EMERGENCE INTO VACUUM OF THE CERENKOV RADIATION PRODUCED FROM LONGITUDINAL WAVES IN A MEDIUM

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The problem treated is that of the radiation from an electron that moves into vacuum from a medium with spatial dispersion. The additional boundary conditions required to take spatial dispersion into account are discussed. It is shown that under certain conditions in addition to the transition radiation there is Cerenkov radiation produced from longitudinal waves in the medium and emerging into the vacuum. For the case of small absorption in the medium the angular distribution of this radiation is determined by the law of refraction together with the condition for Cerenkov radiation in the medium. The main contribution to the transition radiation is the Cerenkov radiation from longitudinal waves, which is produced in the medium near the bounding surface (at a distance of the order of the wavelength). At large distances from the boundary this part of the Cerenkov radiation is a spherical wave and is confined to a narrow range of frequencies lying close to the plasma frequency.

1. INTRODUCTION

WHEN a fast charged particle moves in a region with spatial dispersion there is excitation of longitudinal waves (plasmons).[1,2] The resulting discrete energy loss of the particle is the Cerenkov radiation from the longitudinal waves (cf., e.g., reference 3).

The problem of the emergence of this radiation into vacuum is of undoubted interest, since the radiation has a narrow spectrum that lies near the plasma frequency $\omega_0^2 = 4\pi ne^2/m$ (n, e, and m are the density, charge, and mass of the electrons in the medium), and can be used both for the generation of radio waves (in a plasma) and also for the generation of infrared and ultraviolet rays (in a dielectric). It must be pointed out that Cerenkov radiation from transverse waves actually appears in a medium at relativistic speeds of the particle, whereas Cerenkov radiation from longitudinal waves appears at much smaller speeds. For Cerenkov radiation from longitudinal waves to occur, the speed of the particle must exceed the average thermal speed of the electrons in the medium.[1]

The following are special features of this problem: 1) the passage of a charged particle through the boundary between two media produces a transition radiation, which at large distances from the point of emergence of the particle is indistinguishable from the Cerenkov radiation[4]; 2) when spatial dispersion is taken into account there is a new solution of Maxwell's equations (longitudinal waves), which requires an additional boundary condition.[3]

Subject to a certain assumption regarding the additional boundary condition we find here the radiation fields for the passage of an electron from a medium into vacuum. As Garibyan has shown,[5,6] the transition radiation becomes actually appreciable only at ultrarelativistic speeds. At nonrelativistic particle speeds one can distinguish in the transition radiation a part of the spectrum in which the main contribution comes from the Cerenkov radiation from longitudinal waves.

2. STATEMENT OF PROBLEM AND CHOICE OF BOUNDARY CONDITIONS

Let the motion of the electron be normal to the boundary between the medium and vacuum. We shall regard the radiative energy loss of the electron as negligibly small in comparison with its kinetic energy. We take into account the effects of spatial dispersion for an isotropic medium by writing the connection between **D** and **E** in the form

$$\mathbf{D} = (\varepsilon + \delta\Delta)\,\mathbf{E}, \tag{1}$$

where **D** and **E** are the electric displacement and field strength at the point **r**, $\epsilon = \epsilon_0 + i\gamma$ is the complex dielectric constant of the medium, δ is a parameter that characterizes the spatial dispersion of the medium, and Δ is the Laplacian operator. The magnetic permeability of the medium is unity. The expression (1) is equivalent to the inclusion of a term in the dielectric constant proportional to the square of the wave vector.

We shall look for the field of the moving electron in the form of the sum of a longitudinal field (φ) and a transverse field (**A**) with the gauge div **A** = 0. We solve the problem by the method proposed by Ginzburg and Frank[7] and developed by Garibyan.[5] For the radiation fields $\varphi'(\mathbf{k})$ and $\mathbf{A}'(\mathbf{k})$ we get as the dispersion relations* for the longitudinal waves (for $\mathbf{k} \neq 0$)

$$k_{\parallel}^2 = \varepsilon/\delta, \tag{2}$$

where $k_{\parallel}^2 = \kappa^2 + \lambda_{\parallel}^2$, κ and $\lambda_{\parallel}$ being the tangential and normal components of the wave vector, and for the transverse waves

$$k_{\perp}^2 = \omega^2 c^{-2} \varepsilon_{\perp}, \tag{3}$$

where

$$\varepsilon_{\perp} = \varepsilon/(1 + \delta\omega^2/c^2), \qquad k_{\perp}^2 = \varkappa^2 + \lambda^2.$$

In vacuum $\varphi'(\mathbf{k}) = 0$.

