\cdot(\mathbf{u} - \mathbf{V}_0)$ and integrate over d^3u. The integrals with singular denominators $i\mathbf{q}\cdot(\mathbf{u} - \mathbf{V}_0)$ should, according to Landau,[4] be taken with suitable circuits about the singularities. To take this into account, it is enough to replace $\mathbf{q}\cdot\mathbf{V}_0$ by $\mathbf{q}\cdot\mathbf{V}_0 + i\delta$, where $\delta \to +0$. As a result we obtain

$$\int f'_{\mathbf{q}} d^3u - \frac{e}{M}\varphi_{\mathbf{q}}\int \frac{\partial f_0}{\partial \mathbf{u}}\frac{\mathbf{q}}{\mathbf{q}(\mathbf{u} - \mathbf{V}_0) - i\delta}d^3u = \frac{1}{i}\int \frac{I(\mathbf{u})}{\mathbf{q}(\mathbf{u} - \mathbf{V}_0) - i\delta}d^3u. \tag{19}$$

Solving equations (18) and (19) simultaneously and using the fact that $\partial f_0/\partial \mathbf{u} = -(\mathbf{u}M/kT)f_0$, we get

$$n_{\mathbf{q}} = \frac{\dfrac{1}{iq}\displaystyle\int \frac{I(\mathbf{u})}{\mathbf{n}(\mathbf{u} - \mathbf{V}_0) - i\delta}d^3u}{\left[1 + \left(\dfrac{M}{2\pi kT}\right)^{3/2}\displaystyle\int \frac{\mathbf{n}\mathbf{u}}{\mathbf{n}(\mathbf{u} - \mathbf{V}_0) - i\delta}e^{-Mu^2/kT}d^3u\right]}, \tag{20}$$

where $\mathbf{n} = \mathbf{q}/|\mathbf{q}|$. Writing the integral in the denominator of (20) in Cartesian coordinates with the x axis along $\mathbf{n}$, and using the well known identity

$$\int_{-\infty}^{\infty} \frac{e^{-y^2}}{y - a - i\delta}dy = 2\sqrt{\pi}\left[\frac{i\sqrt{\pi}}{2} - \int_0^a e^{x^2}dx\right]e^{-a^2}, \tag{21}$$

we obtain finally

$$n_{\mathbf{q}} = \frac{1}{iq}\int \frac{I(\mathbf{u})}{\mathbf{n}(\mathbf{u} - \mathbf{V}_0) - i\delta}d^3u \Bigg/ \left[2 - 2a\left(\int_0^a e^{x^2}dx - i\frac{\sqrt{\pi}}{2}\right)e^{-a^2}\right],$$
$$a = \mathbf{n}\mathbf{V}_0\sqrt{\frac{M}{2kT}}. \tag{22}$$

It follows directly from (22) that the electron density decreases as $1/r^2$ with increasing distance from the body, in agreement with the results of Gurevich.* We note that when $V_0 \gg kT/M$ the denominator of (22) has a rather pronounced angular dependence, namely, $a \gg 1$ when $\mathbf{n}\cdot\mathbf{V}_0 \sim V_0$, and the denominator is equal to zero in this case. At angles for which $\mathbf{n}\cdot\mathbf{V}_0\sqrt{M/2kT} = a \ll 1$, the denominator is equal to 2. This result can be readily explained. The frequency of the field component with wave vector $\mathbf{q}$ is obviously equal to $\mathbf{q}\cdot\mathbf{V}_0$ in the coordinate system at rest. If $\mathbf{q}\cdot\mathbf{V}_0 \gg qv_T$, where $v_T \sim \sqrt{kT/M}$ is the thermal velocity of the ions, the ions do not have a chance to screen the potential. On the other hand, when $\mathbf{q}\cdot\mathbf{V}_0 \ll qv_T$, the ions screen the potential so that the Debye radius and the potential φ_q are each reduced to one half. We note also that the denominator of (22) coincides exactly with the expression for the dispersion of ionic plasma waves. Since, however, these waves are strongly damped, no special significance should be attached to this fact.

