TEMPERATURE GREEN'S FUNCTION FOR ELECTRONS IN A SUPERCONDUCTOR

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The temperature Green's function for electrons in a superconductor are computed by the diagram technique for $T \neq 0$. An estimate is made of the region near the critical temperature for which the usual analysis, which makes use of a temperature-dependent gap in the excitation spectrum, is no longer valid. The magnitude of this temperature range is of the order of $(2\Delta(0)/\omega_0)^4 T_c$.

A characteristic of the electron excitation spectrum in superconductors is the appearance of an energy gap. Strictly speaking, the gap exists only at zero temperature. However, if we neglect scattering, then we can determine the temperature-dependent gap $\Delta(T)$ for $T > 0$.[1] Scattering leads to the appearance of finite damping in the excitations, which vanishes at $T = 0$ for minimum energy $\Delta(0)$.[2,3] For $T > 0$, even the excitations with energy $\Delta(T)$ have a finite lifetime. The very concept of the gap loses its meaning when the damping becomes comparable with $\Delta(T)$. Further, it will be shown that the temperature range in which it is not possible to use the gap concept is very small.

The most convenient way for studying this problem is the calculation of the Green's function. Without taking damping into account, Gor'kov[4] determined the damping at $T > 0$ for a superconductor with four-fermion interaction. For the problem under consideration, a model with electron-phonon interaction would be more suitable.

In a paper by the author[3] equations were obtained for the functions $G(\epsilon, p)$ and $F(\epsilon, p)$ for $T = 0$. (Similar equations are contained in the work of Buĭkov.[2] The notation used henceforth is that introduced in reference 3.) Using the diagram technique[5,6] for $T \neq 0$, we can immediately write down the corresponding equations for $T > 0$:

$$G(\varepsilon_n, p) = \frac{\varepsilon_n + \xi(p) + \Sigma_1(-\varepsilon_n, p)}{\Omega(\varepsilon_n, p)}, \qquad F(\varepsilon_n, p) = \frac{\Sigma_2(\varepsilon_n, p)}{\Omega(\varepsilon_n, p)}. \tag{1}$$

$$\Omega(\varepsilon_n, p) = [\varepsilon_n - \xi(p) - \Sigma_1(\varepsilon_n, p)][\varepsilon_n + \xi(p) + \Sigma_1(-\varepsilon_n, p)] - [\Sigma_2(\varepsilon_n, p)]^2; \tag{2}$$

$$\Sigma_1(\varepsilon_n, p) = -\frac{T}{(2\pi)^3}\sum_{n'}\int d\mathbf{p}'\, G(\varepsilon_{n'}, p')\, D(\varepsilon_n - \varepsilon_{n'};\ p - p'),$$

$$\Sigma_2(\varepsilon_n, p) = \frac{T}{(2\pi)^3}\sum_{n'}\int d\mathbf{p}'\, F(\varepsilon_{n'}, p')\, D(\varepsilon_n - \varepsilon_{n'};\ p - p'). \tag{3}$$

Here

$$\varepsilon_n = i(2n+1)\pi T, \qquad \xi(p) = v_0(p - p_0),$$

$$D(\omega_n, q) = \alpha_q^2\, 2\omega_q/(\omega_n^2 - \omega_q^2), \qquad \omega_n = i2n\pi T.$$

For $q \ll q_{max}$, the value $\alpha_q^2 = \lambda_0 \pi^2 sq/\sqrt{1-\lambda_0}\, p_0$, $\omega_q = \sqrt{1-\lambda_0}\, sq$, s is the velocity of sound ($\hbar = m = 1$).[7]