As has been pointed out earlier, when spatial dispersion is taken into account the usual boundary conditions imposed on the normal and tangential components of the fields are insufficient. We can get a general formulation of the missing boundary condition by setting up a connection, containing some arbitrary constants, which is to hold between the amplitudes of the longitudinal fields and their derivatives at the boundary. Possible forms which are invariant with respect to rotation around the z axis and to reflection in the boundary plane are:

$$\varepsilon\delta\partial\varphi/\partial n + \alpha_1\delta\Delta(\partial\varphi/\partial n) = \alpha_2\varepsilon_{\perp}\delta\dot{A}_n + \alpha_3\delta\dot{A}_n + \alpha_4\delta\Delta\dot{A}_n, \tag{4}$$

$$\varepsilon\delta\partial\varphi/\partial\tau + \beta_1\delta\Delta(\partial\varphi/\partial\tau) = \beta_2\varepsilon_{\perp}\delta\dot{A}_\tau + \beta_3\delta\dot{A}_\tau + \beta_4\delta\Delta\dot{A}_\tau, \tag{5}$$

where δ is the parameter that characterizes the spatial dispersion, **n** and τ are the normal and tangent to the boundary plane, and α and β are constants. Dots denote differentiation with respect to time. Ginzburg[3] proposes as the additional boundary condition

$$\mathbf{D}' = \delta\Delta\mathbf{E} = 0. \tag{6}$$

For certain values of the constants α and β the conditions (4) and (5) go over respectively into the normal and tangential components of the condition (6).

If we take Eq. (5) as the additional boundary condition, then the amount of Cerenkov radiation from longitudinal waves emerging into vacuum will be the same for passage of the electron from the medium to vacuum and for its passage from vacuum to the medium. In particular, if we take as the additional boundary condition the tangential component of Eq. (6), then there will be no emergence at all of Cerenkov radiation from the medium into vacuum.

Because of this it seems to us that the condition (5) is physically unjustified. The condition (4), on the other hand, does not have this defect. It can be shown that the amount of Cerenkov radiation emerging into the vacuum has only a weak dependence on the constants α_i that appear in the condition (4). Therefore for a qualitative estimate of the emergence into vacuum of the Cerenkov radiation from longitudinal waves we shall use as the additional boundary condition the normal component of Eq. (6).

3. THE RADIATION FIELDS IN THE VACUUM

The radiation fields in the vacuum are found in exactly the way that has been described by Garibyan.[5] We present the results of the calculation of the radial component of the electric field strength:

$$E'_\rho(R, \theta, t) = \frac{e}{v\pi\sqrt{2\pi R \sin\theta}} e^{-3\pi i/4} \int_{-\infty}^{\infty} I(\omega) e^{-i\omega t}\, d\omega, \tag{7}$$

$$I(\omega) = \int_0^\infty F(\varkappa) e^{f(\varkappa)R}\, d\varkappa, \tag{8}$$

where the notations used are

$$F(\varkappa) = \varkappa^{3/2}\lambda_0\eta(\varkappa)/\zeta(\varkappa), \qquad f(\varkappa) = i\varkappa\sin\theta + i\lambda_0\cos\theta,$$

$$\eta(\varkappa) = \beta\frac{\omega}{c}\frac{(\varepsilon - 1 - \delta k^2)}{\mu\mu_0}\left(\varepsilon\lambda_{\parallel}\lambda + \frac{\delta\omega^2}{c^2}\varepsilon_{\perp}\varkappa^2\right) + \varepsilon_{\perp}\left(\frac{\varepsilon\lambda_{\parallel} + \delta k^2 k_z}{\mu(\varepsilon - \delta k^2)} - \frac{\varepsilon\lambda_{\parallel}}{\mu_0} - \frac{\delta k^2\omega}{c\mu}\beta\right),$$

$$\zeta(\varkappa) = \delta\omega^2 c^{-2}\varepsilon_{\perp}\varkappa^2 + \varepsilon\lambda_{\parallel}(\lambda + \varepsilon\lambda_0), \quad \mu = k^2 - \omega^2 c^{-2}(\varepsilon - \delta k^2),$$

$$\mu_0 = k^2 - \omega^2/c^2, \quad \lambda^2 = \omega^2 c^{-2}\varepsilon_{\perp} - \varkappa^2, \quad \lambda_0^2 = \omega^2/c^2 - \varkappa^2,$$

$$\lambda_{\parallel}^2 = \varepsilon/\delta - \varkappa^2, \quad \omega = k_z v, \quad \beta = v/c, \tag{9}$$