Let us proceed now to clarify the physical meaning of the quantity $I(\mathbf{u})$. For this purpose we turn in (14) to the coordinate space and assume that all the quantities depend (in the frame which is at rest) on the time only via the combination $\mathbf{r} - \mathbf{V}_0 t$, so that $\mathbf{V}_0\partial f'/\partial\mathbf{r} = -\partial f'/\partial t$. We obtain

$$\frac{\partial f'}{\partial t} + \frac{\partial f'}{\partial \mathbf{r}}\mathbf{u} - \frac{e}{M}\frac{\partial f_0}{\partial \mathbf{u}}\frac{\partial \varphi}{\partial \mathbf{r}} = I(\mathbf{u})\,\delta(\mathbf{r} - \mathbf{V}_0 t). \tag{23}$$

The left half of (23) is the total derivative of the distribution function with respect to time. From the physical meaning of the distribution function it is clear that $I(\mathbf{u})d^3u$ is the number of particles per unit time which acquire, by collision with the body, velocities that lie in the interval d^3u about $\mathbf{u}$. Thus $I(\mathbf{u})$ serves so to speak as a sort of a "collision interval" for the ions with the body.

This physical meaning of $I(\mathbf{u})$ enables us, by following the reasoning used to determine the form of the usual collision integral, to recast $I(\mathbf{u})$ in a different form, very useful for approximate calculations. Let us change again to the coordinate system of the body. Let a particle passing near the

*The formulas obtained can also be used, naturally, to calculate the perturbed electron density, etc., at large distances from the moving body. In the present paper, however, we shall not consider the transition to coordinate space.

body with an impact parameter ρ and an azimuth angle φ acquire a velocity $\mathbf{v}$ after scattering (by interaction with the surface of the body and the electric field surrounding it). Then the initial velocity of the ion is $\mathbf{v}_1 = \mathbf{v}_1(\mathbf{v}, \rho, \varphi)$, where the function $\mathbf{v}_1(\mathbf{v}, \rho, \varphi)$ is determined by the scattering law, and the number of particles that acquire a velocity $\mathbf{v}$ in a unit of time is merely the number of incident particles with velocity $\mathbf{v}_1$, i.e.,

$$\rho d\rho\, d\varphi\, v n_0 (M / 2\pi kT)^{3/2} \exp\{-M[\mathbf{v}_1(\mathbf{v}, \rho, \varphi) + \mathbf{V}_0]^2 / 2kT\}. \tag{24}$$

We have assumed that the scattering is elastic ($|\mathbf{v}_1| = |\mathbf{v}|$) and that the incident particles have at infinity a Maxwellian distribution in the frame which is at rest. To find I is also necessary to subtract from (24) the number of particles with velocity $\mathbf{v}$, knocked out by collision with the body

$$\rho\, d\rho\, d\varphi\, v n_0 (M / 2\pi kT)^{3/2} \exp\{-M(\mathbf{v} + \mathbf{V}_0)^2 / 2kT\}.$$

Finally

$$I(\mathbf{v} + \mathbf{V}_0) = n_0 \left(\frac{M}{2\pi kT}\right)^{3/2} v \int \rho d\rho d\varphi \Big[\exp\Big\{-\frac{M[\mathbf{v}_1(\mathbf{v},\rho,\varphi)+\mathbf{V}_0]^2}{2kT}\Big\} - \exp\Big\{-\frac{M(\mathbf{v}+\mathbf{V}_0)^2}{2kT}\Big\}\Big]. \tag{25}$$

Thus, $I(\mathbf{u})$ can be readily calculated if the law of scattering of the ions on the body, with allowance for the electric field, is known. Naturally, formula (25) does not permit $I(\mathbf{u})$ to be calculated in the general case, since the electric field around the body is unknown, if for no other reason. We shall show in the next section, however, that in some important limiting cases this difficulty can be circumvented.