The function $G(\epsilon_n, p)$ coincides with the analytic continuation of the retardation function $G_R(\epsilon, p)$ $(n > 0)$ and the advance function $G_A(\epsilon, p)$ $(n < 0)$.[5] The same applies to the functions $F(\epsilon_n, p)$, $\Sigma_1(\epsilon_n, p)$ and $\Sigma_2(\epsilon_n, p)$. Making use of these properties, we can go from Eqs. (1) — (3) to integral equations. For this purpose, we note that when $n > 0$ we have the equality

$$\int_{C_1} dz\, G_R(z, p')\, D(\varepsilon_n - z, q) \tanh\frac{z}{2T} = 4\pi i T \sum_{n'=0}^{\infty} G(\varepsilon_{n'}, p')\, D(\varepsilon_n - \varepsilon_{n'}; q) + 2\pi i \alpha_q^2\left[G_R(\varepsilon_n + \omega_q; p')\tanh\frac{\varepsilon_n + \omega_q}{2T} - G_R(\varepsilon_n - \omega_q;\ p')\tanh\frac{\varepsilon_n - \omega_q}{2T}\right],$$

$$\int_{C_2} dz\, G_A(z, p')\, D(\varepsilon_n - z, q)\tanh\frac{z}{2T} = 4\pi i T \sum_{n'=-1}^{-\infty} G(\varepsilon_{n'}; p')\, D(\varepsilon_n - \varepsilon_{n'}; q).$$

The contours C_1 and C_2 encompass the upper and lower halfplanes, respectively, while integration over both contours is carried out in a counterclockwise direction. The real axis is a cut.

Since the integrals over the large semicircles are equal to zero, and since we have $G_A(\epsilon') = G_R^*(\epsilon')$ for real $z = \epsilon'$, we can then write

$$\Sigma_1(\varepsilon_n, p) = \frac{i}{2(2\pi)^4}\int_{-\infty}^{\infty} d\varepsilon' \tanh\frac{\varepsilon'}{2T} d\mathbf{p}' [G_R(\varepsilon', p') - G_R^*(\varepsilon', p')] D(\varepsilon_n - \varepsilon', p - p') + \frac{1}{2(2\pi)^3}\int d\mathbf{p}' \alpha^2_{p-p'} [G_R(\varepsilon_n + \omega_{p-p'}; p') + G_R(\varepsilon_n - \omega_{p-p'}, p')] \coth\frac{\omega_{p-p'}}{2T}$$

in place of (2). The first component contains only Im G_R. Making use of the well-known relation

$$G_R(\varepsilon, p) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Im} G_R(\varepsilon', p')}{\varepsilon - \varepsilon' + i\delta} d\varepsilon',$$

we can also express the second component in terms of Im G_R. Continuing both sides of the equation analytically from the upper halfplane to the real axis, we get

$$\Sigma_{1R}(\varepsilon, p) = -\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d\varepsilon' \int d\mathbf{p}' \operatorname{Im} G_R(\varepsilon', p') \times \alpha^2_{p-p'}\left[\frac{\tanh(\varepsilon'/2T) - \coth(\omega_{p-p'}/2T)}{\varepsilon' - \varepsilon - \omega_{p-p'} - i\delta} - \frac{\tanh(\varepsilon'/2T) + \coth(\omega_{p-p'}/2T)}{\varepsilon' - \varepsilon + \omega_{p-p'} - i\delta}\right]. \quad (4)$$

In the isotropic case under consideration, Σ_1 and Σ_2 are practically independent of p for $p \sim p_0$. We divide $\Sigma_{1R}(\epsilon)$ into an odd $f_0(\epsilon)$ and even $\mu_1 + if_1(\epsilon)$. Since the imaginary part of Σ_{1R} is an even function of ϵ, $f_0(\epsilon)$ is real. Let us consider Im $G_R(\epsilon, p)$. Bearing in mind that $\Sigma_1(-\epsilon_n)$ goes over into $\Sigma^*_{1R}(-\epsilon)$ upon analytic continuation from the upper half-plane, we can write

$$G_R(\varepsilon, p) = \frac{\varepsilon - f_0(\varepsilon) - if_1(\varepsilon)}{\Omega(\varepsilon, p)} + \frac{\xi(p) + \mu_1}{\Omega(\varepsilon, p)}.$$