R is the distance from the point where the electron leaves the medium to the point of observation, and θ is the angle between R and the z axis. The real and imaginary parts of λ_0, λ, $\lambda_{\parallel}$ are positive. For $\delta \to 0$ Eq. (7) goes over into Eq. (23) of reference 5.

Besides the pole $\mu(\kappa) = 0$ considered by Garibyan,[5] the function $F(\kappa)$ that appears in Eq. (8) has a pole determined by the equation

*Here and in what follows the intermediate calculations are not presented (cf. reference 5).

$$\varepsilon - \delta(\varkappa^2 + \omega^2/v^2) = 0. \qquad (10)$$

As Garibyan has shown, the residue at the pole $\mu(\kappa) = 0$ gives the Cerenkov radiation from transverse waves. It will be shown below that the residue at the pole (10) gives the Cerenkov radiation arising in the medium from longitudinal waves.

The integral (8) is calculated by means of a formula of Van der Pol,[8] which takes into account the closeness of the pole to the saddle point. The result is as follows:

$$I(\omega) = -2\left(-\frac{2\pi}{f''(\varkappa_0)R}\right)^{1/2} F(\varkappa_0)\, w e^{f(\varkappa_0)R - w^2} \int\limits_{w}^{i\infty} e^{u^2} du, \qquad (11)$$

where $\kappa_0 = \omega c^{-1} \sin\theta$ is the saddle point of the function $f(\kappa)$; $w = \pm s_0 R^{1/2}$; $s_0^2 = f(\kappa_0) - f(\kappa_1)$; and $\kappa_1^2 = \epsilon/\delta - \omega^2/v^2$ is the pole determined from Eq. (10). The sign of the function w is chosen so that its values lie in the upper half-plane. The closeness of the pole to the saddle point must be taken into account for $|w|^2 \lesssim 1$. For $|w|^2 \gg 1$ we can expand the integral in Eq. (11) in powers of $1/w$, and thus get

$$I_1 = e^{f(\varkappa_0)}\left(-\frac{2\pi}{f''(\varkappa_0)R}\right)^{1/2} F(\varkappa_0). \qquad (12)$$

When we deform the original path of integration with respect to κ ($0 - \infty$) into the path of steepest descent, for $\gamma = 0$ and $v^2 \geq 4\delta\omega_0^2$ in a certain range of frequencies it is necessary to take into account the residue at the pole κ_1. We shall assume that the speed of the electron satisfies the condition $v^2 \gg 4\delta\omega_0^2$. Then the frequency range in question is determined by the inequalities

$$\omega_1^2 = \omega_0^2(1 + \delta\omega_0^2/v^2) \leqslant \omega^2 \leqslant \omega_0^2(1 + \delta\omega_0^2 a/v^2) = \omega_2^2, \qquad (13)$$

where $a = 1 + \beta^2 \sin^2\theta$. We have taken into account the fact that the influence of spatial dispersion is most important near the plasma frequency ω_0. For the real part of ϵ for $\omega > \omega_0$ we have used the expression $\epsilon_0 = 1 - \omega_0^2/\omega^2$.

If the imaginary part of ϵ, which determines the absorption, satisfies the condition

$$\gamma^2 \lesssim \delta^2\omega^3/Rc^3, \qquad (14)$$

then we have $|w|^2 \lesssim 1$ in a certain range of frequencies around ω' which is determined by the conditions

$$\frac{\partial |w|^2}{\partial\omega} = 0, \quad \Delta\omega \leqslant (1 - |w|^2_{\omega'})^{1/2}\left(\frac{\partial^2 |w|^2}{\partial\omega^2}\right)^{-1/2}_{\omega'}. \qquad (15)$$