Let us note that $I(\mathbf{u})$ enables us to express in simple fashion the force acting on the moving body if collisions between the body and neutral molecules can be neglected. Namely, multiplying (23) by $M\mathbf{u}$ and integrating over d^3u and d^3r we obtain the time derivative of the total particle momentum, which is equal to the negative of the sought force. As a result

$$\mathbf{F} = -M \int \mathbf{u} I(\mathbf{u})\, d^3u. \tag{26}$$

In the derivation of the formulas obtained in this section we have assumed that the particles do not lose their identity as they collide with the body and, in particular, that the ions do not become neutralized. It is obvious, however, that the final formula (22) is of great significance. In particular, it can also be used when the ions are partially or completely neutralized. All the effects connected with neutralization modify in this case only the form of the function $I(\mathbf{u})$, which will no longer satisfy the law of conservation of the number of particles

$$\int I(\mathbf{u})\, d^3u = 0.$$

The results can be readily generalized to include the case of noticeable photoeffect or secondary emission of electrons from the surface of the body.

3. CALCULATION OF THE "COLLISION INTEGRAL" I FOR DIFFERENT PARTICULAR CASES

The function $I(\mathbf{u})$ can of course not be calculated in general form for arbitrary bodies. In this section we shall determine this function for different limiting cases which are of principal interest. For simplicity we confine ourselves to the motion of a spherically symmetrical body, although in some cases (low and high velocities) the formulas obtained can be applied also to more general cases.

a) Slowly moving body. To illustrate the method we shall consider first the case of the body moving in a plasma at a velocity much lower than the average thermal velocity of the ions (this case is probably only of methodological interest). In this case, in the zeroth approximation in $\mathbf{V}_0$, the potential around the body is spherically symmetrical and independent of $\mathbf{V}_0$. Expanding the exponentials in (25) in powers of $\mathbf{V}_0$ we get

$$I(\mathbf{u}) = n_0 \left(\frac{M}{2\pi kT}\right)^{3/2} \frac{Mu}{kT} e^{-Mu^2/2kT} \int \rho\, d\rho\, d\varphi\, (\mathbf{u} - \mathbf{u}_1) \mathbf{V}_0$$
$$= n_0 \left(\frac{M}{2\pi kT}\right)^{3/2} e^{-Mu^2/2kT} \frac{Mu}{kT} \mathbf{u} \mathbf{V}_0 \sigma^*(u),$$
$$\sigma^* = 2\pi \int (1 - \cos\vartheta)\, \rho\, d\rho, \tag{27}$$

where $\vartheta(\rho)$ is the angle between $\mathbf{u}$ and $\mathbf{u}_1$, $\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{V}_0$, and σ^* is the so-called transport cross section of the body (see, for example, reference 5). For an uncharged sphere from which the ions are specularly reflected, we have

$$\sigma^* = \pi R_0^2 = \sigma_0.$$

Substituting (27) into (26) we obtain for the force acting in the body

$$\mathbf{F} = -(M^2 / 3kT)\, \overline{\sigma^* u^3} n_0 \mathbf{V}_0, \tag{28}$$

where $\overline{\sigma^* u^3}$ is the average value of $\sigma^*(u) u^3$, taken over the Maxwellian distribution. Formula (28) coincides, as it should, with the formula for the force acting on a heavy particle in a light gas.[5]

b) Rapidly moving body with dimensions which are not small compared with the Debye radius. If the velocity of the body is high

$$V_0 \gg \sqrt{kT/M}, \tag{29}$$

the problem can be greatly simplified, for in this case the electric field exerts a small influence on the scattering of the ions by the body. Actually, as was shown by Gurevich,[2] the electric potential of the body is in this case of order kT/e, or more accurately $(kT/e)\ln(R_0/R_D)$, and when $MV_0^2/2 \gg kT$ it is less than the kinetic energy of the ions relative to the body. Consequently it is natural to neglect in the calculation of $I(u)$ the influence of the field and to consider only scattering of the ions by the surface of the sphere.* Further calculations depend on the law governing this scattering. To be specific, we confine ourselves to specular reflection. All other cases can be analyzed analogously.