Upon substitution of the second component in (4), we obtain an expression in which the important region of integration is far from the Fermi surface. This gives for the chemical potential μ a correction that is practically independent of T. The contribution of the first component comes from the region close to p_0. Therefore, we can go from integration over p′ to integration over $q = |\mathbf{p} - \mathbf{p}'|$, $\xi = v_0(p' - p_0)$, and the angle φ. As a result, we obtain the following after integration over ξ:

$$f(\varepsilon) = f_0(\varepsilon) + if_1(\varepsilon) = \frac{1}{8\pi^2 v_0}\int_{-\infty}^{\infty} d\varepsilon' \operatorname{Re}\left\{\frac{\varepsilon' - f(\varepsilon')}{\Omega_1(\varepsilon')}\right\} \operatorname{sign} \operatorname{Im} \Omega_1(\varepsilon') \varphi_1(\varepsilon', \varepsilon);$$

$$\varphi_1(\varepsilon', \varepsilon) = \int_0^{q_1} dq\, q\alpha_q^2 \left[\frac{\tanh(\varepsilon'/2T) - \coth(\omega_q/2T)}{\varepsilon' - \varepsilon - \omega_q - i\delta} - \frac{\tanh(\varepsilon'/2T) + \coth(\omega_q/2T)}{\varepsilon' - \varepsilon + \omega_q - i\delta}\right];$$

$$\Omega_1^2(\varepsilon) = [\varepsilon - f(\varepsilon)]^2 - \Sigma_2^2(\varepsilon), \qquad q_1 = \min\{2p_0, q_{max}\}.$$

We can establish the fact that sign Im $\Omega_1(\epsilon') = \operatorname{sign} \epsilon'$. If we neglect the imaginary parts of the functions f and Σ_2 in Ω_1 under the integral sign, then it is evident that Re $\Omega_1(\epsilon) = 0$ for $|\epsilon| \le \Delta$, where $\pm\Delta$ are the roots of the equation $[\epsilon - f_0(\epsilon)]^2 - [\operatorname{Re} \Sigma_2(\epsilon)]^2 = 0$.

Therefore, in the same approximation,

$$\operatorname{Re}\left\{\frac{\varepsilon - f(\varepsilon)}{\Omega_1(\varepsilon)}\right\} = \begin{cases}(\varepsilon - f_0(\varepsilon))/\Omega_1(\varepsilon), & |\varepsilon| > \Delta,\\ 0, & |\varepsilon| < \Delta,\end{cases}$$

and hence

$$f(\varepsilon) = \frac{1}{8\pi^2 v_0}\int_{-\infty}^{\infty} d\varepsilon' \frac{\varepsilon' - f_0(\varepsilon')}{\Omega_1(\varepsilon')} \theta(\varepsilon'^2 - \Delta^2)\varphi_1(\varepsilon', \varepsilon) \operatorname{sign} \varepsilon',$$

$$\Omega_1(\varepsilon) = [[\varepsilon - f_0(\varepsilon)]^2 - [\operatorname{Re}\Sigma_2(\varepsilon)]^2]^{1/2}. \quad (5)$$

An estimate will be given below of the imaginary parts, from which the limits of applicability of the approximations will be clear.

We note that only the possibility of discarding the imaginary parts for $\epsilon \lesssim \Delta$ is of importance for us. As will be seen, this possibility vanishes only in the immediate vicinity of the transition point. As in reference 3, we define the function $Q(\epsilon) = \Sigma_2(\epsilon)(1 - f(\epsilon)/\epsilon)^{-1}$. From the definition of Δ as a root of the equation $\Omega_1^2(\epsilon) = 0$, it follows that $\Delta = Q(\Delta)$. Replacing $Q(\epsilon)$ under the radical in the expression for Ω_1 by the constant Δ, we rewrite (5) in the following form:

$$f(\varepsilon) = \frac{1}{8\pi^2 v_0}\int_{\Delta}^{\infty} d\varepsilon' \frac{\varepsilon'}{\sqrt{\varepsilon'^2 - \Delta^2}}\int_0^{q_1} dq\, q\alpha_q^2 \left[\left(\tanh\frac{\varepsilon'}{2T} + \coth\frac{\omega_q}{2T}\right)\times\left(\frac{1}{\varepsilon' + \varepsilon + \omega_q + i\delta} - \frac{1}{\varepsilon' - \varepsilon + \omega_q - i\delta}\right) + \left(\tanh\frac{\varepsilon'}{2T} - \coth\frac{\omega_q}{2T}\right)\left(\frac{1}{\varepsilon' - \varepsilon - \omega_q - i\delta} - \frac{1}{\varepsilon' + \varepsilon - \omega_q + i\delta}\right)\right]. \quad (6)$$