The first of these conditions gives the value of ω',

$$(\omega')^2 = \omega_0^2(1 + \delta\omega_0^2 a/v^2) \qquad (16)$$

for $\gamma = 0$. If $\gamma \neq 0$, we get for ω' a value close to the value (16). Using Eq. (16), we get for the second of the conditions (15)

$$\Delta\omega \leqslant \frac{\omega_0}{2}\left[8\left(\frac{\omega_0}{c}\right)^3 \frac{\delta^2}{R} \sin^2\theta\cos^2\theta - \gamma^2\right]^{1/2}. \qquad (17)$$

The values of the integral (8) over the whole range of frequencies are as follows:

ω:	$\omega < \omega_1$	$\omega_1 \leqslant \omega \leqslant \omega' - \Delta\omega$	$\omega' - \Delta\omega \leqslant \omega \leqslant \omega_2$	$\omega_2 \leqslant \omega \leqslant \omega' + \Delta\omega$	$\omega > \omega' + \Delta\omega$
I:	I_1	$I_1 + I_3$	$I_2' + I_3$	I_2	I_1

Here I_1 is the steepest-descent value of the integral for $|w|^2 \gg 1$, I_2 is the value of the integral for $|w|^2 \lesssim 1$; and I_3 is the residue at the point κ_1. In I_2' the function w is taken with the minus sign, and in I_2 with the plus sign.[8] For $\gamma = 0$ the frequency ω' coincides with ω_2. If we take the limiting value $\gamma \sim (\delta^2\omega_0^3/Rc^3)^{1/2}$ for values of R that are small but not in contradiction with the validity of the method of steepest descent ($\omega_0 c^{-1} R \sin^2\theta \gg 1$), then I_2' becomes negligibly small and can be omitted.

The calculation of the normal component E_n' is made in an analogous way. For the total radiation into the vacuum we have

$$E' = -\frac{e}{v\pi R}\cos\theta\sin\theta \int\limits_{-\infty}^{+\infty}\left(\frac{\omega}{c}\right)^3 \frac{\eta(\varkappa_0)}{\zeta(\varkappa_0)} e^{i\omega(R/c - t)}\, d\omega$$

$$-\frac{\sqrt{2}e}{v\pi\sqrt{R}}\sin\theta e^{-3\pi i/4}$$

$$\times \int\limits_{\omega' - \Delta\omega}^{\omega' + \Delta\omega}\left(\frac{\omega}{c}\right)^{5/2} \frac{\eta(\varkappa_0)}{\zeta(\varkappa_0)}[\pm(\varkappa_1 - \varkappa_0)]\left(e^{-w^2}\int\limits_{w}^{i\infty} e^{u^2} du\right) e^{i\omega(R/c - t)}\, d\omega$$

$$-\frac{e}{v}\sqrt{\frac{2}{\pi R\sin\theta}}\, e^{-\pi i/4}\int\limits_{\omega_1}^{\omega_2} \varepsilon_{\perp}\frac{\omega^2}{vc}\frac{\varkappa_1^{1/2}}{\zeta(\varkappa_1)} e^{f(\varkappa_1)R - i\omega t}\, d\omega. \qquad (18)$$

It is easy to show that the magnetic field strength is $H' = E'$. The first term in Eq. (8) corresponds to I_1, the second to I_2, and the third to I_3. The choice of the sign of $(\kappa_1 - \kappa_0)$ in the second term corresponds to the choice of the sign of the function w. The regions of integration have been indicated above.

The third term in Eq. (18) represents the amount of Cerenkov radiation produced in the medium from longitudinal waves that emerge into the vacuum.[3] In fact, by expanding the function in the exponent, we find that the field of frequency ω is propagated at the angle $\vartheta(\omega)$,

$$\sin \vartheta(\omega) = c\omega^{-1}\sqrt{\varepsilon_0/\delta - \omega^2/v^2}, \qquad (19)$$

with the direction of motion of the electron. For the longitudinal waves the index of refraction is $n = c\omega^{-1}(\epsilon_0/\delta)^{1/2}$. By taking the condition for Cerenkov radiation in the medium and using the law of refraction, we get for $\vartheta(\omega)$ in the vacuum the expression (19).