It is now natural to represent expression (25) for $I(u)$ in the form $I = I_1 - I_2$ in accordance with the two terms in the integrand. For a sphere, I_2 is simply

$$I_2 \approx n_0 V_0 \sigma_0 (M/2\pi kT)^{3/2} e^{-Mu^2/2kT}. \quad (30)$$

To calculate I_1 we recognize that

$$(\mathbf{v}_1 + \mathbf{V}_0)^2 = v^2 + V_0^2 + 2vV_0(\cos\vartheta_0\cos\vartheta + \sin\vartheta_0\sin\vartheta\cos\varphi), \quad (31)$$

where ϑ_0 is the angle between $\mathbf{v}$ and $\mathbf{V}_0$ and ϑ is the angle between $\mathbf{v}_1$ and $\mathbf{v}$. In addition, in the case of specular reflection $\rho\, d\rho = \frac{1}{4}R_0^2 \sin\vartheta\, d\vartheta$. Substituting this in (25) we have

$$I_1 = n_0\left(\frac{M}{2\pi kT}\right)^{3/2} v \frac{R_0^2}{4}\int d\varphi \sin\vartheta\, d\vartheta \times \exp\left\{-\frac{M}{2kT}[v^2 + V_0^2 + 2vV_0(\cos\vartheta_0\cos\vartheta + \sin\vartheta_0\sin\vartheta\cos\varphi)]\right\}. \quad (32)$$

Since the exponential function has a large modulus by virtue of (29), we can calculate I_1 by the method of steepest descents. It is easy to verify that the exponent in (32) has a maximum at $\varphi = \pi$, $\vartheta + \vartheta_0 = \pi$. Putting $\varphi = \pi + \varphi'$, $\vartheta + \vartheta_0 = \pi + \vartheta'$, and expanding in φ' and ϑ', we get

$$I_1 = n_0\left(\frac{M}{2\pi kT}\right)^{3/2} v \sin\vartheta_0 \frac{R_0^2}{4}\exp\left\{-\frac{M}{2kT}(v^2 + V_0^2 - 2vV_0)\right\} \times \int \exp\left\{-\frac{MvV_0}{2kT}\varphi'^2\sin\vartheta_0\right\}d\varphi' \int \exp\left\{-\frac{MvV_0}{2kT}\vartheta'^2\sin\vartheta_0\right\} \times d\vartheta' = \frac{n_0}{V_0}\frac{R_0^2}{4}\left(\frac{M}{2\pi kT}\right)^{1/2}\exp\left\{-\frac{M}{2kT}(v - V_0)^2\right\} \quad (33)$$

(We recall that $\mathbf{v} = \mathbf{u} - \mathbf{V}_0$). Thus, the total value of $I(u)$ is

$$I = n_0\sigma_0\left[\left(\frac{M}{2\pi kT}\right)^{1/2}\frac{1}{2V_0}\exp\left\{-\frac{M}{2kT}(v - V_0)^2\right\} - V_0\left(\frac{M}{2\pi kT}\right)^{3/2}\exp\left\{-\frac{Mu^2}{2kT}\right\}\right]. \quad (34)$$

*We emphasize that this does not mean that the electric field is in general unimportant; the effect of the field manifests itself in the angular dependence of the denominator of (22).