The real part of this expression $f_0(\epsilon)$ does not differ appreciably from its value for $T = 0$. We consider the imaginary part $f_1(\epsilon)$ in more detail:

$$f_1(\varepsilon) = -\frac{1}{8\pi^2 v_0}\int_\Delta^\infty d\varepsilon' \frac{\varepsilon'}{\sqrt{\varepsilon'^2-\Delta^2}}\int_0^{q_1} dq\, q\alpha_q^2\Big[\Big(\tanh\frac{\varepsilon'}{2T} + \coth\frac{\omega_q}{2T}\Big)(\delta(\varepsilon'+\varepsilon+\omega_q)+\delta(\varepsilon'-\varepsilon+\omega_q)) + \Big(\coth\frac{\omega_q}{2T}-\tanh\frac{\varepsilon'}{2T}\Big)(\delta(\varepsilon'-\varepsilon-\omega_q) + \delta(\varepsilon'+\varepsilon-\omega_q))\Big]. \quad (7)$$

The first component vanishes for $|\epsilon| \le \Delta$. The second component is different from zero for all ϵ but vanishes for $T = 0$. The greatest interest for us lies in $f_1^{(2)}(\epsilon)$, since it determines the magnitude of the damping of excitations with minimum energy Δ. Moreover, the imaginary part of $G_R(\epsilon, p)$ is connected with $f_1^{(2)}(\epsilon)$ for $\epsilon \le \Delta$.

Since the small $\omega_q \sim T \ll \Theta$ (Θ is the Debye temperature) make a contribution to $f_1^{(2)}$, we can write

$$f_1^{(2)}(\varepsilon) = -\frac{\pi\lambda_0}{(1-\lambda_0)^2}\frac{1}{\omega_0^2}\int_\Delta^\infty d\varepsilon' \frac{\varepsilon'}{\sqrt{\varepsilon'^2-\Delta^2}} \times\Big\{\frac{\varepsilon'^2+\varepsilon^2}{e^{\varepsilon'/T}+1} + \frac{(\varepsilon'-\varepsilon)^2}{e^{(\varepsilon'-\varepsilon)/T}-1} + \frac{(\varepsilon'+\varepsilon)^2}{e^{(\varepsilon'+\varepsilon)/T}-1}\Big\}, \quad (8)$$

where $\omega_0 = 2sp_0$.

Let us consider some limiting cases.

1. $T \ll \Delta$, $(\Delta - \epsilon)/T \gg 1$. In this case, the imaginary part is an exponentially small quantity:

$$f_1^{(2)}(\varepsilon) = -\frac{\pi^{3/2}\lambda_0}{(1-\lambda_0)^2}\Big(\frac{T}{2\Delta}\Big)^{1/2}\frac{1}{\omega_0^2}[\varepsilon^2+\Delta^2 + (\varepsilon+\Delta)^2 e^{-\varepsilon/T} + (\Delta-\varepsilon)^2 e^{\varepsilon/T}]\, e^{-\Delta/T}\Delta. \quad (9)$$

2. $T \ll \Delta$, $(\Delta - \epsilon)/T \ll 1$. For ϵ sufficiently close to Δ, the dependence of $f_1^{(2)}$ on T become a power dependence:

$$f_1^{(2)}(\varepsilon) = -\frac{\pi^{3/2}\lambda_0}{(1-\lambda_0)^2}\Big(\frac{T}{\omega_0}\Big)^2\Big(\frac{T}{2\Delta}\Big)^{1/2}\Delta. \quad (10)$$