The energy flux dW/ds through unit area during the entire time of flight of the electron can be divided into three parts, corresponding to the three terms in Eq. (18) (cf. also reference 5). The transition radiation, which corresponds to the first term in Eq. (18), is a spherical wave. The energy radiated into the solid angle $d\Omega = \sin\theta\, d\theta\, d\varphi$ with its vertex at the point where the electron passes out of the medium is given by

$$\frac{dW_1}{d\Omega} = \frac{ce^2}{v^2\pi^2}\cos^2\theta\sin^2\theta \int_0^{\infty}{}' \left| \left(\frac{\omega}{c}\right)^3 \frac{\eta(\varkappa_0)}{\zeta(\varkappa_0)} \right|^2 d\omega \qquad (20)$$

(the prime on the integral sign means that for values of R that satisfy the condition (14) an interval $\pm\Delta\omega$ around ω' is excluded from the region of integration). For the radiation in the interval $\pm\Delta\omega$ around ω' we get

$$\frac{dW_2}{ds} = \frac{2ce^2}{\pi^2 v^2 R}\sin^2\theta \times \int_{\omega'-\Delta\omega}^{\omega'+\Delta\omega} \left| \left(\frac{\omega}{c}\right)^{5/2} \frac{\eta(\varkappa_0)}{\zeta(\varkappa_0)} (\varkappa_1 - \varkappa_0) e^{-w^2} \int_w^{i\infty} e^{u^2} du \right|^2 d\omega. \qquad (21)$$

As R increases the value of w increases. Then $\Delta\omega$, and consequently also W_2, go to zero, and we must integrate over the entire range of frequencies in Eq. (20).

It must be noted that the condition for distinguishing the second term in Eq. (18) is the same as that for distinguishing the third term, which corresponds to the Cerenkov radiation ($v^2 \geq 4\delta\omega_0^2$). From this we can conclude that the second term in Eq. (18) is a superposition of the transition radiation and the Cerenkov radiation, which is produced near the boundary surface (at a distance of the order of the wavelength). At large distances from the point where the electron comes out, this part of the radiation goes over into a spherical wave. For the Cerenkov radiation, which corresponds to the third term of Eq. (18), the energy emitted between cones of aperature angles θ and $\theta + d\theta$ depends on the distance R and is given by the formula

$$\frac{dW_3}{d\theta} = \frac{4ce^2}{v^2} R \int_{\omega_1}^{\omega_2} \frac{\omega^4}{c^2 v^2} \left| \frac{\varepsilon_\perp \varkappa_1^{1/2}}{\zeta(\varkappa_1)} e^{f(\varkappa_1)R} \right|^2 \cos(\theta - \vartheta(\omega))\, d\omega. \qquad (22)$$

In the case of passage of an electron from vacuum to a medium the radiation field in vacuum is determined in the same way, the only difference being that the poles do not affect the calculations. Thus there remains only the transition radiation W_1; the amount of such radiation can be found from Eq. (20) by replacing v by $-$v.

4. QUALITATIVE ESTIMATE OF THE RESULTS

The case of motion of an electron from a medium into vacuum is interesting, since in this case besides the transition radiation there is Cerenkov radiation caused by the excitation of longitudinal waves in the medium.

The parameter δ that characterizes the spatial dispersion is of the order of the square of the Debye radius[3]: $\delta \sim \lambda_D^2 = kT/4\pi ne^2$. Owing to this the condition $v^2 \geq 4\delta\omega_0^2$ for the appearance of Cerenkov radiation can be written in the form $v^2 \geq \langle v^2 \rangle$ ($\langle v^2 \rangle$ is the mean square thermal speed of the electrons in the medium). As is well known (cf., e.g., references 1 and 2), the motion of an electron through a medium with speed satisfying $v^2 \geq \langle v^2 \rangle$ produces longitudinal waves (plasmons). In the following estimates we assume $v^2 \gg 4\delta\omega_0^2$, $v/c = \beta \ll 1$.

In the absence of absorption ($\gamma = 0$), we have $W_3 \to \infty$ for $R \to \infty$, which corresponds to the emergence of the Cerenkov radiation from a semi-infinite trajectory. The angular distribution is then determined by the law of refraction, Eq. (19).

When there is absorption the pole (10) must be taken into account for angles that satisfy the condition $\sin^2\theta > \gamma c^2/\delta\omega_0^2$. It can be seen from this that the Cerenkov radiation with the energy W_3 will be emitted between the bounding surface and a cone of aperture given by $\sin^2\theta \sim \gamma c^2/\delta\omega_0^2$. Moreover, it will be exponentially damped with increase of the distance.