We note that by including terms of higher order in φ' and ϑ' we can readily determine the subsequent terms in the expansion (32) in $\sqrt{kT/MV_0^2}$. We shall not stop, however, to calculate these corrections.

c) Charged body of small size. Let us consider now a body with dimensions much smaller than the Debye radius, carrying a charge that satisfies the condition

$$\epsilon Q \ll R_D M v_i^2,$$

where v_i is the greater of the two quantities $(V_0^2 kT/M)^{1/4}$ and $\sqrt{kT/M}$. (This is precisely the case considered by Kraus and Watson.[3] We have already pointed out that their approximation does not include the effects which we are investigating). In this case the main contribution to I is made by the impact distances ρ which satisfy the condition $R_D \gg \rho \gg r_0$, where r_0 is the greater of the two quantities R_0 and Q/Mv_i^2. For such values of ρ the field can be considered to be a Coulomb field exclusively, and the scattering angle ϑ is small and given by[6]

$$\vartheta = 2Qe/Mv^2\rho. \quad (35)$$

Expanding the right half of (25) in powers of ϑ, retaining terms up to ϑ^2, and integrating with respect to ρ with cutoff at large when $\rho \sim R_D$ and at small ρ when $\rho \sim r_0$, we obtain

$$I = 2\pi n_0\left(\frac{M}{2\pi kT}\right)^{3/2} e^{-Mu^2/2kT} v\left(\frac{M\mathbf{v}\mathbf{V}_0}{2kT} + \frac{M^2}{4(kT)^2}[V_0^2v^2 - (\mathbf{V}_0\mathbf{v})^2]\right)\int \vartheta^2\rho\, d\rho = \frac{4\pi Q^2e^2}{MkT} n_0\left(\frac{M}{2\pi kT}\right)^{3/2} e^{-Mu^2/2kT}\left(\mathbf{V}_0\mathbf{u} - V_0^2 + \frac{M}{2kT}[u^2V_0^2 - (\mathbf{V}_0\mathbf{u})^2]\right)\frac{1}{v^3}\ln\frac{R_D}{r_0}. \quad (36)$$

We note that when $eQ \ll kTR_D$ this formula is suitable for all body velocities $\mathbf{V}_0$. The formula (36) for I can, of course, be obtained without the use of (25), directly from definition (15) for I. For this purpose it is enough to calculate φ_q and f'_q by perturbation theory and substitute in (15). We note also that if $R_0 \ll R_D$, but at the same time $R_0^2 \sim (Q^2e^2/MkTV_0^2)\ln(R_D/r_0)$, we must add to (36) a term due to the scattering by the body itself, as given by (34) (when $V_0 \gg kT/M$). (The two effects are additive in the approximation considered here.)

4. CALCULATION OF THE EFFECTIVE SCATTERING CROSS SECTION

a) Slowly moving body. Substituting (27) in (22) and neglecting everywhere $\mathbf{V}_0$ compared with $\mathbf{u}$, we obtain

$$n_q = \frac{1}{i2q} n_0(\mathbf{n}\mathbf{V}_0)\frac{M}{kT}\overline{\sigma^* u}, \quad (37)$$

where $\overline{\sigma^* u}$ is the average value of $\sigma^*(u)u$, taken over a Maxwellian distribution. Substituting in (5), we obtain the effective cross section for a smooth sphere

$$d\sigma = \frac{1}{8\pi^3}\left(\frac{\omega_0}{c}\right)^4 \frac{(\mathbf{n}\mathbf{V}_0)^2 M}{kTq^2}\, \sigma_0^2 \sin^2\psi_1\, do. \qquad (38)$$

b) Rapidly moving large body. For this case $I(u)$ is given by formula (34). Let us calculate $n_\mathbf{q}$. We introduce the notation

$$A_1 = \frac{1}{i}\int \frac{I_1(\mathbf{u})\, d^3u}{\mathbf{n}\mathbf{u} - \mathbf{n}\mathbf{V}_0 - i\delta}$$

with A_2 defined analogously. To calculate A_1 we change over from integration over d^3u to integration over d^3v. The integration is best carried out in spherical coordinates with the polar axis along $\mathbf{n}$. In first approximation we can put $\mathbf{v} = \mathbf{V}_0$ in all the factors preceding the exponents and replace the lower limit 0 in the integration with respect to dv by $-\infty$. We then obtain readily

$$A_1 = n_0\sigma_0\pi/2. \qquad (39)$$

The quantity A_2 should be calculated in Cartesian coordinates u_x, u_y, and u_z with the x axis along $\mathbf{n}$. With the aid of (21) we obtain

$$A_2 = -n_0\sigma_0\left(\frac{MV_0^2}{2kT}\right)^{1/2}\left(\sqrt{\pi} + 2i\int_0^a e^{x^2}dx\right)e^{-a^2}. \qquad (40)$$