3. $T \gg \Delta$, $\epsilon \lesssim \Delta$. In this case the expression does not contain Δ and therefore coincides with that part of the damping of the excitations in the normal metal which is connected with the absorption of phonons:

$$f_1^{(2)}(\varepsilon) = -\frac{4\pi\lambda_0}{(1-\lambda_0)^2}\Big(\frac{T}{\omega_0}\Big)^2 T\int_0^\infty dx\frac{x^2e^x}{e^{2x}-1} \approx -\frac{8{,}2\,\pi\lambda_0}{(1-\lambda_0)^2}\Big(\frac{T}{\omega_0}\Big)^2 T. \quad (11)$$

The equation for $\Sigma_2(\epsilon)$ can be obtained by starting out from the same considerations as in the derivation of (6). We write out the result:

$$Q(\varepsilon) \equiv \frac{\Sigma_2(\varepsilon)}{1-f(\varepsilon)/\varepsilon} = \frac{1}{8\pi^2 v_0}\frac{1}{1-f(\varepsilon)/\varepsilon}\int_\Delta^\infty d\varepsilon'\frac{C(\varepsilon')}{\sqrt{\varepsilon'^2-\Delta^2}} \times\int_0^{q_1} dq q\alpha_q^2\Big[\Big(\tanh\frac{\varepsilon'}{2T}+\coth\frac{\omega_q}{2T}\Big)\Big(\frac{1}{\varepsilon'+\varepsilon+\omega_q+i\delta} + \frac{1}{\varepsilon'-\varepsilon+\omega_q-i\delta}\Big) + \Big(\coth\frac{\omega_q}{2T}-\tanh\frac{\varepsilon'}{2T}\Big) \times\Big(\frac{1}{\varepsilon'-\varepsilon-\omega_q-i\delta}+\frac{1}{\varepsilon'+\varepsilon-\omega_q+i\delta}\Big)\Big]. \quad (12)$$

Here $C(\epsilon) = \mathrm{Re}\, Q(\epsilon)$. It is not difficult to prove that the equation can be written in the following form, with accuracy up to quantities of the order of $(T_c/\omega_0)^2$:

$$C(\varepsilon) = \frac{1}{4\pi^2 v_0}\frac{1}{1-f_0(\varepsilon)/\varepsilon}\int_\Delta^\infty d\varepsilon'\frac{C(\varepsilon')}{\sqrt{\varepsilon'^2-\Delta^2}} \tanh\frac{\varepsilon'}{2T}\int dq q\alpha_q^2\Big(\frac{1}{\varepsilon'+\varepsilon+\omega_q}+\frac{1}{\varepsilon'-\varepsilon+\omega_q}\Big). \quad (13)$$

This equation differs from Eq. (37) of reference 3 only in the presence of $\tanh(\epsilon'/2T)$. Thanks to this, the usual relation between T_c and $\Delta(0)$ is preserved. The imaginary part of $\Sigma_2(\epsilon)$, together with Δ, vanishes for $T = T_c$ and therefore is of no interest to us.

In conclusion, let us consider the problem of at what temperatures $f_1(\epsilon)$ becomes comparable with Δ and when the concept of shell loses its meaning. It follows from (11) that $|f_1|/\Delta \sim 1$ for $\Delta/T_c \sim (8\pi\lambda_0/(1-\lambda_0)^2)(T_c/\omega_0)^2$, i.e.,

$$T_c - T \sim \frac{64\lambda_0^2}{(1-\lambda_0)^4}\Big(\frac{T_c}{\omega_0}\Big)^4 \approx \frac{0.4\lambda_0^2}{(1-\lambda_0)^4}\Big(\frac{2\Delta(0)}{\omega_0}\Big)^4.$$

Since $T_c \ll \omega_0$ for all known superconductors, the interval of temperatures close to T_c in which the usual approach becomes unsuitable is very small. Even for lead, for which T_c/ω_0 is comparatively large (~ 0.1), the value $(T_c - T)/T_c \lesssim 0.01$.

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