For $\gamma > \delta\omega_0^2$ the pole (10) need not be considered. Both the emerging Cerenkov radiation and the transition radiation will be determined by the single term W_1, in which the integration is taken over the entire range of frequencies. For small values of γ ($\gamma \ll \delta\omega_0^2/c^2$), however, and values of R that are small but do not come within the zone of formation of the radiation ($\delta^2\omega_0^3/\gamma^2 c^3 \gtrsim R \gg c/\omega_0$), the value of W_3 is large. With this statement of the problem one cannot estimate the yield of Cerenkov radiation W_3 (the radiation losses are comparable with the kinetic energy).

Noting that both the spectrum (13) of the Cerenkov radiation and the range $\Delta\omega$ of frequencies around the value ω' of Eq. (16) lie in the neighborhood of the plasma frequency ω_0, let us use the following method to estimate the amount of Cerenkov radiation caused by the excitation of longitudinal waves in the medium when an electron passes through a plate.

Let us take a plate that is "thick" enough for the formation in it of a wave corresponding to the frequency ω_0 (and for the "one-boundary" approximation to the problem to be justified for this frequency), but "thin" enough for the absorption to be negligible. As Garibyan has shown,[6] a plate is thick if $d \gg c/\omega_0$, where d is the thickness of the plate, and it is assumed that $\beta \ll 1$. The absorption can be neglected for the longitudinal waves if $d \ll 1/k''_{\parallel}$, where $k''_{\parallel}$, is the imaginary part of the wave vector of Eq. (2). At the frequency ω' this condition means that $d \ll \delta\omega_0/v\gamma$. For the plate thickness d to satisfy both conditions it is necessary for the imaginary part of ϵ to satisfy the condition $\gamma \ll \delta\omega_0^2/c^2$. In this case both conditions on the thickness of the plate can be satisfied.

We can estimate the quantity W_3 by calculating for an electron moving in the medium the energy loss per unit path length owing to the Cerenkov radiation from longitudinal waves, and multiplying it by the thickness of the plate. We shall not find the loss here, however, since it has been calculated by many authors. The most exact calculation of the energy loss per unit time of an electron moving in a medium has been made by Larkin.[2] In our notation the formula is

$$dE/dt = e^2\omega_0^2 v^{-1} \ln (v/\omega_0 \sqrt{\delta}), \tag{23}$$

where it is assumed that $v^2 \gg 4\delta\omega_0^2$ (see also reference 1). Assuming that almost all of the energy radiated in the plate emerges into the vacuum, we get for the estimate of the quantity W_3

$$W_3 \approx e^2\omega_0^2 v^{-2} \ln (v / \omega_0 \sqrt{\delta}). \tag{24}$$

Comparison with the kinetic energy gives the ratio

$$\frac{W_3}{W_K} = \frac{e^2\omega_0^2 d}{mv^4} \ln \frac{v}{\omega_0 \sqrt{\delta}}. \tag{25}$$

Let us estimate the amount of Cerenkov radiation W_2 that arises at a distance of the order of a wavelength from the boundary of the film through which the electron emerges into the vacuum. For the given plate thickness ($\delta\omega_0/v\gamma \gg d \gg c/\omega_0$) we can use the expression (21) to estimate the quantity W_2. If γ satisfies the condition (14), we can distinguish W_2 from the transition radiation at distances R_0 in the range

$$\delta^2\omega_0^3 / \gamma^2 c^3 \gtrsim R_0 \gg c / \omega_0. \tag{26}$$

Taking the integrand of Eq. (21) at the point ω' given by Eq. (16) and multiplying by the frequency interval $\Delta\omega$ of Eq. (17), we get for W_2 the value

$$W_2 = \frac{4e^2\omega_0^2}{\pi v^2} R \int_{\theta_1}^{\theta_2} \left[2\delta^2 \left(\frac{\omega_0}{c}\right)^3 \frac{\sin^2\theta \cos^2\theta}{R} - \frac{\gamma^2}{4} \right]^{1/2} \frac{|\xi(\omega')|^2}{\sin\theta} d\theta, \tag{27}$$

where θ_1 and θ_2 are the roots of the radicand and

$$|\xi(\omega')| = |e^{-w^2(\omega')} \int_{w(\omega')}^{i\infty} e^{u^2} du|.$$

For a value of R_0 that satisfies the condition (26), the expression (27) can be estimated in order of magnitude as follows,

$$W_2 \approx \frac{e^2\omega_0}{c} \frac{\langle v^2 \rangle}{v^2} \sqrt{R_0 \frac{\omega_0}{c}}, \tag{27'}$$

where it is assumed that $\delta\omega_0^2 \approx \langle v^2 \rangle$.