Ultimately

$$n_\mathbf{q} = \frac{n_0\sigma_0}{q}\left\{\left[\frac{\pi}{2} - \sqrt{\pi}\left(\frac{MV_0^2}{2kT}\right)^{1/2} e^{-a^2}\right] + i2\left(\frac{MV_0^2}{2kT}\right)^{1/2} e^{-a^2}\int_0^a e^{x^2}dx\right\}$$
$$\times\left[2\left(1 - ae^{-a^2}\int_0^a e^{x^2}dx\right) - ia\sqrt{\pi}e^{-a^2}\right]^{-1} \qquad (41)$$

and

$$d\sigma = \frac{\sin^2\psi_1}{16\pi^2}\left(\frac{\omega_0}{c}\right)^4 \frac{\sigma_0^2}{q^2}\left\{\left[\frac{\pi}{2} - \sqrt{\pi}\left(\frac{MV_0^2}{2kT}\right)^{1/2} e^{-a^2}\right]^2\right.$$
$$\left. + 4\left(\frac{MV_0^2}{2kT}\right)\left(e^{-a^2}\int_0^a e^{x^2}dx\right)^2\right\}\left[4\left(1 - ae^{-a^2}\int_0^a e^{x^2}dx\right)^2\right.$$
$$\left. + a^2\pi e^{-2a^2}\right]^{-1} \qquad (42)$$

(We recall that $a = \mathbf{n}\cdot\mathbf{V}_0\sqrt{M/2kT}$).

The expression (42) (for a given ψ) has a sharp minimum when $\mathbf{n}\cdot\mathbf{V}_0 \lesssim \sqrt{2kT/M}$ or, introducing the angle α between $\mathbf{n}$ and $\mathbf{V}_0$, when $|\alpha - \pi/2| \lesssim \sqrt{2kT/MV_0^2}$. This is as expected, since the trail of the body has a form which is prolate along $\mathbf{V}_0$, and scattering by a prolate body is always a maximum at the angle of specular reflection from its axis, in this case when $\mathbf{q}\cdot\mathbf{V}_0$ is close to zero. (If $\mathbf{q}\cdot\mathbf{V}_0 = 0$, then $\mathbf{k}\cdot\mathbf{V}_0 = \mathbf{k}'\cdot\mathbf{V}_0$, i.e., the angle of incidence is equal to the angle of reflection.) Let us note also that $n_\mathbf{q}$ increases sharply as $\mathbf{q}\to 0$. (Formally $n_\mathbf{q}\to\infty$ as $\mathbf{q}\to 0$. Actually, however, the formula becomes invalid when $q \sim 1/l$, where l is the mean free path.) This means, in particular, that the trail is capable of "focusing" all possible noises radiated in the plasma above the body.

Formula (41) is valid for sufficiently large wavelengths. It is easy to show that in this particular case this implies the requirement

$$qR_0 = 2\frac{\varepsilon\omega}{c}R_0\sin\frac{\psi}{2} \ll 1. \qquad (43)$$

It is seen from (42) that when ψ is not close to zero, and the angle α is not close to $\pi/2$, the order of magnitude of $d\sigma$ is

$$d\sigma \sim \left(\frac{\omega_0}{c}\right)^4 \frac{\sigma_0^2 c^2}{\varepsilon\omega^2}\, do.$$

Comparing this with Eq. (6) for the cross section of the scattering by the body itself, we find that the ratio of the intensity of scattering from the trail to the intensity of scattering from the body itself has an order of magnitude

$$d\sigma/d\sigma_m \sim (\omega_0/\omega)^4 c^2/\varepsilon\omega^2\sigma_0. \qquad (44)$$