Taking the ratio of W_2 to the kinetic energy of the electron, we find

$$\frac{W_2}{W_K} = \frac{e^2\omega_0}{mv^4} \frac{\langle v^2 \rangle}{c} \sqrt{R_0 \frac{\omega_0}{c}}. \tag{28}$$

The ratio W_3/W_2 is given in order of magnitude by

$$W_3 / W_2 \approx \frac{\omega_0 c}{\langle v^2 \rangle} \frac{d}{\sqrt{R_0\omega_0 / c}} \gg \frac{c^2}{\langle v^2 \rangle} \frac{1}{\sqrt{R_0\omega_0 / c}}. \tag{29}$$

Though at small distances R_0 the energy W_3 of the Cerenkov radiation is much larger than the quantity W_2, still, as was pointed out above, its value falls off exponentially with the distance. On the other hand the part of the Cerenkov radiation that corresponds to W_2 is converted at large distances into an undamped spherical wave.

We can estimate the value of γ for which such a physical picture can be observed, by starting from the condition (14) and noting that the condition $R\omega_0/c \gg 1$ for the applicability of the method of steepest descent must be satisfied. Assuming for dielectrics $\delta \sim 10^{-16}$, $\omega_0 \sim 10^{14}$ and taking $R\omega_0/c \sim 10^4$, we get the value $\gamma \sim 10^{-10}$. For a plasma, assuming $\delta \sim 10^{-6}$, $\omega_0 \sim 10^{10}$, $R\omega_0/c \sim 10^2$, we get $\gamma \sim 10^{-7}$.

Assuming $\beta \sim 10^{-2}$, we get for the thickness of the plate the conditions

$1 \gg d \gg 10^{-4}$ (dielectric) $10^3 \gg d \gg 1$ (plasma)

The quantity W_2 of Eq. (27), like the W_3 of Eq. (24), has a tendency to increase with decreasing speed of the electron. In this case, however, the method that has been presented does not apply, since the radiation losses become comparable with the kinetic energy W_K of the electron [the expressions (25) and (28) approach unity].

The transition radiation, which corresponds to the quantity W_1, has been qualitatively described by Garibyan.[5] Here we shall estimate its value for $\beta \ll 1$ for comparison with the Cerenkov radiation from longitudinal waves.

It is important to take spatial dispersion into account only in a frequency interval close to the plasma frequency ω_0. Since this frequency interval has been taken into account by the term W_2, we can make the estimate of the transition radiation (20) by neglecting spatial dispersion. Then Eq. (20) goes over into the formula of Garibyan [Eq. (28) of reference 5]:

$$W_1 = \frac{2e^2\beta^2}{\pi c}\int_0^{\pi/2} d\theta \frac{\sin^3\theta\cos^2\theta}{(1-\beta^2\cos^2\theta)^2}$$
$$\times \int_0^\infty \left(\frac{(\varepsilon - 1)(1 - \beta^2 - \beta\sqrt{\varepsilon - \sin^2\theta})}{(\varepsilon\cos\theta + \sqrt{\varepsilon - \sin^2\theta})(1 - \beta\sqrt{\varepsilon - \sin^2\theta})} \right)^2 d\omega. \quad (30)$$

Setting $\epsilon_0 = \text{const} \sim 1$, $\epsilon_0 - 1 \sim 1$ for $0 < \omega < \omega_0$ and $\epsilon_0 = 1 - \omega_0^2/\omega^2$ for $\omega_0 < \omega < \infty$, we get the following estimate:

$$W_1/W_K \approx e^2\omega_0/mc^3. \quad (31)$$

It can be seen from a comparison of Eq. (31) with Eqs. (25) and (28) that for $\beta \ll 1$ the transition radiation is much smaller than the emerging Cerenkov radiation that has been produced in the medium from longitudinal waves.

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