The scattering cross section in the directions for which $|\alpha - \pi/2| \ll \sqrt{kT/MV_0^2}$ is much greater:

$$d\sigma \sim \frac{MV_0^2}{kT}\left(\frac{\omega_0}{c}\right)^4 \frac{\sigma_0^2 c^2}{\varepsilon\omega^2}\, do.$$

It is seen from (44) that actually, for ordinary body dimensions and not too long waves, the main contribution to the scattering in practically all directions is made by the body itself and not by the trail. We emphasize, however, that this conclusion has been obtained by neglecting the magnetic field, which can change the entire scattering picture appreciably. The influence of the magnetic field is the subject of another investigation. We indicate only that the results obtained in the present article are valid if

$$eH/Mc \ll q_z\sqrt{kT/M},$$

where q_z is the projection of $\mathbf{q}$ on the direction of $\mathbf{H}$.

Let us see now whether the perturbation-theory formula (5) is valid in our case. In order for (5) to be valid it is necessary to have $E' \ll E_0$ over significant distances. It can be verified that in our case the significant distances are $R \sim 1/q$, over which $E' \sim (\omega_0/c)^2\sigma_0 E_0$. This leads to the condition

$$(\omega_0/c)^2\sigma_0 \ll 1$$

(for analogous arguments to justify of the applicability of the Born approximation in quantum mechanics see reference 7). Since the only frequencies that can be used to investigate the scattering are $\omega > \omega_0$, the condition for the validity of formula (5) coincides in our case with the main condition (2).

c) Rapidly moving small body. Since the integral of I, as defined by (36) and contained in the expression for $n_{\mathbf{q}}$, cannot be calculated for an arbitrary body velocity, we confine ourselves here only to the case when $V_0 \gg \sqrt{kT/M}$. Neglecting u compared with V_0, we obtain

$$I = -\frac{2\pi Q^2 e^2}{MkT}\frac{1}{V_0^3}\ln\frac{R_D}{R_0} n_0 \left(\frac{M}{2\pi kT}\right)^{3/2} e^{-Mu^2/2kT} \times \left\{V_0^2 - \frac{M}{2kT}\left[V_0^2 u^2 - (\mathbf{V}_0\mathbf{u})^2\right]\right\}.$$

Calculating the integral of I with the aid of (21), we obtain

$$n_{\mathbf{q}} = \frac{2\pi Q^2 e^2 n_0}{q\,(2k^3T^3MV_0^2)^{1/2}} \ln\frac{R_D}{r_0}\left[V_0^2 - (\mathbf{V}_0\mathbf{n})^2\right]\left[(1-2a^2)\left(\sqrt{\pi} + 2i\int_0^a e^{x^2}dx\right)e^{-a^2} + 2ia\right]\left[2 - 2a\left(\int_0^a e^{x^2}dx - i\frac{\sqrt{\pi}}{2}\right)e^{-a^2}\right]^{-1}. \quad (45)$$

It is easy to verify that this formula is suitable when

$$q \ll 1/r_1, \qquad r_1 = (MkT)^{1/2} kTV_0/Qe^3 n_0 \ln(R_D/r_0).$$

In the case when $eQ/R_D \ll kT$, the formula for $n_{\mathbf{q}}$ for $q \gg 1/r_1$ can be obtained in the usual manner from the linearized kinetic equation, i.e., in the same approximation as used by Kraus and Watson[3] (a practically analogous calculation for the Fourier components of the potential was also made by Sitenko and Stepanov[8]). For the sake of completeness, we cite the corresponding formula:

$$n_{\mathbf{q}} = \frac{Q}{e}\left\{2\,(R_D q)^2 + \left[2 - 2a\left(\int_0^a e^{x^2}dx - \frac{i\sqrt{\pi}}{2}\right)e^{-a^2}\right]\right\}^{-1}. \quad (46)$$

Substituting (45) and (46) in (5) we obtain $d\sigma$. We shall not write out these formulas.

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